Asset Prices and Liquidity in an Exchange Economy

Ricardo Lagos*
Federal Reserve Bank of Minneapolis
and
New York University

October 1, 2005

Abstract

I develop an asset-pricing model in which financial assets are valued for their *liquidity*—the extent to which they are useful in facilitating exchange—as well as for being claims to streams of consumption goods. The implications for average asset returns, the equity-premium puzzle, and the risk-free rate puzzle, are explored both analytically and quantitatively in a version of the model that nests Mehra and Prescott (1985).

JEL Classification: E42, E52, G12
Keywords: asset pricing, liquidity, exchange, equity premium, risk-free rate.

*First draft: May 2005. Preliminary version, please do not circulate without permission. I am grateful to Ellen McGrattan and Fabrizio Perri for many helpful conversations and feedback at various stages. I also thank V.V. Chari, Patrick Kehoe, Nobu Kiyotaki, Narayana Kocherlakota, Hanno Lustig, Erzo G. J. Luttmer, Monika Piazzesi, Ed Prescott, Guillaume Rocheteau, Tom Sargent, Martin Schneider, and Randy Wright for their comments. Daniil Manaenkov, Carlos Serrano, and Jing Zhang provided research assistance. Financial support from the C.V. Starr Center for Applied Economics at NYU is gratefully acknowledged. The views expressed herein are those of the author and not necessarily those of the Federal Reserve Bank of Minneapolis or the Federal Reserve System.

1 Introduction

In this paper I develop an asset-pricing model in which financial assets are valued not only as claims to streams of consumption goods but also for their liquidity. By *liquidity* I mean the degree to which an asset is valued as a medium of exchange at the margin. Specifically, I study a class of exchange economies in which agents sometimes trade goods and financial assets as in Walrasian theory (in well-organized markets, at market-clearing prices), and sometimes as in search theory (in a decentralized manner, with the terms of trade determined by bargaining). Decentralized trade combined with an exchange motive generates the need for a medium of exchange. The equilibrium price and rate of return of a financial asset are partly determined by the asset's usefulness to facilitate exchange. When an asset is held partly for its exchange value, the demand for the asset and its price tend to be higher than if the asset were not held for exchange. Its intrinsic rate of return—which takes into account only the stream of consumption goods that the asset represents—will be lower.

In Sections 2–4 I consider an economy with two assets: an equity share and a one-period government-issued risk-free real bill. In the basic setup, assets differ only in their payoffs, and agents are free to choose which assets to use as means of payment in decentralized trades. In this case, the theory unambiguously predicts that someone testing an agent's Euler equation for the risk-free bill using its intrinsic rate of return would find that, at the margin, this agent can gain from transferring consumption from the future to the present. That is, there would appear to be a *risk-free rate puzzle*.

In Section 5 I analyze versions of the economy in which legal or institutional restrictions give bonds an advantage over equity as a medium of exchange. In this case, it is possible to show that there are degrees of these restrictions for which someone testing an agent's Euler Equation for the intrinsic excess returns would find that, at the margin, the agent can gain from disinvesting in bills and investing in equity: There would appear to be an *equity-premium puzzle*. For this class of economies, the risk-free rate would still seem too low to an outside observer. In fact, the risk-free rate will be even lower than it would be in the absence of the legal or institutional restrictions. Without these restrictions, the theory may still be consistent with an equity-premium puzzle, depending on parameter values. These results are discussed in Section 6.

In Section 7 I calibrate the model economies and study the extent to which they are able to generate average equity returns and risk-free rates that are in line with U.S. data. (For the empirical implementation, the model must be extended so that it is stationary in the growth rate of the aggregate endowment. This extension is worked out in the Appendix.) Since the class of model economies I consider nests the one studied by Mehra and Prescott (1985), I can quantify the degree to which the liquidity mechanism considered here can help explain the anomalies they identified. Mehra and Prescott's test of their theory essentially consisted of experimenting with different values of the curvature of the agent's utility function (call it σ) to find out for which values the average risk-free rate and equity premium in the model matched those in the U.S. economy. I carry out a similar exercise.

First, I consider the economy with no legal or institutional differences between equity shares and bills, and assess the ability of the model to produce risk-free rates and equity premia that match the data, for values of σ ranging from 1 to 10. I find that for values of σ up to 7, the liquidity mechanism is inactive, and the equilibrium looks just like the one in Mehra–Prescott. For values of σ equal to or greater than 8, equity shares and bills are valuable in decentralized exchange at the margin, which lowers the return on equity and the risk-free rate from what they would be in the Mehra–Prescott economy and brings them closer to the data. However, relative to the data, for this range of σ the equity return is a bit too low and the risk-free rate a bit too high, so the average equity premium is still too low.

