Episodic Liquidity Crises:

Cooperative and Predatory Trading

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Abstract

We develop a theoretical model to explain how episodic illiquidity can arise from a breakdown in cooperation between traders and be associated with predatory trading. In a multi-period framework, and with a continuous-time stage game with an asset-pricing equation that accounts for transaction costs, we describe an equilibrium where traders cooperate most of the time through repeated interaction and provide 'apparent liquidity' to each other. Cooperation can break down, especially when the stakes are high, and lead to predatory trading and episodic illiquidity. Equilibrium strategies involving cooperation across markets can cause the contagion of illiquidity.

Why is illiquidity rare and episodic? Pastor and Stambaugh (2003) detect only 14 aggregate low-liquidity months in the time period 1962-1999. The origin of this empirical observation still remains a puzzle. In this paper, we develop a theoretical model in which episodic illiquidity results from a breakdown in cooperation between traders in the market and manifests itself in predatory trading. This mechanism explains how sudden illiquidity appears, even in the absence of observable market distress.

We develop a dynamic model of trading based on liquidity needs. During each period, a *liquidity event* may occur in which a trader is required to liquidate a large block of an asset in a relatively short time period. This need for liquidity is observed by a tight oligopoly, whose members may choose to predate or cooperate. Predation involves racing and fading the *distressed* trader to the market, causing an adverse price impact for the trader[1]. Cooperation involves refraining from predation and allows the distressed trader to transact at more favorable prices. In our model, traders cooperate most of the time through repeated interaction, providing 'apparent liquidity' to each other. However, episodically this cooperation breaks down, especially when the stakes are high, leading to opportunism and loss of this apparent liquidity.

The following quote provides a recent example of an episodic breakdown in

[1]Predatory trading has been defined by Brunnermeier and Pedersen (2004) as trading that induces and/or exploits another investor's need to change their position. It is important to distinguish predatory trading from front-running. Front-running is an illegal activity in which a specialist, acting as an agent of an investor, trades on his own account in the same direction as his client before he fulfills his client's order. In this way, the specialist profits but violates his legal obligation as an agent of the investor. Predatory activity occurs in the absence of such a legal obligation.

cooperation between cooperative periods in the European debt market (New York Times Sept. 15, 2004):

> "*...The bond sale, executed Aug. 2, caused widespread concern in Europe's markets. Citigroup sold 11 billion euros of European government debt within minutes, mainly through electronic trades, then bought some of it back at lower prices less than an hour later, rival traders say. Though the trades were not illegal, they angered other bond houses, which said the bank violated an unspoken agreement not to flood the market to drive down prices.*" [2]

This suggests that market participants cooperate, though there is episodic predation which leads to acute changes in prices. Note that predatory behavior can involve either exploiting a distressed trader's needs or inducing another trader to be distressed.

There exists empirical evidence that cooperation affects price evolution and liquidity in financial markets. Cocco, Gomes, and Martins (2003) detect evidence in the Interbank market that banks provide liquidity to each other in times of financial stress. They find that banks establish lending relationships in this market to provide insurance against the risk of shortage or excess of funds during the reserve maintenance period. Cooperation and reputation have been documented to affect liquidity costs on the floor of the New York Stock Exchange (NYSE). Battalio, Ellul, and Jennings (2004) show an increase in liquidity costs in the trading days surrounding a stock's relocation on the floor

[2] On February 2, 2005 the Wall Street Journal reported that this predatory trading plan was referred to as "Dr. Evil" by traders working at Citicorp.

of the exchange.[3] They find that brokers who simultaneously relocate with the stock and continue their long-term cooperation with the specialist obtain a lower cost of liquidity, which manifests in a smaller bid-ask spread.[4]

In our *predatory stage game*, each trader faces a differential game with other strategic traders, that is a dynamic game in continuous time. Our stage game is related to the model by Brunnermeier and Pedersen (2004), but distinct in many respects. Because we use a pricing equation that accounts for the effect of trading pressure on price, the strategic traders, as a group, suffer surplus loss when predatory trading is present. This surplus loss motivates the traders to cooperate and provide liquidity to each other in our repeated game. In the formulation by Brunnermeier and Pedersen, no transaction costs are incurred in the equilibrium solution and all gains by the predators are exactly offset by losses by distressed traders, so that there would be no feasible Pareto improvement in a repeated game.[5] [6] In the equilibrium of our stage game, traders 'race' to market, selling quickly in the beginning of the period, at an exponentially decreasing rate. Also in equilibrium, predators initially race the distressed

[3]This is an exogenous event that changes long-run relationships between brokers and the specialist.

[4]Other articles in this literature include Berhardt, Dvoracek, Hughson, and Werner 2004; Desgranges and Foucault 2002; Reiss and Werner 2003; Ramadorai 2003; Hansch, Naik, and Viswanathan 1999; Massa and Simonov 2003

[5]There are other substantial differences between the models. Brunnermeier and Pedersen impose exogenous holding limits ($[-\bar{x}, \bar{x}]$) for traders, whereas we do not make this restriction. Our model involves a stochastic price process, while in Brunnermeier and Pedersen the asset pricing relationship is deterministic. Finally note that Brunnermeier and Pedersen's model predicts "price-overshooting", whereas our model does not. However, this is a consequence of, in our stage game model, all traders having an identical time horizon. If we relax this as in Brunnermeier and Pedersen to allow predators a longer horizon, price overshooting is also observed in our model.

[6]Attari, Mello, and Ruckes (2004) also describe predatory trading behavior with a two-period model. They show that predators may even lend to others that are "financially fragile" because they can obtain higher profits by trading against them for a longer period of time. Our paper generalizes their model in a multi-period framework, with each period in a continuous-time setting.

traders to market, but eventually 'fade' them and buy back. This racing and fading behavior is well-known in the trading industry and has been previously modelled by Foster and Viswanathan (1996). The associated trading volumes are also consistent with the U-shaped daily trading volume seen in financial markets.

We model cooperation by embedding the predatory stage game in a dynamic game. We first consider an infinitely-repeated game in which the magnitude of the liquidity event is fixed. In this framework, there exists an extremal equilibrium which is Pareto superior for the traders. We show that traders are more likely to cooperate in markets where assets are thinly-traded (*i.e.*, thin corporate bond issues, exotic options, credit derivatives) and in markets with low transaction costs. Further, we show that the need for liquidity over time needs to be sufficiently symmetric between the traders for cooperation to be maintained. The existence of asymmetric distress probabilities leads to abandonment of cooperation in equilibrium.

We extend the model to episodic illiquidity by allowing the exogenous magnitude of the liquidity event in the repeated game to be stochastic. Given such stochastic liquidity shocks, we provide predictions as to the magnitude of liquidity event required to trigger liquidity crises and describe how a breakdown in cooperation leads to price volatility. Finally, we allow for multimarket contact in the stochastic version of our dynamic game. This increases cooperation across markets, but leads to contagion of predation and liquidity crises across all markets.[7]

We note a few of the empirical implications of our model. First, our model predicts that non-anonymous markets should be stable most of the time with high ‘apparent’ liquidity, but will experience illiquidity in an episodic fashion. This fact is consistent with the observed rareness of liquidity events in financial markets (Pastor and Stambaugh 2003; Gabaix, Krishnamurthy, and Vigneron 2004). Second, in markets with thin issues and low transaction costs, the disappearance of apparent liquidity should be the most marked when these episodes occur. Third, occasions where one player faces extreme financial distress are associated with periods of reduced liquidity. Finally, our model suggests that illiquidity is usually observed across markets (*i.e.*, contagion occurs), and not in isolation.

The paper is organized as follows. Section 1 introduces the pricing relationship and sets up the stage game. We derive closed-form solutions for the trading dynamics and quantify the surplus loss due to competitive trading. Section 2 uses the stage game with one predator and one distressed trader as a basis and provides the solution to the supergame with the magnitude of the liquidity event fixed. Section 2 also models the relationship between insiders and outsiders in these markets. Section 3 models episodic illiquidity by having the magnitude of the liquidity shocks be stochastic. Contagion of illiquidity across markets

[7] Our mechanism for contagion is different from that of Brunnermeier and Pedersen (2004). The contagion in Brunnermeier and Pedersen (2004) is caused by a wealth effect. As prices in the market drop, additional traders are induced into a state of distress and a market-wide sell-off is observed. While this is a reasonable description of contagion during market distress, it does not explain why low-liquidity periods occur even in the absence of market downturns (Pastor and Stambaugh 2003). Our model predicts that contagion of illiquidity can occur in the absence of wealth effects.

is also addressed in this section. Section 4 concludes. Appendix A contains proofs. While the stage-game solution in Section 1 is based on the equilibrium over *open-loop* strategies, in Appendix B we consider the equilibrium over *closed-loop* strategies and argue that results are not qualitatively different as a consequence.

1 Trading and Predation

1 A Asset price model

The economy consists of two types of participants. The *strategic traders*, $i = 1, 2, ..., n$, are risk-neutral and maximize trading profits. These traders form a tight oligopoly over order flow in financial markets. Large traders are usually present in markets as proprietary trading desks, trading both on their own accounts as well as for others. The strategic traders observe the order flow and have inside information regarding transient liquidity needs within the market. They attempt to generate profits through their ability to forecast price moves, and to affect asset prices.

The other players are the *long-term investors* who form the competitive fringe. The long-term investors usually trade in the interest of mutual funds or private clients and exhibit a less aggressive trading strategy. Long-term investors are more likely to take a "buy and hold strategy", limit the number of transactions that they undertake, and avoid taking over-leveraged positions. The long-term investors trade according to fundamentals. The primary difference between the two types of traders is that the long-term investors are not aware of transient liquidity needs in the economy.

There exist a risk-free asset and a risky asset, traded in continuous-time. The aggregate supply $S > 0$ of the risky asset at any time t is divided between the strategic investors' holdings X_t and the long-term investors' holdings Z_t such that $S = X_t + Z_t$. The return on the risky asset is stochastic. The yield on the risk-free asset is zero.

The asset is traded at the price

$$P_t = U_t + \gamma X_t + \lambda Y_t, \tag{1}$$

where

$$dX_t = Y_t dt, \tag{2}$$

and U_t is the stochastic process $dU_t = \sigma(t, U_t)dB_t$, with B_t some one-dimensional Brownian motion on $(\Omega, \mathcal{F}, P)$.

A similar pricing relationship was previously derived by Vayanos (1998), and as well as by Gennotte and Kyle (1991) who show that it arises from the equilibrium strategies between a market maker and an informed trader when the position of the noise traders follows a smoothed Brownian motion. Likewise, Pritsker (2004) obtains a similar relationship for the price impact of large trades when institutional investors transact in the market. [8] [9]

[8] Directly linking trading pressure and price distinguishes our model from Brunnermeier and Pedersen (2004), where transaction costs are modeled via an exogenous parameter A (the maximum trading rate at which transaction costs are avoided), which does not directly affect prices in the market. This results in there being no surplus to be gained from cooperation.

[9] Further motivation for our use of this pricing relationship are the empirical and theoretical studies that link trading pressure and asset prices (Keim and Madhavan 1996; Kaul, Mehrotra, and Morck 2000; Holthausen, Leftwich, and Mayers 1990; Chan and Lakonishok 1995; Bertsimas and Lo 1998; Fedyk 2001; DeMarzo and Uroševic 2000; Almgren and Chriss 2000; Almgren and Chriss 1999; Huberman and Stanzl 2000).

The pricing equation is composed of three parts. U_t represents the expected value of future dividends and is modeled as a martingale stochastic diffusion process. The diffusion does not include a drift term, which is justified by the short-term nature of the events modeled. The results described here can be derived with the inclusion of a drift term in the diffusion, but with considerable loss in clarity of exposition.[10]

The second and third terms decompose the effect of liquidity on the asset price into permanent and temporary components. The effect of liquidity risk on asset prices has been studied by Acharya and Pedersen (2005) and others, and a related decomposition into short- and long-term effects has been considered by Sadka (2005).