I then go on to consider the specification with legal or institutional restrictions, in which equity shares are not accepted as a means of payment in a fraction θ of decentralized exchanges. For this specification, the question I pose is, for a given value of σ, how large does θ (the relative assumed "illiquidity" of equity) have to be for the model to generate an average risk-free rate of

1% and an average equity premium that matches the long-term average for the U.S. economy? The answer is, quite small. Section 7.3 offers a way to assess the absolute size of θ by relating it to relative trade volumes of equity and bonds.

In Section 8 I discuss how the liquidity mechanism operates in this model, and what it adds to Mehra–Prescott, by contrasting how the mean and standard deviation of their respective stochastic discount factors fare against the bounds of Hansen and Jagannathan (1991). In this section, I also use the model to decompose the equity premium into two components: a pure risk premium for bearing nondiversifiable aggregate consumption risk, and an illiquidity premium related to bonds being easier to trade away if a decentralized trade opportunity arises.

By now a vast literature seeks to solve the puzzle identified by Mehra and Prescott (1985). As they framed it, the puzzle is the observation that the restrictions that a particular class of general equilibrium models place upon average returns of equity and Treasury bills are violated by U.S. data. This particular class of models has: (*i*) agents who maximize the expected discounted value of a stream of utilities generated by a power utility function; (*ii*) "frictionless" trading (e.g., no brokerage fees or other trading or transaction costs); and (*iii*) complete asset markets (agents can write insurance contracts against any contingency).

The literature spurred by Mehra and Prescott can be classified depending on which of these ingredients it alters.[1] From this angle, this paper relaxes (*ii*) and (*iii*). There are trading frictions in the sense that agents sometimes trade bilaterally instead of in a Walrasian marketplace. Markets are incomplete in that agents cannot make binding commitments, and trading histories are private in a way that precludes any borrowing and lending between people. Therefore, all trade—in both the centralized and decentralized markets—must be quid pro quo. In terms of this broad taxonomy, this paper is related to Aiyagari and Gertler (1991). They consider an economy with equity and government bonds in which agents face idiosyncratic shocks, and markets are incomplete in a way that individual agents must self-insure. In their model, agents hold assets not only for the stream of dividends they yield but also as a vehicle of

[1] I will not attempt to list all the relevant work in the area. See Mehra and Prescott (2003) for references.

self-insurance. This alone can help to lower the risk-free rate. The basic logic of this mechanism is similar to the one I am emphasizing, except that here, the additional motive for holding assets is their role in transactions rather than self-insurance considerations.[2] At a conceptual level, the paper also shares the basic premise of Bansal and Coleman (1996) and Kiyotaki and Moore (2005), namely that an asset's role in facilitating some form of exchange will manifest itself in the equity premium and as a risk-free rate puzzle.

This paper is also closely related to the literature that provides micro foundations for monetary economics based on search theory, as pioneered by Kiyotaki and Wright (1989). This approach has proven useful for understanding the nature of monetary exchange by making explicit the frictions—e.g., the configuration of meetings, specialization patterns, information structure, and so on—that make monetary exchange an equilibrium. Put differently, this approach has proven useful in pricing the most elusive among financial assets: fiat money, an asset that is a formal claim to nothing yet sells at a positive price. Somehow, this literature and the mainstream asset-pricing literature have managed to stay disconnected.[3] Recently, Duffie, Gârleanu, and Pedersen (2005a,b), Vayanos and Wang (2005), Vayanos and Weill (2005), and Weill (2005a,b) have begun to build some interesting connections between both fields. They model asset trading as a decentralized exchange process that resembles the original vintages of the equilibrium search models of Diamond (1982) and Kiyotaki and Wright (1989). This paper also bridges these fields, in the precise sense that the model can be viewed as a blend of Lagos and Wright (2005)—a recent vintage of the search-based model of exchange—and Lucas (1978).[4]

[2] To widen the spread between the risk-free rate and the return on equity, Aiyagari and Gertler (1991) introduce differential (proportional) trading costs across equity and bonds. If transaction costs on bonds are lower than on equity, then in equilibrium equity must pay a premium, which they refer to as a "transactions/liquidity premium." They also emphasize the model implications for the volumes of trade for bonds and equity as a way of assessing the plausibility of the magnitudes of the trading costs that they feed into the model. Heaton and Lucas (1996) analyze an economy similar to the one in Aiyagari and Gertler, but they allow for aggregate uncertainty. Other papers that consider various combinations of transaction costs and short-sale constraints include He and Modest (1995), Lucas (1994), Luttmer (1996), and Telmer (1993). See Heaton and Lucas (1995) for a survey.

[3] Mehra and Prescott (1985) were the first to point out the similarities between the equity premium puzzle and the "rate-of-return-dominance puzzle" that pervades the pure theory of money. Kocherlakota (1996) picked up on this in his concluding section. I will return to this issue in mine.

[4] Contemporaneously, Ravikumar and Shao (2005) are working on a related model that instead combines features of Lagos and Wright (2005) and Shi (1997) with Lucas (1978). Our papers clearly share much ground:

2 The Environment

There is a $[0,1]$ continuum of agents, time is discrete and the horizon infinite. Each period is divided into two subperiods where different activities take place. There are three nonstorable and perfectly divisible consumption goods at each date: fruit, general goods, and special goods. ("Nonstorable" here means that the goods cannot be carried from one subperiod to the next.) The only durable commodity in the economy is a set of "Lucas trees." The number of trees is fixed and equal to the number of agents. Trees yield (the same amount of) fruit in the second subperiod of every period. Production of fruit is entirely exogenous: no resources are utilized and it is not possible to affect the output at any time.

In each subperiod every agent is endowed with $\bar{n}$ units of time that can be employed as labor services. In the second subperiod, each agent has access to a production technology that allows him to transform labor services into general goods one-for-one. In the first subperiod, each agent has access to another production technology that transforms his own labor input, n, into q units of a particular variety of the special good that he himself does not consume, according to $q = l(n)$, with $l(0) = 0$ and $l' > 0$.

Each agent produces a subset and consumes a different subset of the special goods. Specialization is modeled as follows. Given two agents i and j drawn at random, there are three possible events. The probability that i consumes something j produces but not vice-versa (a single coincidence) is denoted α. Symmetrically, the probability that j consumes something i produces but not vice-versa is also α. In a single-coincidence meeting I call *buyer* the agent who wishes to consume and *seller* the agent who produces. The probability neither wants anything the other can produce is $1 - 2\alpha$, which means $\alpha \leq 1/2$. In contrast to special goods, fruit and general goods are consumed by all agents.

In the first subperiod, agents participate in a *decentralized market* where trade is bilateral

both investigate how the role that an asset plays in the exchange process affects its equilibrium price. As for differences—aside from several in terms of modeling—their agents trade a single asset (equity), as the model is designed to address the *excess volatility puzzle*, rather than the *equity premium puzzle* and the *risk-free rate puzzle*. So all in all, our work is decidedly complementary.

(each meeting is a random draw from the set of pairwise meetings), and the terms of trade are determined by bargaining. The specialization of agents over consumption and production of the special good combined with bilateral trade, give rise to a double-coincidence-of-wants problem in the first subperiod. In the second subperiod, agents trade in a *centralized market*. Agents cannot make binding commitments, and trading histories are private in a way that precludes any borrowing and lending between people, so all trade—both in the centralized and decentralized markets—must be *quid pro quo*.

Each tree has outstanding one durable and perfectly divisible equity share that represents the bearer's ownership of a tree and confers him the right to collect the fruit dividends. Later I will introduce a second perfectly divisible asset, a one-period risk-free government-issued real bill. All assets are perfectly recognizable, cannot be forged, and can be traded among agents in the centralized and decentralized markets. At $t = 0$ each agent is endowed with a_0^s equity shares, and possibly also with a_0^b units of the bond.

Let $u(q)$ be the utility of consuming quantity q of special goods, and $\hat{e}(x)$ be the disutility from working x hours in the first subperiod. Since producing quantity q of special goods requires $x = l^{-1}(q)$ hours, the disutility from producing q is $\hat{e}\left[l^{-1}(q)\right] \equiv e(q)$. Let $U(c)$ be the utility from consuming quantity c of fruit, $v(y)$ be the utility from consuming quantity y of general goods, and $A > 0$ be the marginal disutility from working in the second subperiod. Each agent wishes to maximize

$$E\left\{\sum_{t=0}^{\infty} \beta^t \left[u(q_t) - e(\tilde{q}_t) + v(y_t) + U(c_t) - Ah_t\right]\right\},$$

where $\beta \in (0, 1)$, q_t and $\tilde{q}_t$ are the quantities of special goods consumed and produced in the decentralized market, y_t denotes consumption of general goods, c_t consumption of fruit, h_t the number of hours worked in the second subperiod, and E is an expectations operator. Assume $u(0) = e(0) = 0$, $u' > 0$, $e' > 0$, $v' > 0$, $U' > 0$, $u'' < 0$, $e'' \geq 0$, $v'' \leq 0$, and $U'' < 0$. Also, suppose there exists $q^* \in (0, \infty)$ defined by $u'(q^*) = e'(q^*)$, with $l^{-1}(q^*) \leq \bar{n}$; and $y^* \in (0, \infty)$, defined by $v'(y^*) = A$, with $v(y^*) > Ay^*$.[5]