In the second term, X_t is the inventory variable in the economy, which measures the amount of the asset that the strategic traders hold at time t. As X_t increases, the supply available to the long-term investors decreases and the price at which they can access the asset increases. The model parameter γ measures the permanent liquidity effects of trading. That is, it measures the change in price of the asset which is independent of the rate at which the asset is traded. Note that the level of asymmetric information in an asset is likely to be a major determinant of γ, as demand for the asset will then play a more important role in price formation. For instance, we expect a AAA-rated

[10] More precisely, the assumption is that the difference between the drift coefficient and the continuous-time discount factor is zero. For the multi-period game which we will later discuss, the assumption is that T is relatively small, that is the distress and predation events develop over short periods of time, and the discounting over each period is therefore not significant. The period-to-period discount factor is then also close to one. Since each period is short, the multi-stage game will consist of many short periods, where the probabilities of a player being distressed in any given period are small, so that the period-to-period discount factor is significant to the problem.

corporate bond to have lower asymmetric information associated with it then a B-rated bond. In our model, the AAA-rated bond should then have a lower γ than the B-rated bond. Likewise, an asset with concentrated ownership should have a higher level of asymmetric information (and therefore a higher γ) than an asset with a more dispersed ownership structure. For an asset with more asymmetric information, the market will more strongly adjust the asset price based on the net change in the supply of the asset.

The third term measures the instantaneous, reversible price pressure that occurs as a result of trading. Y_t is the aggregate rate of trading of the asset by the strategic traders. The faster the traders sell, the lower a price they will realize. This leads to surplus loss effects, which are discussed in the following subsections. The price-impact parameter λ measures the temporary, reversible asset price change that occurs during trading. Trading volume, concentration of ownership, and shares outstanding are all likely to play a role in the level of λ.

In the following sections, we use γ and λ to predict which securities in the market are more prone to predatory and to cooperative trading, and to episodic illiquidity. We show that the magnitude of the ratio $\frac{\gamma}{\lambda}$ is the key determinant of predatory or cooperative trading behavior.

1 B Stage Game: Trading Dynamics

The stage game that we consider is a game of complete information. Many assets that are prone to illiquidity are traded in non-anonymous markets in which a few large dealers dominate order flow. Further, roughly half of the trading volume at the New York Stock Exchange is traded in blocks over $10,000$ shares (Seppi 1990) and much of that occurs in the "upstairs" market, which is non-anonymous. As a result, the liquidity needs of large traders are usually observed quickly by others.

To understand the intuition behind this choice of structure for the game, consider a thinly-traded corporate bond issue that is traded by a small number of broker-dealers. Trading occurs either by direct negotiation over the phone, or by "sunshine" trading in which a mini-auction is held. The players are well-known to each other because they deal repeatedly with each other. Their trading habits and strategies are common knowledge.[11] When one trader needs to trade a large block of shares of an asset, this need is observed by others in the market and the optimal trading strategies solve a game of complete information.

In the stage game, strategic traders are either *distressed* or are *predators*. A liquidity event occurs at time $t = 0$, whereby the distressed traders are required buy or sell a large block of the asset Δx in a short time horizon T (say, by the end of the trading day). Forced liquidation usually arises because of the need to offset another cash-constrained position such as an over-leveraged position, or it occurs as a result of a risk management maneuver. The predators are informed of the trading requirement of the distressed traders and compete strategically in the market to exploit the price impact of the distressed trader's selling. For

[11] A particularly clear example of this is the Mortgage market developed by Salomon Brothers. Once this market was established and profitable, many of Salomon's mortgage traders were hired by other investment banks to run their mortgage desks. As a consequence, the trading habits of all of the desks were especially well-known to each other.

clarity of exposition, we assume that the opportunistic traders must return to their original positions in the asset by the end of the trading period, and that the distressed traders are informed of this requirement.[12] Further, we assume that, except for their trading targets, all strategic traders are identical. That is, the only difference between the two types of strategic traders is that the trading target for the distressed traders is Δx, and zero for the predatory traders.

At the start of the stage game, every trader chooses a trading schedule (Y_t^i) over the period $[0, T]$ to maximize their own expected value, assuming the other traders will do likewise. Subject to their respective initial and terminal holding constraints, they solve the following dynamic program

$$\begin{aligned} \text{maximize} \quad & \mathbf{E} \textstyle\int_0^T -P_t Y_t^i \, dt \\ \text{subject to} \quad & X_0^i = x_{0i} \\ & X_T^i = x_{Ti} \end{aligned} \tag{3}$$

by choosing a trading function Y_t^i. The expectation is over the Brownian motion's measure. We restrict our analysis of equilibrium solutions of this differential game to open-loop Nash equilibria, in which each trader chooses *ex ante* a time-dependent trading strategy that is the best response to the other traders' expected actions. As noted in proof in appendix, convexity ensures that the problem is well-posed.[13] The equilibrium solution is weakly time consistent.

[12] A variation on this model would be to allow the opportunistic traders to trade over a longer time period than the distressed traders. The solution for such a model is similar. However, the predatory trader will now choose what position to have by the end of the distressed seller's deadline. This choice is made by maximizing the expected value from trading over the distressed seller's period, plus the expected value from selling the position at the end of that period at a constant rate over the additional time. (See also Footnote 5.)

[13] Convexity ensures that any solution is identical *almost everywhere* to the one given here (*i.e.*, has the same integrals). Further restrictions can be imposed on the Y_t if desired, such as

The solution to the subproblem over the inteval $[t_1, T]$ (with initial conditions as given by the solution of the $[0, T]$-problem at time t_1) is the truncation of the $[0, T]$-solution over that sub-interval.[14] Once trading is underway, and at any given point in the trading schedule, no trader has a rational reason to deviate from the chosen trading function, as long as other traders do not deviate in response (either because they are able to credibly commit, or because they cannot observe the first player's deviation). Traders are not able to benefit from altering their trading strategy part-way through the game, as long as other players stick to their strategies.[15]

The following result outlines the Nash equilibrium solution for the traders. This formulation will serve a basis for deriving the equilibrium strategies when several distressed traders are present without opportunism and when there are both opportunistic and distressed traders present in the economy. It will also allow for analysis of trader surplus, which will motivate cooperation between traders in the repeated game.

Result 1 *(General Solution) Consider N traders that choose a time-dependent trading rate Y_t^i to solve the optimization problem in Equation 3, subject to the*

being of bounded variation. Since the traders solve an open-loop problem, Y_t^i can be defined as a functional rather than a process (by the same argument randomized solutions can be excluded, and the Brownian motion is not in the information set). Alternatively, smoothness can be ensured by defining the solution to the continuous-time problem as the limit of the solutions to a sequence of discrete-time problems, which is the approach we will use to analyze the closed-loop version of the problem in Appendix B.

[14] See Theorem 6.12 in Basar and Olsder (1999) and related discussion.

[15] In Appendix B we analyze closed-loop strategies, where players take into account the other players' response functions and are able to change their trading strategies part-way through the game. Solutions to the closed-loop problem are strongly time consistent. Our analysis, which we also confirmed with numerical experiments, suggests that the equilibrium trading strategies are qualitatively similar, and that the welfare loss is somewhat higher than what we obtain with open-loop strategies in Section 1 C. In the repeated-game analysis of Section 2, this would make the players more likely to cooperate.

asset price given by Equation 1. Then, the unique open-loop Nash equilibrium in this game is for trader i to trade according to the function

$$Y_t^i = a\, e^{-\frac{n-1}{n+1}\frac{\gamma}{\lambda}t} + b_i\, e^{\frac{\gamma}{\lambda}t}, \tag{4}$$

with $a \in \mathbf{R}$, and $b_i \in \mathbf{R}$, $i = 1, \ldots, n$, such that $\sum_{i=1}^{n} b_i = 0$. The coefficients a and b_i are uniquely determined from the trading constraints to be

$$\begin{aligned} a &= \frac{n-1}{n+1}\frac{\gamma}{\lambda}\left(1 - e^{-\frac{n-1}{n+1}\frac{\gamma}{\lambda}T}\right)^{-1} \frac{\sum_{i=1}^{n} \Delta x_i}{n} \\ b_i &= \frac{\gamma}{\lambda}\left(e^{\frac{\gamma}{\lambda}T} - 1\right)^{-1} \left(\Delta x_i - \frac{\sum_{j=1}^{n} \Delta x_j}{n}\right), \end{aligned} \tag{5}$$

where $\Delta x_i = x_{Ti} - x_{0i}$.

Proof. See appendix. ■

The equilibrium trading strategy in Equation (4) is composed of two parts. For small t, the first component dominates the trading strategy and for larger t within the interval T, the second part dominates. The first term, $a\, e^{-\frac{n-1}{n+1}\frac{\gamma}{\lambda}t}$, describes how fast traders 'race' to the market during a sell-off or a buying frenzy. The second term, $b_i e^{\frac{\gamma}{\lambda}t}$, describes the magnitude of 'fading' by each trader. Fading refers to the rate at which a trader reverses his initial position and, for the cases we consider in our stage game, is only present when opportunistic traders are present. For example, consider that a trader needs to sell a block of shares of an asset and that there are predatory traders present in the market. The first component describes the rate at which they all trade when they race each other to the market initially, and the second component describes the trading dynamic when the predators buy back.

Note that the constant a in Equation (5) is a function of the average trading target over all traders. All traders race to the market in similar fashion, based on the common knowledge of their overall trading target. Towards the end of the period, traders 'fade' based on their particular trading targets. The constants b_i are a function of how each trader's trading target is different from the average. A 'distressed trader', in the sense that he has a higher-than-average trading target, towards the end of the period will trade in the same direction as the racing. A 'predatory trader', in the sense that she has a smaller-than-average trading target, will 'fade' in the opposite direction, that is reverse the direction in which she is trading.

For a single trader, the equations reduce to $Y_t^1 = \frac{\Delta x_1}{T}$, that is to a trading rate schedule. To develop more intuition regarding Result 1, we evaluate Equation (4) for special cases, which we will use when we consider the repeated game in Sections 2 and 3.

Case 1 (Symmetric Distressed Traders). First, consider the optimal trading policy when a trader has monopoly power and buys or sells in the absence of other strategic traders. For $n = 1$, the optimal trading policy (4) for a single trader is to trade at a constant rate, $Y_t = a = \frac{\Delta x}{T}$, where Δx is the block of shares that the trader needs to buy or sell.

Now, consider n symmetric traders, each needing to sell an identical amount of shares $\frac{\Delta x}{n}$. From Equation (4), the unique equilibrium trading strategy is

$$Y_t^i = a\, e^{-\frac{n-1}{n+1}\frac{\gamma}{\lambda}t},\ i = 1, \ldots, n, \tag{6}$$

where a is as in (5) with $\sum_{i=1}^{n} \Delta x_i = \Delta x$. Figure 1 plots these trading policies with $\Delta x = 1$, $T = 1$, $\gamma = 10$ and $\lambda = 1$. If the market is deeper (small γ), trading will occur comparatively later. If the short-term price impact of trading is smaller (small λ), trading will occur comparatively earlier. For $n = 1$ we obtain the constant selling rate (*solid horizontal line*). If there are more traders, everybody will trade earlier. Note that the rate of trade goes to $e^{-\frac{\gamma}{\lambda}t}$ as $n \to \infty$ (*dotted line*). That is, there is an upper bound on how fast traders will sell their position, regardless of how many traders are in the race. Note that the shape of the curve depends on γ and on λ only through the ratio λ/γ. However, the scale of the γ and λ parameters does otherwise matter in relation to U_t, for instance the expected long-term fractional loss in value of the asset is proportional to γ/U_t.

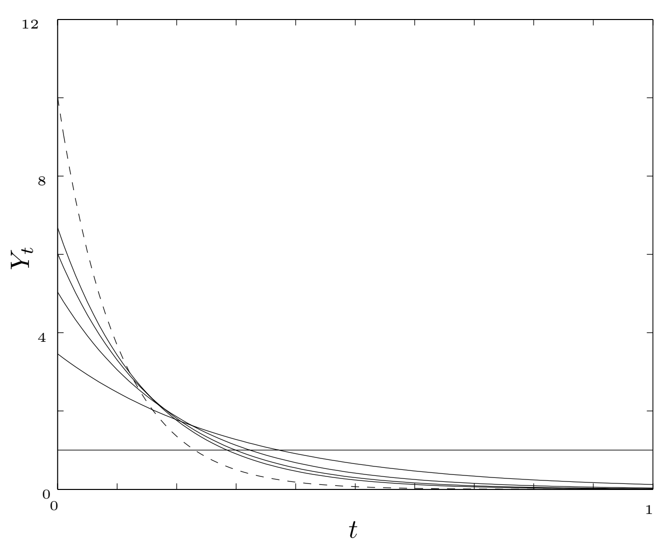

Figure 1: Trading rate for multiple traders with identical targets (solid for $n = 1, 2, 3, 4, 5$, dashed for $n = \infty$): Competition among traders leads to a 'race to trade'. The parameters for this example are $\Delta x = 1,\ T = 1,\ \gamma = 10,\ \lambda = 1$.

Figure 2 plots the corresponding price process (for a constant $U_t = 5$). For a single trader, the price changes linearly over the trading period. By trading at a constant rate, the single trader is able to "walk down the demand curve" and not incur a loss in surplus due to excessive short-term price pressure from the trading intensity (*straight solid line*). For a large number of traders, the price function over $t \in [0, T]$ quickly approaches a constant value. The information regarding the trader's target position in the asset quickly becomes incorporated in the asset price (*dotted line*). There is a surplus loss to the strategic traders as trading pressure depresses prices quickly. We quantify these surplus changes in the next subsection.

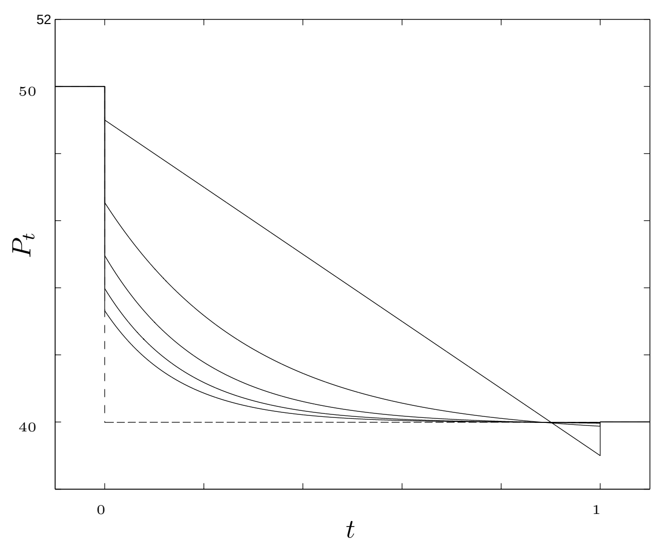

Figure 2: Expected price for multiple traders with identical targets (solid for $n = 1, 2, 3, 4, 5$, dashed for $n = \infty$). The parameters for this example are $\Delta x = -1,\ T = 1,\ \gamma = 10,\ \lambda = 1$, with constant $U_t = 50$.

Case 2 (Distressed Trader and Predatory Trader). We now set up and analyze the two-player predatory stage game, which will form the basis for the infinitely-repeated game in Section 2. Consider that there exists one distressed and one opportunistic trader. Each trader chooses a trading schedule (Y_t^d and Y_t^p) over the period $[0,T]$ to maximize his own expected value, assuming the other trader will do likewise. From Result 1, the unique equilibrium trading policies are

$$\begin{aligned} Y_t^d &= a\,e^{-\frac{1}{3}\frac{\gamma}{\lambda}t} + b\,e^{\frac{\gamma}{\lambda}t} \\ Y_t^p &= a\,e^{-\frac{1}{3}\frac{\gamma}{\lambda}t} - b\,e^{\frac{\gamma}{\lambda}t}, \end{aligned} \tag{7}$$

where

$$a = \frac{\gamma}{6\lambda}\left(1 - e^{-\frac{1}{3}\frac{\gamma}{\lambda}T}\right)^{-1}\Delta x, \qquad b = \frac{\gamma}{2\lambda}\left(e^{\frac{\gamma}{\lambda}T} - 1\right)^{-1}\Delta x. \tag{8}$$

The shape of the trading strategy depends on the parameters of the market. Figure 3 gives an example, with $\Delta x = 1$, $T = 1$, $\gamma = 10$ and $\lambda = 1$. The strategy involves the opportunistic trader initially racing the distressed trader to the market in an exponential fashion, and then fading the distressed trader towards the end of the period, also exponentially. If the first trader needs to sell, that is $\Delta x < 0$, the predatory trader sells short at beginning and buys back in later periods to cover his position. If the distressed trader is required to buy a block of the asset, the opposite strategy by the predator ensues. In general, we see that the presence of the predator will lead the distressed trader to increase his trading volume at the beginning and at the end of the trading period. This leads to a U-shaped trading volume over the period, a pattern observed in most markets.

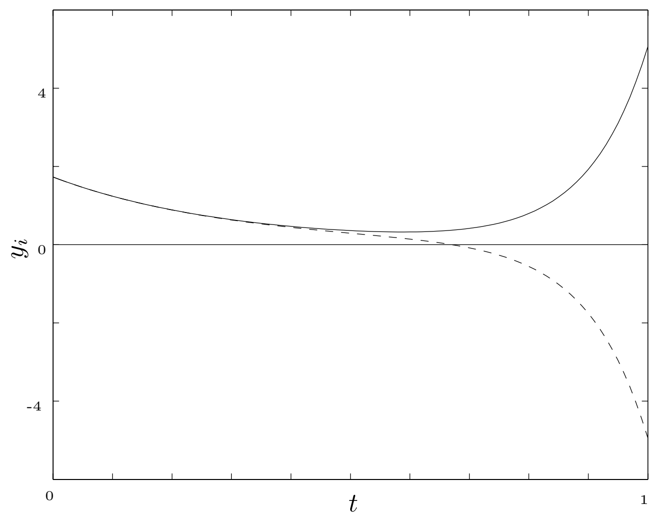

Figure 3: One trader with position target (solid) and one 'opportunistic' trader (dashed). The parameters for this example are $\Delta x = 1, T = 1, \gamma = 10, \lambda = 0.1$.

1 C Stage Game: Surplus Effects

Based on the trading dynamics in Section 1 B, we quantify the surplus changes that occur when traders race to market and when predatory trading occurs. The surplus values that we derive will be used in the following sections.

First, consider the expected value for a single trader with monopoly power. Given the price (1) and the optimal trading rate $Y_t = \frac{\Delta x}{T}$, the expected value for the single trader is easily seen to be

$$V_1 = -U_0\,\Delta x - \left(\frac{\gamma}{2} + \frac{\lambda}{T}\right)\Delta x^2. \tag{9}$$

This is the trader's first best when there are no other competing traders informed of the trader's trading requirement Δx. The costs due to short-term trading pressure on the price are minimized. When other players trade strategically at the same time, the value that the trader can derive is strictly lower that V_1. We will also see that when multiple traders compete in a sell-off or if there is predatory trading, the total surplus available to all traders is decreased.

Define V_n as the total expected value for the strategic traders when n traders play this game and define ΔV_n as the change in total surplus that occurs compared to the expected value when all participants trade at a constant rate (V_1). The following result provides expressions for V_n and ΔV_n, and shows that the loss in surplus is increasing with the number of traders. It will lay some groundwork for the surplus results for the case where there is a distressed and a predatory trader, and is also of interest on its own for the monotonicities.

Result 2 *(Expected Total Surplus and Loss for Multiple Traders) The total expected value for n traders with a combined trading target Δx is*

$$V_n \;=\; -\,U_0\,\Delta x - \frac{\gamma}{2}\left(1 + \frac{n-1}{n+1}\cdot\frac{e^{\frac{n-1}{n+1}\frac{\gamma}{\lambda}T}+1}{e^{\frac{n-1}{n+1}\frac{\gamma}{\lambda}T}-1}\right)\Delta x^2. \qquad (10)$$

The expected loss in total surplus from competition is

$$\Delta V_n \;=\; V_1 - V_n \;=\; \gamma\left(\frac{1}{2}\cdot\frac{n-1}{n+1}\cdot\frac{e^{\frac{n-1}{n+1}\frac{\gamma}{\lambda}T}+1}{e^{\frac{n-1}{n+1}\frac{\gamma}{\lambda}T}-1} - \frac{1}{\frac{\gamma}{\lambda}T}\right)\Delta x^2. \qquad (11)$$

ΔV_n is positive, monotonic increasing in γ, T and n, and monotonic decreasing in λ.

Proof. See appendix. ■

Now we apply Result 2 to the two-trader case and derive a surplus result that we will use in Section 2. We define V_2 as the total expected value for the strategic traders when two traders play this game, and we define V_d and V_p as the expected values to the distressed trader and to the opportunistic trader (as defined in Section 1 B). Likewise, we define ΔV_2 as the change in surplus that occurs compared to the expected value V_1 that is obtained when the participants trade at a constant rate.

Result 3 *(Expected Total Surplus and Loss for Two Traders) The total expected value for the distressed trader and the predatory trader is*

$$V_2 \;=\; V_d + V_p \;=\; -\,U_0\,\Delta x - \frac{\gamma}{3}\cdot\frac{2e^{\frac{1}{3}\frac{\gamma}{\lambda}T}-1}{e^{\frac{1}{3}\frac{\gamma}{\lambda}T}-1}\,\Delta x^2. \tag{12}$$

The expected value is divided as

$$\begin{aligned} V_d &= -U_0\,\Delta x - \frac{\gamma}{6}\,\frac{5e^{\frac{\gamma}{\lambda}T}+e^{\frac{2}{3}\frac{\gamma}{\lambda}T}+e^{\frac{1}{3}\frac{\gamma}{\lambda}T}-1}{e^{\frac{\gamma}{\lambda}T}-1}\,\Delta x^2, \\ V_p &= \frac{\gamma}{6}\cdot\frac{e^{\frac{2}{3}\frac{\gamma}{\lambda}T}-1}{e^{\frac{2}{3}\frac{\gamma}{\lambda}T}+e^{\frac{1}{3}\frac{\gamma}{\lambda}T}+1}\,\Delta x^2. \end{aligned} \tag{13}$$

The expected loss due to predation for the distressed trader is

$$\Delta V_d = V_1 - V_d = \gamma\left(\frac{1}{6}\cdot\frac{2e^{\frac{\gamma}{\lambda}T}+e^{\frac{2}{3}\frac{\gamma}{\lambda}T}+e^{\frac{1}{3}\frac{\gamma}{\lambda}T}+2}{e^{\frac{\gamma}{\lambda}T}-1}-\frac{1}{\frac{\gamma}{\lambda}T}\right)\Delta x^2, \tag{14}$$

and the expected loss from predation in total surplus for the strategic traders is

$$\Delta V_2 \;=\; V_1 - V_2 \;=\; \gamma\left(\frac{1}{6}\cdot\frac{e^{\frac{1}{3}\frac{\gamma}{\lambda}T}+1}{e^{\frac{1}{3}\frac{\gamma}{\lambda}T}-1}-\frac{1}{\frac{\gamma}{\lambda}T}\right)\Delta x^2. \tag{15}$$

V_p is monotonically increasing in γ and in T, and monotonically decreasing in λ.

The ratio of gains to the predator to the losses to the distressed trader, $\frac{V_p}{\Delta V_d}$, is monotonically decreasing in γ and in T, monotonically increasing in λ, and is bounded (tightly) by $\frac{4}{5} > \frac{V_p}{\Delta V_d} > \frac{1}{2}$. Note that it follows from Result 2 that ΔV_2 is positive, monotonic increasing in γ and T, and monotonic decreasing in λ. From the monotonicity of V_p it also follows that $\Delta V_d = \Delta V_2 + V_p$ is monotonic.

Proof. See appendix. ■

From the solutions for the rate of trading, we can see that a larger $\frac{\gamma}{\lambda}$ ratio (less market depth and lower transaction costs) creates conditions for more aggressive predation, in the sense that trading will be relatively more concentrated at the beginning and at the end of the period. Racing is more aggressive, and fading occurs closer to the end of the trading period.

Since $\frac{V_p}{\Delta V_d}$ is bounded in the interval $[\frac{1}{2}, \frac{4}{5}]$, the losses to the distressed trader are strictly higher than the gains by the predator. Even though the monotonicity of V_p implies that market conditions that lead to more aggressive predation (larger γ or lower λ or both) will lead to more gains from predation, since $\frac{V_p}{\Delta V_d}$ decreases in γ and increases in λ, the losses to the distressed trader grow faster than the gains to the predator. In this one-shot stage game, this represents a significant surplus loss to the traders.[16] In a dynamic setting, which we model in the next section, if both traders have a possible liquidity need in each period, there exists a potential for Pareto improvement if the traders can cooperate. As we will see, Δx, λ, and γ are important determinants in predicting whether cooperation is possible.