3 Equity

I begin by considering the case where the equity share is the only asset. Let d_t be the quantity of fruit, or "dividends," produced by each tree in period t, and let a_t be an individual agent's share holdings. Let $W(a_t, d_t)$ denote the value function of an agent who enters the centralized market holding a_t shares in a period when dividends are d_t, and let $V(a_t, d_t)$ denote the corresponding value when he enters the decentralized market. These value functions satisfy the following Bellman equation:

$$W(a_t, d_t) = \max_{c_t, y_t, n_t, h_t, a_{t+1}} \{U(c_t) + v(y_t) - Ah_t + \beta EV(a_{t+1}, d_{t+1})\}$$

$$\text{s.t. } c_t + w_t n_t + \phi_t a_{t+1} = (\phi_t + d_t) a_t + w_t h_t$$

$$0 \le c_t,\ y_t = n_t,\ 0 \le n_t,\ 0 \le h_t \le \bar{n},\ 0 \le a_{t+1}.$$

The agent chooses consumption of fruit (c_t), of general goods (y_t), how many hours of work to demand (n_t) and supply (h_t), and an end-of-period portfolio (a_{t+1}). Dividends $\{d_t\}$ follow a Markov process with $\Pr(d_{t+1} \le x' | d_t = x) = F(x', x)$. The conditional expectation E will be defined with respect to this transition probability. Let Ξ denote the support of F. The realization of $d_t \in \Xi$ is known when agents enter the decentralized market at the beginning of period t. Dividends are paid to the bearer of the (equity) share after decentralized trade, but before the time-t centralized trading session. Fruit is used as numeraire: w_t is the real wage, and ϕ_t is the price of a share (ex-dividend).[6] Substituting the constraints that hold with

[5] This formulation with three consumption goods is in some sense the most parsimonious integration of the asset pricing model of Lucas (1978) with the model of exchange in Lagos and Wright (2005). I have considered other formulations, for example, one where general and special goods are the only consumption goods, while fruit—together with labor—is a necessary input in the production of the general consumption good. (See the Appendix to Lagos (2005) for details.)

[6] The agent also has perceived pricing functions that map the current state d_t (or more generally the whole sequence of realizations $\{d_j\}_{j=0}^{t}$) into current prices ϕ_t and ω_t. More on this below.

equality,

$$W(a_t, d_t) = \max_{c_t, n_t, a_{t+1}} \left\{ U(c_t) + v(n_t) - (c_t + w_t n_t + \phi_t a_{t+1}) \frac{A}{w_t} + \lambda_t a_t + \beta E V(a_{t+1}, d_{t+1}) \right\},$$

where $\lambda_t \equiv \frac{A}{w_t}(\phi_t + d_t)$.

Given the buyer and the seller have share holdings a and $\tilde{a}$ respectively, the terms at which they trade in the decentralized market are $[q(a,\tilde{a}), p(a,\tilde{a})]$, where $q(a,\tilde{a}) \in \mathbb{R}_+$ is the quantity of special good traded, and $p(a,\tilde{a}) \in \mathbb{R}_+$ represents the transfer of assets from the buyer to the seller. The value of an agent who enters the decentralized market with share holdings a in a period when the dividend realization is d, satisfies

$$\begin{aligned} V(a,d) &= \alpha \int \{u[q(a,\tilde{a})] + W[a - p(a,\tilde{a}), d]\} \, dG(\tilde{a}) + \\ & \alpha \int \{-e[q(\tilde{a},a)] + W[a + p(\tilde{a},a), d]\} \, dG(\tilde{a}) + (1-2\alpha) W(a,d), \end{aligned}$$

where G denotes the distribution of share holdings.