Finally, for some insight into the magnitude of the available Pareto improvement, consider the limit case when $\lambda \ll \gamma T$. These are the conditions under which predation is most aggressive, that is when racing and fading are fastest as a consequence of the low transaction costs. Takingh the limit $\lambda \to 0$ (which,

[16]In our stage game, we do not allow for ex-post renegotiation. Surplus losses are common in many models in non-cooperative game theory (*i.e.*, Prisoner's Dilemma and Centipede Game) and motivate cooperation in repeated play.

by change of units, is immediately seen to be equivalent to $T \to \infty$), L'Hôpital's rule yields for the overall surplus loss, the distressed trader's losses from predation, and the predator's gains:

$$\begin{aligned} \Delta V_2 &\to \frac{\gamma}{6}\,\Delta x^2, \\ \Delta V_d &\to \frac{\gamma}{3}\,\Delta x^2, \\ V_p &\to \frac{\gamma}{6}\,\Delta x^2. \end{aligned} \tag{16}$$

In the limit, under market conditions that favor the most aggressive predation, the predator gains (V_p) half of what the distressed trader loses (ΔV_d). This is the lower bound for $\frac{V_p}{\Delta V_d}$.

2 Cooperation and Liquidity (Repeated Game)

The repeated game is based on the case in which there are two strategic traders, as well as a large number of long-term investors. Each player faces a common discount factor δ, and the common asset price determinants U_0, γ, and λ. At the beginning of each stage, nature moves first, assigns a type to each of the traders and both traders know each other's type in each round.[17] In each round, each trader, with probability p_i $i = 1, 2$, must liquidate a large position of size Δx, and may act as a predator with probability $1 - p_i$. We assume that the distress probabilities p_1 and p_2 are mutually independent and the magnitude of the shock Δx is constant. An alternative approach, which we take in Section 3,

[17] In models of implicit collusion under imperfect information (Green and Porter 1984; Abreu, Pearce, and Stacchetti 1986), players never deviate in equilibrium, but enter punishment phases because of exogenous price changes. Our game is one of complete information, where deviations cannot arise from such exogenous factors. This allows for a simpler model wich captures the key issues of interest, namely how apparent liquidity arises from incentives to cooperate.

is to model Δx as a random variable, and compute the value of the supergame by expectation over future liquidity events.

In each time period one of the following four events occurs: neither of the two players is distressed, with probability $(1 - p_1)(1 - p_2)$; the second player is distressed but the first is not, with probability $(1 - p_1)p_2$; the first player is distressed but the second is not, with probability $p_1(1 - p_2)$; both players are distressed, with probability $p_1 p_2$. The four probabilities add to one. Cooperation is possible when either there exists one predator and one distressed trader (with probability $p_1 + p_2 - 2p_1 p_2$), or when both players are distressed (with probability $p_1 p_2$). If only one of the players is distressed and needs to liquidate a position, cooperation involves the other refraining from engaging in predatory trading. If both traders are distressed, cooperation involves both traders selling at a constant rate and refraining from racing each other to the market for their own gain.

Cooperation provides the players with the ability to quickly sell large blocks of shares, for the price that would be obtained by selling them progressively over time. That is, while cooperation is ongoing, the distressed trader is allowed to 'ride down' the demand curve, rather than having the information regarding the trading target quickly incorporated into the asset price, ahead of most of his trading. In this sense, that large blocks of shares can be moved for a better price, the market will appear more liquid. It will also avoid the volatility and potential instability from the large trading volume peaks associated with the racing and fading.

The punishment strategy considered is a trigger strategy in the spirit of Dutta and Madhavan (1997) and Rotemberg and Saloner (1986). That is, for a given discount factor δ, the value of a perpetuity of cooperation must exceed a one-time deviation plus a perpetuity of non-cooperation. Based on the Folk Theorem, a convex set of subgame perfect Nash equilibria may exist in which intermediate levels of cooperation occur. For clarity of exposition, we focus on the extremal equilibrium in which intermediate levels do not exist.

The purpose of this set-up is to evaluate when the traders will abandon a cooperative effort, thereby leading to illiquidity in the market. Comparative statics regarding the nature of this cooperative effort can be generated by comparing the level of δ required in different scenarios. Given any particular punishment scheme, such as the more complicated penal codes in Abreu 1988, such a critical δ can be derived. For this analysis, we do not allow the players to change punishment schemes to achieve cooperation. Therefore, we focus on trigger stratgies because they lead to the same economic results, while maintaining clarity of the model.

The following result describes the extremal equilibrium of our repeated game with fixed liquidity needs (Δx) in each period, using a trigger strategy. Refer to Table I for the expected values derived in Section 1.

Result 4 *(Repeated Game with Two Symmetric Traders) Define the expected values as in Section 1. When a trigger strategy (punishment strategy) is used, the discount factor required to support collusion is*

$$\delta \geq \delta_{\min} = max \left\{ \delta_1, \delta_2 \right\}, \tag{17}$$

Surplus to distressed trader, no predation	V_1	$-U_0\,\Delta x - \left(\frac{1}{2} + \frac{1}{\frac{\gamma}{\lambda}T}\right)\gamma\,\Delta x^2$
Surplus to distressed trader during predation	V_d	$-U_0\,\Delta x - \frac{1}{6}\,\frac{5e^{\frac{\gamma}{\lambda}T}+e^{\frac{2}{3}\frac{\gamma}{\lambda}T}+e^{\frac{1}{3}\frac{\gamma}{\lambda}T}-1}{e^{\frac{\gamma}{\lambda}T}-1}\,\gamma\,\Delta x^2$
Change in surplus to distressed trader ($V_1 - V_d$)	ΔV_d	$\left(\frac{1}{6}\cdot\frac{2e^{\frac{\gamma}{\lambda}T}+e^{\frac{2}{3}\frac{\gamma}{\lambda}T}+e^{\frac{1}{3}\frac{\gamma}{\lambda}T}+2}{e^{\frac{\gamma}{\lambda}T}-1} - \frac{1}{\frac{\gamma}{\lambda}T}\right)\gamma\,\Delta x^2$
Surplus to predator	V_p	$\frac{1}{6}\cdot\frac{e^{\frac{2}{3}\frac{\gamma}{\lambda}T}-1}{e^{\frac{2}{3}\frac{\gamma}{\lambda}T}+e^{\frac{1}{3}\frac{\gamma}{\lambda}T}+1}\,\gamma\,\Delta x^2$
Total surplus to predator and distressed trader	V_2	$-U_0\,\Delta x - \frac{1}{3}\cdot\frac{2e^{\frac{1}{3}\frac{\gamma}{\lambda}T}-1}{e^{\frac{1}{3}\frac{\gamma}{\lambda}T}-1}\,\gamma\,\Delta x^2$
Change in overall surplus with predatory trading	ΔV_2	$\left(\frac{1}{6}\cdot\frac{e^{\frac{1}{3}\frac{\gamma}{\lambda}T}+1}{e^{\frac{1}{3}\frac{\gamma}{\lambda}T}-1} - \frac{1}{\frac{\gamma}{\lambda}T}\right)\gamma\,\Delta x^2$

Table I: Expected Values from Strategic Trading

where the δ_i are the lowest discount factors for which each player does not have an incentive to predate given the opportunity to do so (and given that the same is true for the other player), which are

$$\begin{aligned} \delta_1 &= \frac{V_p}{p_1(1-p_2)\Delta V_d + 2p_1p_2\Delta V_2 + (1-(1-p_1)p_2)V_p}, \\ \delta_2 &= \frac{V_p}{(1-p_1)p_2\Delta V_d + 2p_1p_2\Delta V_2 + (1-p_1(1-p_2))V_p}. \end{aligned} \tag{18}$$

The δ_1 and δ_2 bounds on the discount factor (and therefore $\delta_{\min}$) are monotonically increasing in λ and in T, and monotonically decreasing in γ.

Proof. See appendix. ■

Result 4 predicts that cooperation is more likely in markets with thinly-traded assets (high γ). Since the δ required to support cooperation is monotonically decreasing in γ, it easier to support cooperation in these thin markets. In Section 1 C, we showed that the surplus loss to predatory trading is monotonically increasing in γ. Since with higher γ there exists a higher Pareto improvement available, it becomes more desirable for the traders to maintain cooperation. Therefore, in markets with thin issues, such as the markets for corporate bonds, exotic options, and credit derivatives, our model predicts that there should be more apparent liquidity because of cooperation.

In contrast, in markets with high transaction costs (high λ), we would expect a lower level of cooperation. Since the Pareto improvement available monotonically decreases in λ, the level of cooperation is lower for markets with high transaction costs. Therefore, we can focus on γ-λ combinations to predict whether there will exist more or less aggressive predation. For large $\frac{\gamma}{\lambda}$, we expect cooperation to dominate predation, thereby producing more apparent liquidity. For small $\frac{\gamma}{\lambda}$, we expect predation to be more prevalent.

To analyze how δ relates to the probabilities of distress, consider the case where the probabilities of distress are independent. Before looking at comparative statics, consider the example graphed in Figure 4. The figure plots the minimum δ required for which cooperation is feasible as a function of γ. The other parameters are set as $\Delta x = 1$, $U_0 = 10$, $T = 10$, and $\lambda = 1$. The base case is the solid-line, in which $p_1 = 0.5$ and $p_2 = 0.5$. For all values of γ, the δ required for cooperation is less than 1, which means that it is possible to cooperate in all markets. Now consider the case in which $p_1 = 0.1$ and $p_2 = 0.1$. It is still possible to cooperate in all markets (all γ), but the level of δ is higher. This implies that as the probability of distress increases for both traders, the ability to cooperate also increases. Now consider the case in which $p_1 = 0.5$ and $p_2 = 0.3$. In this case, it is not possible for the traders to cooperate in markets with low γ. Since $\delta \in [0, 1]$, it is possible to find a bound on γ, below which

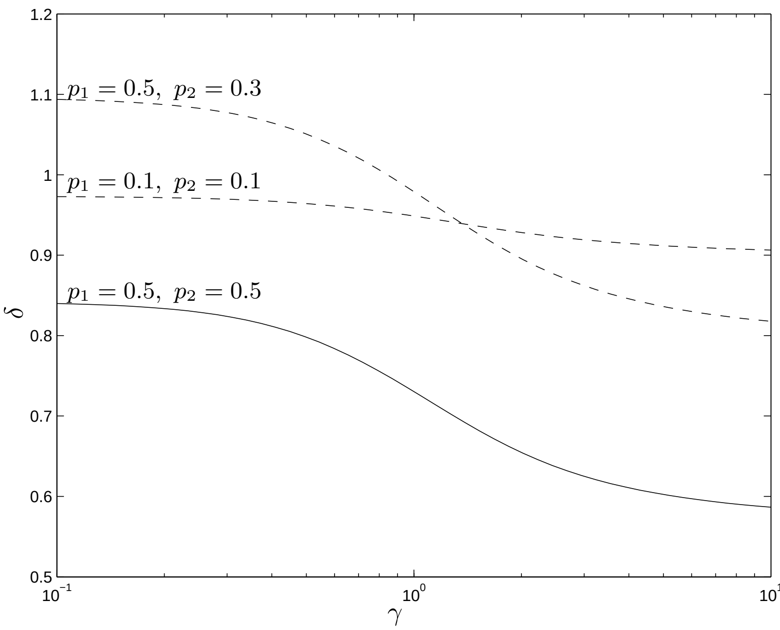

Figure 4: Minimum δ for which cooperation is feasible as a function of γ, with logarithmic scale on the γ-axis. Three different cases of the distress probabilities of each trader are plotted (the probabilities are assumed independent). Other parameters are $\Delta x = 1$, $U_0 = 10$, $T = 10$, and $\lambda = 1$.

we should not observe cooperation. It is interesting to note that symmetry (in distress probabilities) between the traders is an important driver of cooperation. In markets with a low γ, a small decrease in one trader's distress probability may be enough to cause the traders to abandon their cooperative relationship.

Without loss of generality, assume $\delta_1 = \delta_{\min}$ (that is, $p_1 < p_2$). Equation (18) can be rewritten as

$$\delta_1 = \frac{V_p}{p_2 \left[p_1 \Delta V_2 - V_p\right] + p_1 \Delta V_d + V_p}. \tag{19}$$

This shows that δ_1 is monotonically increasing in p_2 (since we can establish that $V_p \geq \Delta V_2$ from $V_p / \Delta V_d > 1/2$ in Result 3 and $\Delta V_d = \Delta V_2 + V_p$). This means that a larger probability for trader 2 makes cooperation more difficult (that is, cooperation is possible only under a narrower range of market conditions). We can also rewrite Equation (18) as

$$\delta_1 = \frac{V_p}{p_1 \left[(1-p_2)\Delta V_d + 2p_2 \Delta V_2 + p_2 V_p\right] + (1-p_2) V_p}, \tag{20}$$

which shows that δ_1 is monotonic decreasing in p_1. This means that a bigger probability for trader 1 makes cooperation easier (that is, possible under a wider range of market conditions).

If traders are infinitely patient ($\delta = 1$), and without loss of generality $p_1 < p_2$, to support cooperation it must be that

$$p_1 \geq p_2 \frac{V_p}{\Delta V_d} - p_1 p_2 \frac{\Delta V_2}{\Delta V_d}. \tag{21}$$

Evaluating Equation (21) under extreme market conditions provides more insight into the relative values of the distress probabilities which are conducive to cooperation. Taking the limits $\gamma/\lambda \to 0$ and $\gamma/\lambda \to \infty$, Equation (21) becomes

$$p_1 \geq \frac{4}{5}p_2 - \frac{1}{5}p_1 p_2 \quad \text{and} \quad p_1 \geq \frac{1}{2}p_2 - \frac{1}{2}p_1 p_2. \tag{22}$$

If distress events are infrequent ($p_1, p_2 \ll 1$), then the size of the smaller player relative to larger player is bounded by

$$\frac{p_1}{p_2} \geq \frac{V_p}{\Delta V_d}. \tag{23}$$

In the limit cases ($\gamma/\lambda \to 0$ and $\gamma/\lambda \to \infty$) this is

$$\frac{p_1}{p_2} \geq \frac{4}{5} \quad \text{and} \quad \frac{p_1}{p_2} \geq \frac{1}{2}. \tag{24}$$

The traders' distress probabilities need to be sufficiently symmetric or else cooperation is not possible. Equation (24) provides the lower bounds on trader 1's relative distress probabilities in order to sustain cooperation. An important consequence of this is that if the probability of distress is linked to the market share of external clients that a trader services, it may benefit a large trader to allow a smaller trader to grow in size so that a Pareto superior outcome for the strategic traders can be achieved or maintained. We briefly address this relationship as follows.

Consider the following scenario in which there exists two strategic traders and a representative outsider who seeks to trade a block of an asset. The strategic traders have two alternatives when an outsider needs to trade. They may initiate a predatory strategy, race and fade the external player to the market, and earn a profit by affecting the price of the asset. Alternatively, the outsider may become a client of the traders so that the traders may exact rents for use of their services (these rents may arise in the form of a bid-ask spread). The fact that there exists a cooperative outcome in this market between the insiders provides a means by which a relatively stable, albeit widened, bid-ask spread may exist, and we do not necessarily observe price volatility when a non-member needs liquidity. The amount of the surplus available between the traders and the client is ΔV_d, since the outsider is indifferent between receiving V_d, and paying ΔV_d in order to receive V_1 when using the services of the strategic traders. [18]

The external client uses each trader with probabilities p_1 and p_2. Equation (21) implies that the relative market shares of the traders should be reasonably symmetric to support cooperation. Consider the example in Figure 5 in which the minimum δ necessary to support cooperation is plotted as a function of p_1. Two scenarios are demonstrated: $p_1 = p_2$ (*bold line*) and $p_1 + p_2 = 0.5$ (*dotted line*). When the traders' distress probabilities (market shares) are symmetric, cooperation is always possible, as long as traders are sufficiently patient. However, when the market shares are asymmetric and $p_1 < 0.18$ (36% market share) or $p_1 > 0.32$ (64% market share), cooperation is not possible. Therefore, if one trader has a larger than 64% market share, he may find it beneficial to allow his opponent to gain market share so that their ongoing Pareto superior relationship may continue. This finding may explain why deviations in the bid-ask spread may be observed in practice without resulting in price wars.

[18] To determine the division of this surplus between the insiders and the external player, it is possible to use a generalized Nash bargaining solution in which the insiders receive fraction τ of the surplus and the client receives fraction $1-\tau$. In the example that we consider (Figure 5), we assume $\tau = 1$, without loss of generality. Relaxing this assumption does does not lead to different comparative statics.

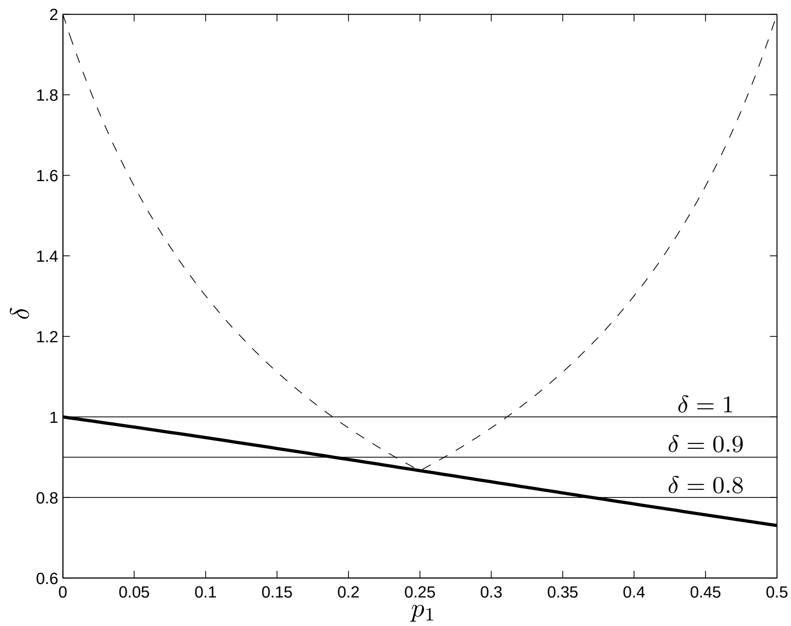

Figure 5: Minimum δ as a function of distress probabilities, the cases of $p_1 = p_2$ (bold), and $p_2 = 0.5 - p_1$ (dashed). Parameters are $\Delta x = 1$, $U_0 = 10$, $T = 10$, $\gamma = 1$, and $\lambda = 1$. The horizontal reference lines correspond to the iso-δ lines in Figure 6.

Figure 6 illustrates, for three different values of δ, the (p_1, p_2) pairs that support cooperation. The values of δ plotted are 1, 0.9 and 0.8. The other parameters are $\Delta x = 1$, $U_0 = 10$, $T = 10$, $\gamma = 1$, and $\lambda = 1$. The shaded region corresponds to the (p_1, p_2) pairs for which cooperation is possible if both traders use a discount factor $\delta = 0.8$. Note that the boundaries of the sets are not straight lines due to the bilinear terms $p_1 p_2$, but are nearly so for small values of p_1 and p_2. For $\delta = 1$, and for small probabilities, the set boundaries go to the origin with slope $V_p/\Delta V_d$ and $\Delta V_d/V_p$. As the sum of the two probabilities becomes smaller, traders are required to be of more similar sizes for a cooperative outcome to be feasible. Note that the bold and dashed lines correspond to the cases plotted in Figure 5. Considering cases with smaller overall frequency of events $p_1 + p_2$ corresponds to moving the dashed line to the lower-left. Figure 5 and Figure 6, therefore, demonstrate that the traders' market shares need to be sufficiently symmetric or else cooperation is not possible.

In the next section, we consider these relationships when the liquidity event (Δx) is stochastic across time. Also, we evaluate the effect of multi-market contact and contagion of illiquidity.

3 Episodic Illiquidity and Contagion

3 A Shocks of Random Magnitude and Episodic Illiquidity

In Section 2, we evaluated the requirements for cooperation given that Δx is a fixed amount of the asset. In that formulation, if cooperation is possible (based

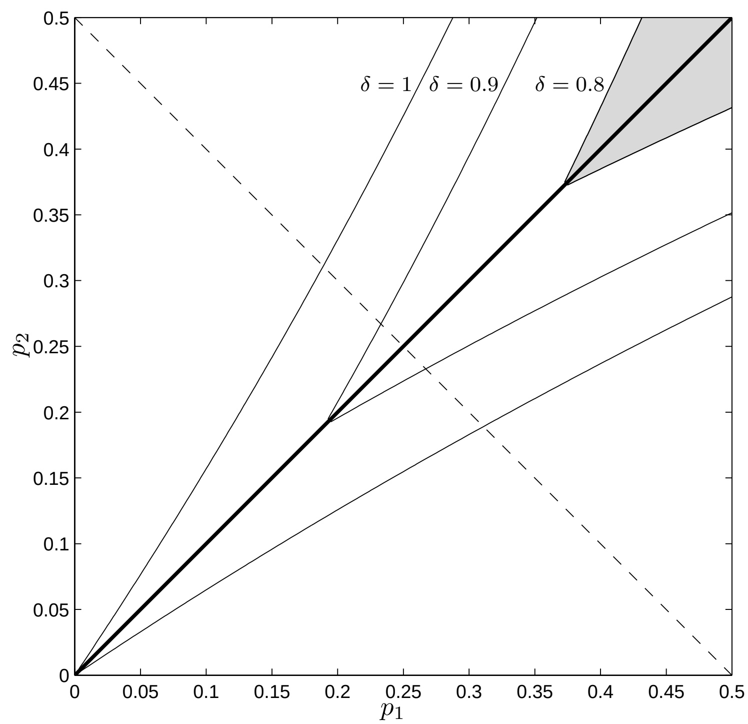

Figure 6: Values of p_1 and p_2 for which cooperation can be sustained, given δ (ploted for the cases $\delta = 1$, $\delta = 0.9$, and $\delta = 0.8$). For instance, for $\delta = 0.8$ the cooperative equilibrium exists if p_1 and p_2 fall in the shaded region. The bold and dashed lines indicate the values of p_1 and p_2 that are plotted in Figure 5.

on the market parameters and δ), the traders never deviate. To characterize episodic illiquidity, Δx is better modeled as a random variable. In the event of a large Δx it is more profitable for the traders to deviate for a one-time gain. However, instead of initiating the grim-trigger strategy outlined in Section 2, there are more profitable strategies available to the cartel. The approach that we take is along the lines of Rotemberg and Saloner (1986).

The large traders implicitly agree to restrain from predating when the magnitude of the shock is below some threshold $\widetilde{\Delta x}$ and, conversely, not to punish other players in future periods for predating when the shock is above that threshold. That is, when a player has trading requirement that exceeds $\widetilde{\Delta x}$, the other player will predate, but cooperation is resumed in subsequent periods. This equilibrium behavior results in episodically increased volatility.[19]

Another way to describe this equilibrium is that each trader agrees to restrain from predating on the other, but only as long as they behave 'responsibly' in their risk management. This creates a natural restriction on the exposure that each trader can take without a substantial increase in the risk of their portfolio.

The value of $\widetilde{\Delta x}$ which is optimal for the cartel (in the sense of leading to the highest expected value for its members) can be computed for any distribution of the trading requirement for each player. In general, $\widetilde{\Delta x}$ can only be characterized implicitly.