Consider a meeting in the decentralized market between a buyer holding a_t and a seller holding $\tilde{a}_t$. The terms of trade (q_t, p_t) are determined by Nash bargaining where the buyer has all the bargaining power:

$$\max_{q_t, p_t \le a_t} [u(q_t) + W(a_t - p_t, d_t) - W(a_t, d_t)] \text{ s.t. } -e(q_t) + W(\tilde{a}_t + p_t, d_t) \ge W(\tilde{a}_t, d_t).$$

The constraint $p_t \le a_t$ indicates that the buyer cannot spend more assets than he owns. Note that $W(a_t + p_t, d_t) - W(a_t, d_t) = \lambda_t p_t$, so the bargaining problem reduces to:

$$\max_{q_t, p_t \le a_t} [u(q_t) - \lambda_t p_t] \text{ s.t. } -e(q_t) + \lambda_t p_t \ge 0.$$

If $\lambda_t a_t \ge e(q^*)$ then the buyer exchanges $p_t = e(q^*)/\lambda_t \le a_t$ of his shares for the first-best quantity q^*. Else, he gives the seller all his shares, $p_t = a_t$, in exchange for the q_t that solves $e(q_t) = \lambda_t a_t$. Note that the outcome depends only on $\lambda_t a_t$, and in particular, it is independent of the seller's asset holdings. Hence, the solution to the bargaining problem is

$$q(\lambda_t a_t) = \begin{cases} q^* & \text{if } \lambda_t a_t \ge e(q^*) \\ e^{-1}(\lambda_t a_t) & \text{if } \lambda_t a_t < e(q^*). \end{cases} \tag{1}$$

Using this bargaining solution and the linearity of $W(a,d)$, the value of search can be written more compactly as

$$V(a,d)=\mathcal{S}(\lambda a)+W(a,d)\,, \tag{2}$$

where $\mathcal{S}(\lambda a)=\alpha\{u[q(\lambda a)]-e[q(\lambda a)]\}$. Observe that $\mathcal{S}$ is twice differentiable everywhere, with $\mathcal{S}'\geq 0$ and $\mathcal{S}''\leq 0$ (both inequalities are strict for $\lambda a<e(q^*)$). Having characterized the terms of trade in decentralized exchange, I turn to the agent's individual optimization problem in the centralized market.

The agent's problem in the second subperiod is summarized by

$$W(a_t,d_t)=\lambda_t a_t+\max_{c_t,n_t}\left[U(c_t)+v(n_t)-(c_t+w_t n_t)\frac{A}{w_t}\right]+\max_{a_{t+1}}\left[\beta EV(a_{t+1},d_{t+1})-\frac{A}{w_t}\phi_t a_{t+1}\right].$$

Given that U and v are strictly concave, the unique solution (c_t,n_t) satisfies

$$U'(c_t)=\frac{A}{w_t} \tag{3}$$

$$v'(n_t)=A. \tag{4}$$

The first-order condition for the choice of a_{t+1} is

$$\frac{A}{w_t}\phi_t=\beta E_t V_1(a_{t+1},d_{t+1})\,.$$

From (2), $V_1(a_{t+1},d_{t+1})=\left[1+\alpha\left(\frac{u'[q(\lambda_{t+1}a_{t+1})]}{e'[q(\lambda_{t+1}a_{t+1})]}-1\right)\right]\lambda_{t+1}$, and $V_{11}(a_{t+1},d_{t+1})\leq 0$ (<0 for $\lambda_{t+1}a_{t+1}<e(q^*)$). Note that none of the agent's choices depend on his individual asset holdings.[7] I will consider an equilibrium in which all prices are time-invariant functions of the aggregate state: $w_t=w(d_t)$, $\phi_t=\phi(d_t)$, and therefore, $\lambda_t=\lambda(d_t)\equiv\frac{A}{w(d_t)}[\phi(d_t)+d_t]$.

[7] Also, if $a_{t+1}<e(q^*)/\lambda_{t+1}$ for some realizations of the dividend process at date $t+1$, the portfolio choice problem at date t has a unique solution, implying that the distribution of assets is degenerate at the beginning of each decentralized round of trade. Regarding the constraints, the agent's maximization is subject to $0\leq n_t$, and $0\leq c_t$, but these will not bind if, for example, $U'(0)=v'(0)=+\infty$. Similarly, in equilibrium shares will be valued and somebody has to hold them, so $0\leq a_{t+1}$ will not bind either. On the other hand, the constraints $0\leq h_t\leq\bar{n}$, or equivalently,

$$0\leq\frac{1}{w_t}\left[c_t+\omega_t n_t+\phi_t a_{t+1}-(\phi_t+d_t)\,a_t\right]\leq\bar{n},$$

may bind. This is relevant because the result that the distribution of assets is degenerate at the beginning of each decentralized round of trade relies on these constraints being slack. I proceed under the assumption that they are slack, and in the Appendix derive parametric conditions such that this is indeed the case along the equilibrium path.