Result 5 *(Shocks of Random Magnitude) Consider that the trading require-*

[19]Episodic illiquidity also occurs during extreme financial distress. During extreme distress, a member of the oligopoly becomes a finite concern. Because the horizon of this game is finite, the players work out their strategy profiles by backwards induction and cooperation disappears.

ments for each of two players are random shock magnitudes Δx *that are distributed i.i.d. according to the density* $f(y)$*, which we assume to be*

(i) symmetric, $f(y) = f(-y)$*,*

(ii) with unbounded support, $f(y) > 0, \forall y \in \mathbb{R}$*,*

(iii) and with finite variance, $\int_{-\infty}^{\infty} y^2 f(y)dy < \infty$*.*

A strategy with episodic predation with threshold $\widetilde{\Delta x}$ *is feasible with any* $\widetilde{\Delta x}$ *that satisfies*

$$2C \int_0^{\widetilde{\Delta x}} y^2 f(y)dy \geq K\widetilde{\Delta x}^2. \tag{25}$$

The supremum of $\widetilde{\Delta x}$ *such that the inequality is satisfied exists, and we designate it by* $\overline{\Delta x}$*. The following strategy profile constitutes a sub-game perfect Nash equilibrium. At time* $t = 0$*, we predate if* $|\Delta x| > \widetilde{\Delta x}$*, and otherwise cooperate. At time* $t \neq 0$*,*

1. *If the history of play* h^{t-1} *is such that for every period in which* $|\Delta x| < \widetilde{\Delta x}$ *there was no predation, then*
   (a) *If* $|\Delta x| > \widetilde{\Delta x}$*, predate this period.*
   (b) *If* $|\Delta x| < \widetilde{\Delta x}$*, cooperate.*
2. *If* h^{t-1} *is such that for* $|\Delta x| < \widetilde{\Delta x}$*, there was predation, then predate.*

The constants above are

$$C = \frac{\delta}{1-\delta} \left(p_{10}K_d + 2p_{11}K_2 - p_{01}K\right) \tag{26}$$

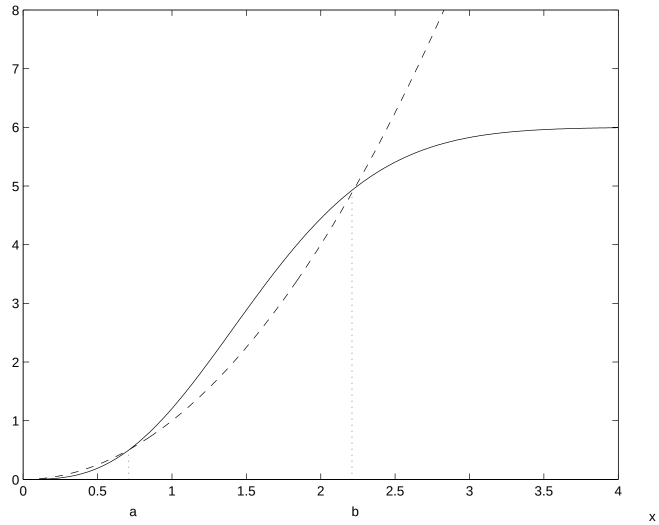

Figure 7: Left- (solid) and right-hand-side (dashed) of Equation (25). The curves intersect at zero, $\underline{\Delta x}$, and $\overline{\Delta x}$.

and $K = V_p/\Delta x^2$, *with* $K_d = \Delta V_d/\Delta x^2$ *and* $K_2 = \Delta V_2/\Delta x^2$ *(that is,* K, K_d *and* K_2 *are the factors multiplying* Δx^2 *in the expected values* V_p, ΔV_d *and* ΔV_2*; note that* K *in (25) is the factor in the expected gain to the predator).*

Proof. See appendix. ■

Figure 7 shows an example with a normally distributed shock. The left-hand-side of the inequality is plotted with the solid-line and the right-hand-side is plotted with the dashed-line. For $\widetilde{\Delta x}$ in an interval $[\underline{\Delta x}, \overline{\Delta x}]$, it is possible to sustain the subgame-perfect Nash equilibrium. For $\widetilde{\Delta x} > \overline{\Delta x}$, the value gained for deviation is too high, and cooperation cannot be maintained. The supremum $\overline{\Delta x}$ defines the most profitable strategy for the cartel. (Note that $\overline{\Delta x}$ might be zero, in which case traders never cooperate.)

The nature of the solutions is essentially independent of the scale parameter of the distribution. Consider a family of distributions $f_a(y) = af(ay)$. If we consider solutions in terms of $\widetilde{\Delta x}/a$, the set of feasible thresholds is independent of the asset parameters (here C and K). The inequality is equivalent to

$$2\frac{C}{K}\int_0^{\frac{\widetilde{\Delta x}}{a}} y^2 f_a(y)dy \geq \left(\frac{\widetilde{\Delta x}}{a}\right)^2, \tag{27}$$

so that, after the corresponding scaling, the solutions to the inequality are constant with scaling of the distribution.

As an example, consider the zero-mean normal distribution

$$f(y) = \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\frac{y^2}{\sigma^2}}. \tag{28}$$

Rewriting the inequality as

$$\frac{K}{C}\left(\frac{\widetilde{\Delta x}}{\sigma}\right)^2 \leq 2\int_0^{\frac{\widetilde{\Delta x}}{\sigma}} y^2 \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}y^2}dy, \tag{29}$$

we see that any such problem can be parameterizing over C/K and $\widetilde{\Delta x}/\sigma$. Figure 8 plots $\underline{\Delta x}/\sigma$ and $\overline{\Delta x}/\sigma$ as a function of C/K. The solid-line represents $\overline{\Delta x}/\sigma$ and the dotted-line represents $\underline{\Delta x}/\sigma$. For any value of C/K, the vertical segment between the lines is the set of $\widetilde{\Delta x}$ (in standard deviations) such that cooperation is possible. Note that there exists a critical value for C/K (represented by the small circle), below which it is impossible to support cooperation because the gains from deviation are too great.

We now consider the problem of determining the minimum C/K for which there is a non-zero $\overline{\Delta x}$, that is for which an episodic predation strategy is feasible. Consider the normal distribution and the inequality in Equation (29).

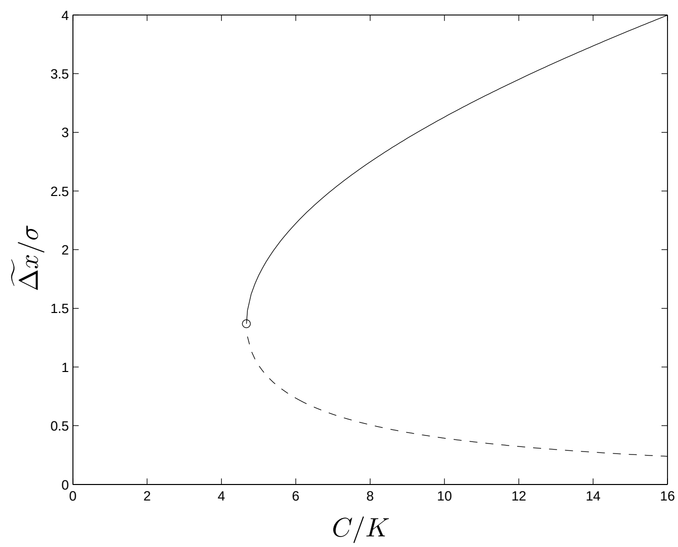

Figure 8: Normally distributed shocks, upper and lower bounds for $\widetilde{\Delta x}/\sigma$ as a function of the asset parameters.

Taking the derivative of the difference between the two sides of the inequality with respect to $\widetilde{\Delta x}$ and equating to zero leads to

$$\frac{K}{C} = \frac{1}{\sqrt{2\pi}} \frac{\widetilde{\Delta x}}{\sigma} e^{-\frac{1}{2}\left(\frac{\widetilde{\Delta x}}{\sigma}\right)^2}. \tag{30}$$

This characterizes the points in Figure 7 where the solid and dashed lines have the same derivative. Using this at the supremum (*i.e.*, with equality holding in (29)), we obtain the case where the lines touch at a single point rather that having two intersections (other than zero). After a change of variable in the integral, we obtain

$$\int_0^{\frac{\widetilde{\Delta x}}{\sigma}} y^2 \frac{1}{2\pi} e^{-\frac{1}{2}y^2} dy = \frac{1}{2} \left(\frac{\widetilde{\Delta x}}{\sigma}\right)^3 \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{\widetilde{\Delta x}}{\sigma}\right)^2}, \tag{31}$$

which is straightforward to solve numerically for $\widetilde{\Delta x}/\sigma$. Since C/K only depends on $\widetilde{\Delta x}$ through $\widetilde{\Delta x}/\sigma$, the minimum C/K ratio for which there is a feasible strategy of the episodic predation type does not depend on the scale parameter of the distribution. For the normal distribution, the minimum C/K for which there is a non-zero $\overline{\Delta x}$, that is for which an episodic predation strategy is feasible, is $C/K = 4.6729$ for any σ. The threshold for this C/K ratio is $\underline{\Delta x} = \overline{\Delta x} = 1.3688\,\sigma$.

3 B Contagion across markets

Suppose that the members of the oligopoly can cooperate in more than one market. For example, consider institutional traders who dominate mortgage markets are also strategic traders in other fixed income markets. If a liquidity event is large enough to disturb cooperation in one market, it may also affect cooperation in the others. According to Bernheim and Whinston (1990), if markets are not identical, multimarket contact supports cooperation. In our case, and since most assets are not perfectly correlated, multi-market contact makes it easier to maintain cooperation. In this section, we demonstrate the effects of multi-market contact on the episodic illiquidity that occurs across markets.

Consider that the traders participate in n markets, where the trading requirement in each market is a stochastic random variable. We define a liquidity event to be such that all trading targets for each of the n assets have the same sign (*i.e.*, liquidity shocks occur in the same direction in all markets). The trading targets are modeled as jointly normal and conditionally independent given that they are either all positive or all negative. We also use the simplifying assumption that trading targets in all the assets have the same variance. The density in the positive orthant (y such that $y_i \geq 0$, all i) and in the negative orthants (y such that $y_i \leq 0$, all i) is

$$f(y) = \frac{2^{n-1}}{\sigma^n (2\pi)^{n/2}} e^{-\frac{y^T y}{\sigma^2}}, \tag{32}$$

and zero elsewhere.

The shape of the optimal region for cooperation is spherical. This is the region in which the temptation to predate, which is proportional to $\sum_{i=1}^{n} \Delta x_i^2$, is constant.

The inequality for n assets involves an integral in n dimension which, using the radial symmetry of the normal distribution can be written as

$$2C \int_0^r S_n y^{n+1} f(y) dy \geq K r^2, \tag{33}$$

where r is the radius of the cooperation region, and

$$S_n = \frac{1}{2^n} \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})} \tag{34}$$

is the area of the intersection of the sphere of unit radius in n dimension with the positive orthant. It can easily be verified that, as for the one-asset case, the nature of the solutions is essentially independent of the scale parameter of the distribution (the standard deviation).

Figure 9 is the multi-market version of Figure 8 in that it plots $\underline{\Delta x}$ and $\overline{\Delta x}$ for episodic illiquidity over n markets, $n = 1, 2, \ldots, 8$. The minimum value of C/K that is required to support cooperation decreases as the number of markets increases. Adding markets can make cooperation possible where it

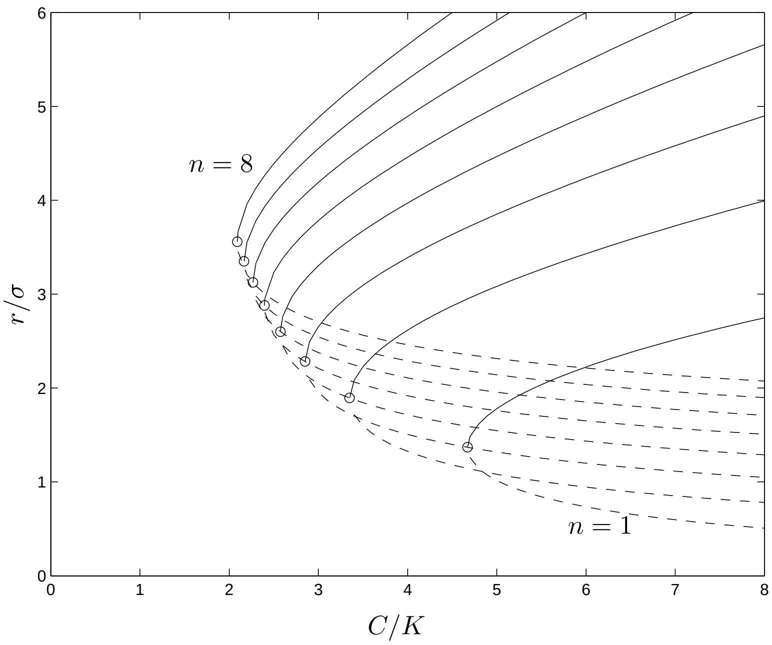

Figure 9: Lowest and highest r such that the strategy (episodic predation with contagion over n assets, $n = 1, 2, \ldots, 8$) is an equilibrium, plotted as a function of the asset parameters, common to all assets. The shocks are independent and normally distributed conditional on shocks being either all positive or all negative.

would otherwise not be possible. (Consider, for instance, $C/K = 4.0$. With these parameters, traders are unable to cooperate over one market, but are able to do so over 2 or more markets.) Also note that the supremum of r increases with n. The probability that an episode of predation will occur is in fact seen to decrease with n. We expect episodes of predation to be more significant, since they now affect n markets, but less frequent with contagion strategies. The minimum values of C/K for cooperation over multiple markets, $n = 1, 2, \ldots, 20$, are listed in Table II.

n	C/K	n	C/K	n	C/K	n	C/K
1	4.6729	6	2.2664	11	1.9299	16	1.7740
2	3.3509	7	2.1687	12	1.8907	17	1.7517
3	2.8507	8	2.0913	13	1.8563	18	1.7314
4	2.5747	9	2.0280	14	1.8259	19	1.7127
5	2.3950	10	1.9751	15	1.7986	20	1.6954

Table II: Multi-market contact across n-markets. The minimum values of C/K for which cooperation is possible over multiple markets, $n = 1, 2, \ldots, 20$.

4 Conclusion

The breakdown of cooperation in financial markets leads to episodic illiquidity. In this paper, we describe an equilibrium strategy in which traders cooperate most of the time through repeated interaction, providing 'apparent liquidity' to each other. However, episodically this cooperation breaks down, especially when the stakes are high, leading to opportunism and loss of this apparent liquidity. Our model explains why episodic liquidity breakdowns do not occur more often and also predicts that markets with thinly-traded assets (thin corporate bond issues, exotic options, and credit derivatives) and low transaction costs should have more 'apparent liquidity'.

We solve a competitive trading game by posing a continuous-time, dynamic programming problem for our traders, using an asset pricing equation which accounts for transaction costs. According to our model, traders 'race' to market, selling quickly in the beginning of the period. In the equilibrium strategy traders sell-off at a decreasing exponential rate. Also in equilibrium, predators initially race distressed traders to market, but eventually 'fade' them and buy back. The presence of predators in the market leads to a surplus loss to liquidity providers in the market.

We then model cooperation in the market by embedding this predatory stage game in a dynamic game with infinite horizon. Cooperation allows for the trading of large blocks of the asset at more favorable prices. We show how traders can cooperate to avoid the surplus loss due to predatory trading and provide predictions as to what magnitude of liquidity event is required to trigger an observable shock in the market. Further, we make predictions on how market depth and transaction costs will affect the ability to cooperate.

In conclusion, we believe our work presents a strong argument for the level of predation or cooperation in financial markets being a determinant of the amount of liquidity available, and a factor causing the episodic nature of illiquidity.

A Derivations

Proof of Result 1 (General Solution)

Given that each player's objective is linear in U_t, and that the strategies are open-loop, we can bring the expectation inside the integral and consider the equivalent problem with a deterministic asset pricing equation, in which U_t is replaced by a constant $u = U_0$,

$$P_t = u + \gamma \sum_{j=1}^{n} X_t^j + \lambda \sum_{j=1}^{n} Y_t^j. \tag{35}$$

With the multiplier function Z_t^i associated with the constraint $dX_t^i = Y_t^i dt$, necessary optimality conditions for the problem faced by trader i are

$$\begin{aligned} & u + \gamma \textstyle\sum_{j=1}^{n} X_t^j + \lambda \sum_{j=1}^{n} Y_t^j + \lambda Y_t^i + Z_t^i = 0 \\ & dZ_t^i = -\gamma Y_t^i dt. \end{aligned} \tag{36}$$

Differentiating the first equation with respect to t, and substituting the second,

$$\gamma \sum_{j=1}^{n} Y_t^j dt + \lambda \sum_{j=1}^{n} dY_t^j + \lambda dY_t^i - \gamma Y_t^i dt = 0. \tag{37}$$

The n such equations for each trader can be collected together as

$$\lambda(I + \mathbf{1}\mathbf{1}^T) dY_t = \gamma(I - \mathbf{1}\mathbf{1}^T) Y_t dt, \tag{38}$$

where I is the $n \times n$ identity matrix, $\mathbf{1}$ is the n-vector with all entries equal to one, and $\mathbf{1}\mathbf{1}^T$ is an $n \times n$ matrix with all elements equal to one. From the formula for the inverse of the rank-one update of a matrix, the inverse of $I + \mathbf{1}\mathbf{1}^T$ is $I - \frac{1}{n+1}\mathbf{1}\mathbf{1}^T$, which we use to write the linear dynamic system in the form

$$dY_t = \frac{\gamma}{\lambda} A Y_t dt, \quad \text{where} \quad A = I - \frac{2}{n+1}\mathbf{1}\mathbf{1}^T. \tag{39}$$

Since $A\mathbf{1} = \mathbf{1} - \frac{2}{n+1}n\mathbf{1} = -\frac{n-1}{n+1}\mathbf{1}$, the vector of ones is an eigenvector of the matrix A, with associated eigenvalue $-\frac{n-1}{n+1}$. Likewise, vectors in the null-space of $\mathbf{1}$ are eigenvectors of A, with eigenvalue 1: for v orthogonal to the vector of ones, that is satisfying $\mathbf{1}^T v = 0$, we find that $Av = v$. The dimension of this sub-space, and multiplicity of the eigenvalue 1, is $n - 1$. Since the matrix A has a full set of n independent eigenvectors, all Jordan blocks are of size 1 and solutions to the system of linear differential equations are as stated in (4). This characterizes any continuous policy (with continuous dual functional) which is an extremal of the problem. Since a (unique) continuous extremal exists, from problem convexity it is the only extremal of the problem. The n trading target constraints and $\mathbf{1}^T b = 0$ uniquely determine the n free parameters in the solution (integrate the Y_t^i, equate to Δx_i, and solve for a and b).

Convexity ensures that any solution is identical *almost everywhere* to the one given in this result (*i.e.*, has the same integrals). Further restrictions can be imposed on the Y_t if desired, such as being of bounded variation. Alternatively, smoothness can be ensured by defining the solution to the continuous-time problem as the limit of the solutions to a sequence of discrete-time problems, which is the approach we will use to analyze the closed-loop version of this problem in Appendix B. ■

We next show a Lemma and Corollary, which will be of use in proving Result 2.

Lemma: The function $f : \mathbf{R}^+ \mapsto \mathbf{R}$

$$f(y) = \frac{1 + e^{-y}}{1 - e^{-y}} - \frac{2}{y} \tag{40}$$

is positive increasing.

Proof: We first show $\lim_{y\to 0} f(y) = 0$. Applying l'Hôpital's rule, we find

$$\lim_{y\to 0} f(y) = \lim_{y\to 0} \frac{-ye^{-y} + 1 + e^{-y} - 2e^{-y}}{1 - e^{-y} + ye^{-y}} = \lim_{y\to 0} \frac{ye^{-y}}{2e^{-y} - ye^{-y}} = 0. \tag{41}$$

We now show $f'(y) > 0$.

$$f'(y) = \frac{-2e^{-y}}{(1-e^{-y})^2} + \frac{2}{y^2} = \frac{2}{(1-e^{-y})^2 y^2} \left(-y^2 e^{-y} + 1 + e^{-2y} - 2e^{-y}\right). \tag{42}$$

For $y > 0$, the denominator is positive. We show that the numerator is also positive, $g(y) = -y^2 e^{-y} + 1 + e^{-2y} - 2e^{-y} > 0$, from $g(0) = 0$ and $g'(y) > 0$:

$$g'(y) = -2ye^{-y} + y^2 e^{-y} - 2e^{-2y} + 2e^{-y} = e^{-y}\left(-2y + y^2 - 2e^{-y} + 2\right). \tag{43}$$

Likewise, we show $h(y) = -2y + y^2 - 2e^{-y} + 2 > 0$, from $h(0) = 0$ and $h'(y) > 0$:

$$h'(y) = -2 + 2y + 2e^{-y}, \tag{44}$$

which is positive if $e^{-y} > 1 - y$, which is true for any $y \neq 0$ (from the intercept and derivative at zero and from the convexity of the exponential). ∎

Corollary: The function $f : \mathbf{R}^+ \mapsto \mathbf{R}$

$$f(y) = y\frac{1+e^{-y}}{1-e^{-y}} \tag{45}$$

is positive increasing.

Proof: Write

$$f(y) = y\tilde{f}(y) + 2, \tag{46}$$

where $\tilde{f}$ is as in the previous Lemma. The product of two positive increasing functions is positive increasing. ∎

Proof of Result 2 (Expected Total Surplus and Loss for Multiple Traders)

The expected surplus is obtained by integration of

$$-P_t \sum_{i=1}^{n} Y_t^i = -P_t\, n\, a\, e^{-\frac{n-1}{n+1}\frac{\gamma}{\lambda}t} \tag{47}$$

over $t \in [0, T]$, followed by algebraic simplification. The proofs of the monotonicities are direct applications of the Lemma above or of its Corollary, using $y = T$, $y = \gamma$, $y = \frac{1}{\lambda}$, and $y = \frac{n-1}{n+1}$ (with n relaxed to be in $\mathbf{R}$). For λ and n, we also need the fact that the composition of two monotonic functions is monotonic. ■

Proof of Result 3 (Expected Total Surplus and Loss for Two Traders)

By Equation (7), $Y = Y_t^d + Y_t^p = 2ae^{-\frac{1}{3}\frac{\gamma}{\lambda}t}$. By integration of $-P_tY$ over $t \in [0, T]$, followed by algebraic simplification, the results in Equations (12) and (15) are derived. The monotonicities are verified by differentiation of Equation (15).

We define $V_- = V_d - V_p$ and $Y_- = Y_t^d - Y_t^p = 2be^{\frac{\gamma}{\lambda}t}$. Integrate $-P_tY_-$ over $t \in [0, T]$ and simplify to obtain

$$V_- \;=\; V_d - V_p \;=\; -\,u\,\Delta x - \gamma\,\frac{e^{\Gamma}}{e^{\Gamma} - 1}\,\Delta x^2. \tag{48}$$

V_d and V_p are obtained by simplification of $(V_2 + V_-)/2$ and $(V_2 - V_-)/2$.

The proof for the monotonicity of $\frac{V_p}{\Delta V_d}$ is along the same lines as for Result 2 (with lengthier algebra). The bounds on $\frac{V_p}{\Delta V_d}$ are the limits at 0 and $+\infty$, obtained by applying l'Hôpital's rule as needed. ■

Proof of Result 4 (Repeated Game with Two Symmetric Traders)

For trader 1, the gains from cooperation must exceed those of one-time deviation and infinite non-cooperation or

$$\frac{\delta_1}{1-\delta_1}[p_{10}V_1(\Delta x)+\frac{1}{2}p_{11}V_1(2\Delta x)] \geq V_p+\frac{\delta_1}{1-\delta_1}[p_{01}V_p+p_{10}V_d+\frac{1}{2}p_{11}V_2(2\Delta x)]. \tag{49}$$

Since $\Delta V_2 = V_1 - V_2$ is quadratic in Δx, we have that $\frac{1}{2}\,\Delta V_2(2\Delta x) = 2\,\Delta V_2(\Delta x)$ (we then omit the argument when it is Δx). The first equation in (18) follows by solving for δ_1. The second equation is derived similarly for trader 2. For both traders to cooperate, it must be that $\delta \geq max\{\delta_1, \delta_2\}$. The monotonicities can be proved algebraically, along the same lines as for Result 2. ■

Proof of Result 5 (Shocks of Random Magnitude)

Equation (49) can be written in this context as Equation (25), where f is the density of Δx and C and K are as defined. For values of C sufficiently large or values of K sufficiently small, Equation (25) will be satisfied and there will exist a $\widetilde{\Delta x}$ such that cooperation is possible. As long as $|\Delta x| < \widetilde{\Delta x}$, the traders will cooperate since the value of cooperating exceeds that of a one-time deviation and subsequent grim-trigger play. If $|\Delta x| \geq \widetilde{\Delta x}$, the traders will predate and resume cooperation in the next period if possible. If Equation 25 is not satisfied, then cooperation is not possible and the traders will always predate. Thus, there exists a subgame perfect Nash equilibrium as described. The left-hand side of Equation 25 is bounded since f has finite variance and the right-hand-side is unbounded. Hence the supremum of $\widetilde{\Delta x}$ is bounded. For existence of $\overline{\Delta x}$, note that zero is a solution. ■

B Closed-loop equilibrium

In our analysis of the stage game, the strategies considered are deterministic. They are *open-loop*, in the sense that traders choose their strategies at time $t = 0$. It is assumed that traders would not respond to other traders' deviations from their optimal strategies. We would have obtained the same solution had we defined the Y_t^i to be $\mathcal{F}_t$-adapted (where $\mathcal{F}_t$ is the filtration of the σ-algebras generated by B_t). Note however that this is in the strict sense that traders are aware of the underlying process U_t that defines price fundamentals, but not of the actual price P_t and of the other players' trading rates Y_t. (The argument for the solution to this variation on the open-loop problem to be the same as in Result 1 is based on the value functions' linearity in U_t, as is done below in this section.)