A *recursive equilibrium* is a collection of individual decision rules $c_t = c(d_t)$, $n_t = n(d_t)$, $a_{t+1} = a(d_t)$, pricing functions $w_t = w(d_t)$ and $\phi_t = \phi(d_t)$, and bilateral terms of trade $q_t = q(d_t)$ and $p_t = p(d_t)$ such that: *(a)* the decision rules $c(\cdot)$, $n(\cdot)$, and $a(\cdot)$, solve the agent's problem in the decentralized market, given prices; *(b)* the terms of trade are determined by Nash bargaining, i.e., $q[\lambda(d_t)a_t]$ is given by (1)) and $p(d_t) = \min\left[a_t, \frac{e(q^*)}{\lambda(d_t)}\right]$; and *(c)* prices are such that the centralized market clears, i.e., $c(d_t) = d_t$, and $a(d_t) = 1$.

Condition (4) immediately implies $n(x) = y^*$ for all realizations x of the aggregate dividend process, and according to (3), the equilibrium wage function is $w(x) = \frac{A}{U'(x)}$. (This implies $\lambda(x) = U'(x)[\phi(x) + x]$.) The Euler equation for share holdings implies the pricing function for shares satisfies

$$U'(x)\phi(x) = \beta \int L[\phi(x')] U'(x') [\phi(x') + x'] dF(x', x), \tag{5}$$

where

$$L[\phi(x')] \equiv 1 + \alpha \left(\frac{u'\{q[U'(x')(\phi(x') + x')]\}}{e'\{q[U'(x')(\phi(x') + x')]\}} - 1 \right).$$

This can be rewritten as

$$\begin{aligned} U'(x)\phi(x) &= \beta \int_\Omega L[\phi(x')] U'(x') [\phi(x') + x'] dF(x', x) + \\ &\quad \beta \int_{\Omega^c} U'(x') [\phi(x') + x'] dF(x', x), \end{aligned} \tag{6}$$

where $\Omega = \{x' \in \Xi : U'(x')[\phi(x') + x'] < e(q^*)\}$, and Ω^c denotes its complement. The set Ω contains the realizations of the aggregate dividend process for which—at the margin—the asset has value to the agent for its role as a medium of exchange, in addition to its "intrinsic" value, i.e., that which stems from the right it confers to collect future dividends. So there is a precise sense in which L in (6) can be thought of as a stochastic *liquidity premium*. Notice that equation (6) reduces to equation (6) in Lucas (1978) if either $\Omega = \varnothing$ (there is no liquidity premium in any state of the world, i.e., $L[\phi(x)] = 1$ for all x), or $\alpha = 0$ (agents have no liquidity needs). In what follows, it will often prove convenient to express (5) as a functional equation in λ:

$$\lambda(x) - xU'(x) = \beta \int \left[1 + \alpha \left(\frac{u'\{q[\lambda(x')]\}}{e'\{q[\lambda(x')]\}} - 1 \right)\right] \lambda(x') dF(x', x). \tag{7}$$

In applications, one will typically have to solve (7) numerically, and I will do this in Section 7. But some useful insights, in particular regarding the properties of $\phi(x)$ and the structure of the set Ω, can be gained by first considering some special cases that can be solved by paper-and-pencil methods.

3.1 Examples

Suppose $\{d_t\}$ is a sequence of independent random variables: $F(d_{t+1}, d_t) = F(d_{t+1})$. In this case (7) implies $\lambda(x) - xU'(x) = \beta\Delta$, where Δ satisfies

$$\Delta = \int \left[1 + \alpha\left(\frac{u'\{q[\beta\Delta + zU'(z)]\}}{e'\{q[\beta\Delta + zU'(z)]\}} - 1\right)\right]\left[\beta\Delta + zU'(z)\right] dF(z). \tag{8}$$

(Lemma 1 in the Appendix shows that (8) has a unique solution.) Therefore,

$$\phi(x) = \frac{\beta\Delta}{U'(x)}. \tag{9}$$

The set of realizations of the dividend process for which there is a liquidity premium is:

$$\Omega = \{x \in \Xi : xU'(x) < e(q^*) - \beta\Delta\}. \tag{10}$$

In general, the *intrinsic gross return to equity* between states x_i and x_j is defined as

$$\hat{R}^s(x_i, x_j) = \frac{\phi(x_j) + x_j}{\phi(x_i)}.$$

And for this i.i.d. case,

$$\hat{R}^s(x_i, x_j) = \left[1 + \frac{x_j U'(x_j)}{\beta\Delta}\right]\frac{U'(x_i)}{U'(x_j)}. \tag{11}$$

One can think of q^* as indexing the economy's liquidity needs. If $e(q^*) \leq \frac{\beta}{1-\beta}\int zU'(z)$, then $\Omega = \varnothing$, and $\Delta = \frac{1}{1-\beta}\int zU'(z)$. That is, if q^* is relatively low, then asset prices reduce to those in the i.i.d. example in Lucas (1978). Clearly, the same happens if one simply specifies that the asset is completely illiquid, say by setting $\alpha = 0$. Lemma 1 in the Appendix shows that Δ is increasing in α, so the price of equity will be higher, and its (state-by-state) return lower in every state of the world, the higher the probability the asset can be used in exchange. Examples 1 and 2 study two further specializations of this i.i.d. case.

Example 1 *Suppose $U(c) = \log c$. Then either $\Omega = \varnothing$ or $\Omega^c = \varnothing$ for all realizations of the dividend process, d_t. In fact, if $e(q^*) \leq \frac{1}{1-\beta}$, then $\Omega = \varnothing$ (there is no liquidity premium in any state), (8) implies $\Delta = \frac{1}{1-\beta}$, and (9) implies $\phi(x) = \frac{\beta}{1-\beta}x$. State-by-state equity returns are $\hat{R}^s(x_i, x_j) = \beta^{-1}\frac{x_j}{x_i}$. Conversely, if $\frac{1}{1-\beta} < e(q^*)$, then $\Omega^c = \varnothing$, and $\phi(x) = \beta\Delta x$, where Δ solves*

$$\frac{u'[q(1+\beta\Delta)]}{e'[q(1+\beta\Delta)]} = 1 + \frac{(1-\beta)\Delta - 1}{\alpha(1+\beta\Delta)}. \tag{12}$$

One can show that $\frac{1}{1-\beta} < \Delta < \frac{e(q^*)-1}{\beta}$. The first inequality means that asset prices are higher in every state in the economy where agents have relatively large liquidity needs (the economy with high q^*). The liquidity premium is constant in all states: $L = 1 + \alpha\left\{\frac{u'[q(1+\beta\Delta)]}{e'[q(1+\beta\Delta)]} - 1\right\}$, and $L > 1$ since $\Delta < \frac{e(q^*)-1}{\beta}$. The intrinsic return on equity between any two states, $\hat{R}^s(x_i, x_j) = \left(1 + \frac{1}{\beta\Delta}\right)\frac{x_j}{x_i}$, is lower in the economy where the asset bears a liquidity premium. An increase in the probability the asset can be used in exchange, α, increases Δ and hence increases the price of the asset and reduces its state-by-state rate of return. It is also possible to show that the liquidity premium L is increasing in α.

Example 2 *Suppose $U(c) = \frac{c^{1-\sigma}-1}{1-\sigma}$, with $\sigma > 0$ and $\sigma \neq 1$. (Example 1 corresponds to the special case of $\sigma = 1$.) Note that (10) becomes $\Omega = \left\{x \in \Xi : x^{1-\sigma} < e(q^*) - \beta\Delta\right\}$, where Δ solves (8). If $e(q^*) \leq \frac{\beta}{1-\beta}\int z^{1-\sigma}dF(z)$, then $\Omega = \varnothing$ and there is never a liquidity premium: $L = 1$ in all states. Asset prices and state-by-state returns are*

$$\phi(x) = \beta\Delta x^{\sigma} \tag{13}$$

$$\hat{R}^s(x_i, x_j) = \left(1 + \frac{x_j^{1-\sigma}}{\beta\Delta}\right)\frac{x_j^{\sigma}}{x_i^{\sigma}}, \tag{14}$$

with $\Delta = \Delta_0 \equiv \frac{1}{1-\beta}\int z^{1-\sigma}dF(z)$. If $e(q^*) > \frac{\beta}{1-\beta}\int z^{1-\sigma}dF(z)$ (and the support Ξ is wide enough), then $\Omega \neq \varnothing$ and $\Omega^c \neq \varnothing$, i.e., there will be a liquidity premium in some but not all states. Asset prices and returns are still as in (13) and (14), except that $\Delta = \Delta_1$, where Δ_1 solves

$$\Delta = \int \left[1 + \alpha\left(\frac{u'\left[q\left(\beta\Delta + z^{1-\sigma}\right)\right]}{e'\left[q\left(\beta\Delta + z^{1-\sigma}\right)\right]} - 1\right)\right]\left(\beta\Delta + z^{1-\sigma}\right)dF(z).$$

Note that $\Delta_0 < \Delta_1$, so asset prices are higher, and state-by-state returns lower in this economy than in the one with $e(q^*) \leq \frac{\beta}{1-\beta} \int z^{1-\sigma} dF(z)$. The set Ω takes a simple form: Let $x^* = [e(q^*) - \beta\Delta_1]^{\frac{1}{1-\sigma}}$; then, $\Omega = \{x \in \Xi : x > x^*\}$ if $\sigma > 1$, and $\Omega = \{x \in \Xi : x < x^*\}$ if $\sigma < 1$. So if the Arrow-Pratt coefficient of relative risk aversion is larger (smaller) than one, there is a liquidity premium for large (small) realizations of the dividend process.