We now consider *closed-loop* strategies, in that traders know or can infer the other traders' rate of trading, and respond accordingly. Under such strategies, Y_t^i is adapted to all the information existing at time t, and players can revise their trading decisions at any time based on such information. This means that earlier decisions must account for other players' response function at later times. This innability to commit ahead of time to not deviate from a given strategy over the entire $[0, T]$ period leads to more aggressive strategies than in the open-loop case, including faster racing.

Closed-loop strategies are substantially more difficult to analyze than open-loop, and we do not provide a closed-form solution. We do provide a description of the equilibrium optimal strategies in terms of a value function with two scalar parameter which satisfy a triangular system of nonlinear differential equations. Our numerical simulations have shown the open- and closed-loop solutions not to be substantially different. We provide some analytical justification for this observation. In particular, from a fixed-point analysis of the Riccati equations, we provide closed-form expressions for the expected value for each of the players when T is large.

We derive the closed-loop result for the deterministic case and then show that it holds for the stochastic pricing equation. We consider strategies where the trading rate is constant over time increments of length Δt. The reward for trader i over each time increment is

$$\begin{aligned} r_i(u, y, \Delta t) &= \int_0^{\Delta t} -\left(u + \gamma\, \mathbf{1}^T y\, \tau + \lambda \mathbf{1}^T y\right) y_i d\tau && (50) \\ &= -\left(u + \left(\frac{\gamma}{2}\Delta t + \lambda\right) \mathbf{1}^T y\right) y_i \Delta t. && (51) \end{aligned}$$

We formulate the problem as a dynamic game, with an $n+1$-dimensional state, composed of $u(t) \in \mathbf{R}$ and $\phi(t) \in \mathbf{R}^n$. The first component, $u(t)$, is the expected price at time t, including the permanent price impact of previous trades (in previous notation, $u(t) = U_t + \gamma \sum_{i=1}^n X_t^i$). The n components of $\phi(t)$ are the remaing trading targets for each trader, that is, the amount they still need to trade by T (in previous notation, $\phi_i(t) = x_{Ti} - X_t^i$). The state transition over a period of length Δt with each player trading at a constant rate y_i is

$$\begin{aligned} u'(u, y, \Delta t) &= u + \gamma\, \mathbf{1}^T y\, \Delta t, && (52) \\ \phi'(\phi, y, \Delta t) &= \phi - y \Delta t. && (53) \end{aligned}$$

The value functions for each of the n traders must simultaneously satisfy

$$V^i(u,\phi,t) = \max_{y_i}\; r_i(u,y,\Delta t) + V^i(u'(u,y,\Delta t), \phi'(\phi,y,\Delta t), t+\Delta t), \qquad i = 1,\ldots,n. \tag{54}$$

We will show that the value functions can be represented in the form

$$V^i(u,\phi,t) = -u\phi_i - \alpha(t)\phi_i\mathbf{1}^T\phi + \beta(t)\left(\mathbf{1}^T\phi\right)^2, \tag{55}$$

and derive Riccati equations for $\alpha(t), \beta(t) \in \mathbf{R}$. We take the limit $\Delta t \to 0$ to find the differential equations for the continuous-time case.

Substituting (55), (51), (52), and (53) in (54) and letting $\Delta t \to 0$, we obtain

$$\begin{aligned} -\frac{d\alpha(t)}{dt}\phi_i\mathbf{1}^T\phi + \frac{d\beta(t)}{dt}\left(\mathbf{1}^T\phi\right)^2 = \\ = \max_{y_i} -\lambda\, y_i\mathbf{1}^Ty - \gamma\,\phi_i\mathbf{1}^Ty + \alpha\,\phi_i\mathbf{1}^Ty + \alpha\, y_i\mathbf{1}^T\phi - 2\beta\,\mathbf{1}^Ty\,\mathbf{1}^T\phi. \end{aligned} \tag{56}$$

Differentiating with respect to y_i and equating to zero, we obtain the optimality condition

$$\lambda(y_i + \mathbf{1}^Ty) = (\alpha-\gamma)\phi_i + (\alpha - 2\beta)\mathbf{1}^T\phi, \tag{57}$$

which we collect over $i = 1,\ldots,n$ as

$$\lambda(I + \mathbf{1}\mathbf{1}^T)y = \left((\alpha-\gamma)I + (\alpha-2\beta)\mathbf{1}\mathbf{1}^T\right)\phi. \tag{58}$$

Multiplying on the left by $\frac{1}{\lambda}(I - \frac{1}{n+1}\mathbf{1}\mathbf{1}^T)$, we solve for the equilibrium trading rates as a function of the state,

$$y = \frac{1}{\lambda}\left((\alpha-\gamma)I + \frac{1}{n+1}(-2\beta+\gamma)\mathbf{1}\mathbf{1}^T\right)\phi, \tag{59}$$

which can equivalently be written individually as

$$y_i = \frac{1}{\lambda}\left((\alpha-\gamma)\phi_i + \frac{1}{n+1}(-2\beta+\gamma)\mathbf{1}^T\phi\right),\; i = 1,\ldots,n. \tag{60}$$

The sum of the trading rates is found to be

$$\mathbf{1}^T y = \frac{1}{\lambda}\left(\alpha - 2\frac{n}{n+1}\beta - \frac{1}{n+1}\gamma\right)\mathbf{1}^T\phi. \tag{61}$$

Substituting (60) and (61) in (56), we verify that the structure of the value function as postulated in (55) is in fact preserved over time (more strongly, it can be verified that the structure is preserved before letting $\Delta t \to 0$). Collecting terms and simplifying, we obtain the following triangular representation for the Riccati equations,

$$\frac{d\alpha}{dt} = \frac{1}{\lambda}\alpha(\alpha - \gamma), \tag{62}$$

$$\frac{d\beta}{dt} = \frac{1}{\lambda}\left(2\alpha\beta - \frac{1}{(n+1)^2}(2n\beta + \gamma)^2\right). \tag{63}$$

So far we have assumed a deterministic u. The objectives for the deterministic case are linear in u, the corresponding optimal policies do not depend on u, and the value functions were found to be linear in u. Using these value functions for the stochastic case, the $\partial^2 V^i/\partial u^2$ terms in the HJB equations are zero, and all other terms are as in the HJB equations for the deterministic case. That is, if given value functions and optimal policies satisfy the HJB equations for the deterministic case, they also satisfy the HJB equations for the stochastic case.

While the system of nonlinear equations (62) and (63) is difficult to solve in closed form, a number of its properties can be studied. As $T \to \infty$, and since the expected values are bounded due to the overall convexity of the problem and the triangular structure precludes oscillatory behavior, $\alpha(0)$ and $\beta(0)$ must converge to fixed points of the differential equations. The fixed points for the first equation are 0 and γ, of which only $\alpha = \gamma$ is stable (note that (62) is convex-quadratic in α, and we are considering integration backwards in time). With $\alpha = \gamma$, we solve for the fixed points of β, which are found to be $\frac{1}{2}\gamma$ and $\frac{1}{2n^2}\gamma$, of which only $\beta = \frac{1}{2n^2}\gamma$ is stable (note that (63) is concave-quadratic in β).

In the case $n = 1$, we have $\phi_i(0)\mathbf{1}^T\phi(0) = (\mathbf{1}^T\phi(0))^2 = \Delta x^2$, so that

$$V = -U_0\Delta x - \gamma\Delta x^2 + \frac{1}{2}\gamma\Delta x^2 = -U\Delta x - \frac{1}{2}\gamma\Delta x^2. \tag{64}$$

As expected, we recover the same value as for the open-loop case (which corresponds to a constant trading rate).

Consider now the case $n = 2$, with the distressed trader needing to trade $\phi_1(0) = \Delta x$, and the predatory trader's target being $\phi_2(0) = 0$. For the distressed trader, $\phi_1(0)\mathbf{1}^T\phi(0) = (\mathbf{1}^T\phi(0))^2 = \Delta x^2$. For the predatory trader, $\phi_2(0)\mathbf{1}^T\phi(0) = 0$ and $(\mathbf{1}^T\phi(0))^2 = \Delta x^2$. Table III summarizes the same expected values as in Table I, but for T large ($T \to \infty$ or, equivalently by change of units, λ small, *i.e.*, $\lambda \to 0$). Overall, the expected values are similar. The loss to the distressed trader is somewhat larger in the closed-loop case, the gain to predatory trader somewhat small, and the overall welfare loss somewhat greater. The ratio of gains to the predatory trader per losses to the distressed trader decreases from 1/2 in the open-loop case to 1/3 in the closed-loop case (*cf.* lower bound in Result 3). Cooperation between traders will therefore be more likely in the closed-loop case.

For n traders, and still under the assumption of a long trading horizon, we can derive the shape of the racing under closed-loop strategies. During the racing stage, that is for small t, $\alpha(t)$ and $\beta(t)$ are approximately constant (under

		Open-Loop	Closed-Loop
Surplus to distressed trader, no predation	V_1	$-U_0\,\Delta x - \frac{1}{2}\gamma\,\Delta x^2$	$-U_0\,\Delta x - \frac{1}{2}\gamma\,\Delta x^2$
Surplus to distressed trader during predation	V_d	$-U_0\,\Delta x - \frac{5}{6}\,\gamma\,\Delta x^2$	$-U_0\,\Delta x - \frac{7}{8}\,\gamma\,\Delta x^2$
Change in surplus to distressed trader ($V_1 - V_d$)	ΔV_d	$\frac{1}{3}\,\gamma\,\Delta x^2$	$\frac{3}{8}\,\gamma\,\Delta x^2$
Surplus to predator	V_p	$\frac{1}{6}\,\gamma\,\Delta x^2$	$\frac{1}{8}\,\gamma\,\Delta x^2$
Total surplus to predator and distressed trader	V_2	$-U_0\,\Delta x - \frac{2}{3}\,\gamma\,\Delta x^2$	$-U_0\,\Delta x - \frac{3}{4}\,\gamma\,\Delta x^2$
Change in overall surplus with predatory trading	ΔV_2	$\frac{1}{6}\,\gamma\,\Delta x^2$	$\frac{1}{4}\,\gamma\,\Delta x^2$

Table III: Expected Values from Strategic Trading when $T \to \infty$ or $\lambda \to 0$, for Open-Loop and Closed-Loop Strategies

the assumption of large T or small λ). Using the stable fixed-point values $\alpha = \gamma$ and $\beta = \frac{1}{2n^2}\gamma$ in (60), we obtain

$$y_i = \frac{\gamma}{\lambda} \cdot \frac{n-1}{n} \cdot \frac{\mathbf{1}^T \phi}{n}, \quad i = 1, \ldots, n. \tag{65}$$

Since $y_i = -d\phi_i/dt$, we conclude that, with closed-loop strategies, the racing is of the form

$$y_i = a\, e^{-\frac{\gamma}{\lambda}\frac{n-1}{n} t}, \tag{66}$$

with the constant $a \in \mathbf{R}$ a function of the average trading target. This is slightly faster than what we found for the open-loop case, which was $y_i = a\, e^{-\frac{\gamma}{\lambda}\frac{n-1}{n+1} t}$.

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