Next, generalize the dividend process by allowing it to be serially correlated over time, but specialize preferences over *special* goods by assuming $u(q) = \log q$, and $e(q) = q$.[8] In this case $q^* = 1$, so $\frac{u'\{q[\lambda(x)]\}}{e'\{q[\lambda(x)]\}} = \max\left[1, \lambda(x)^{-1}\right]$ and (7) becomes

$$\lambda(x) = \beta \int \left\{(1-\alpha)\lambda(x') + \alpha \max\left[\lambda(x'), 1\right]\right\} dF(x', x) + xU'(x). \tag{15}$$

Lemma 2 in the Appendix shows that there exists a unique continuous and bounded function λ that solves (15). In general, the liquidity constraint $\lambda(x) \leq 1$ may bind in some states and not in others, but to illustrate, consider two special cases. First, if the constraint never binds, i.e., $\lambda(x) \geq 1$ for all $x \in \Xi$, then (15) reduces to

$$\lambda(x) = \beta \int \lambda(x') dF(x', x) + xU'(x), \tag{16}$$

which is identical to equation (6) in Lucas (1978), after substituting $\lambda(x) = U'(x)[\phi(x) + x]$.

Alternatively, if the constraint binds in every state of the world, i.e., $\lambda(x) < 1$ for all $x \in \Xi$, then (15) becomes

$$\lambda(x) = \beta(1-\alpha) \int \lambda(x') dF(x', x) + \beta\alpha + xU'(x). \tag{17}$$

Let $x = x_t$, $x' = x_{t+1}$, $\phi(x) = \phi_t$, revert to a sequential formulation and iterate on (17), to arrive at

$$\phi_t = \frac{\alpha\beta}{1-\beta(1-\alpha)} \frac{1}{U'(x_t)} + E_t \sum_{j=1}^{\infty} \frac{[\beta(1-\alpha)]^j U'(x_{t+j})}{U'(x_t)} x_{t+j}. \tag{18}$$

[8] Strictly speaking, standard CRRA preferences do not satisfy the maintained assumption $u(0) = 0$. But consider instead $u(q) = \frac{(q+b)^{1-\gamma} - b^{1-\gamma}}{1-\gamma}$ with $\gamma > 0$ and $b > 0$. Note that $-\frac{qu''(q)}{u'(q)} = \frac{\gamma}{1+b/q}$. As $\gamma \to 1$, $u(q) \to \ln(q+b) - \ln(b)$, and $-\frac{qu''(q)}{u'(q)} \to \frac{1}{1+b/q}$. If $b \approx 0$, then this is essentially the utility function in the text. (See also footnote 39.)

If one shuts down the decentralized market, say by setting $\alpha = 0$, then (18) reduces to a standard textbook asset pricing equation (e.g., equation (3.11) in Sargent (1987), p.96). Note that (18) was derived assuming that $\lambda(x_t) < 1$ for all x_t, or equivalently, that $U'(x_t)(\phi_t + x_t) < 1$ for all x_t. And this is indeed the case in equilibrium if the following parametric restriction is satisfied for all x_t:

$$\frac{\alpha\beta}{1-\beta(1-\alpha)} + E_t \sum_{j=1}^{\infty} [\beta(1-\alpha)]^j U'(x_{t+j}) x_{t+j} + U'(x_t) x_t < 1. \tag{19}$$

So far I have not parametrized preferences over fruit, but consider the following example.

Example 3 *Suppose $U(c) = \varepsilon \log c$ (in addition to $u(q) = \log q$, and $e(q) = q$), then (18) becomes*

$$\phi(x) = \frac{\beta[\alpha + (1-\alpha)\varepsilon]}{\varepsilon[1-\beta(1-\alpha)]} x, \tag{20}$$

and condition (19) reduces to $\varepsilon < 1 - \beta$. In this case, $\hat{R}^s(x_i, x_j) = \frac{\alpha\beta+\varepsilon}{\beta[\alpha+(1-\alpha)\varepsilon]} \frac{x_j}{x_i}$. Notice that $\frac{\partial\phi(x)}{\partial\alpha} = \frac{\beta(1-\beta-\varepsilon)}{\varepsilon[1-\beta(1-\alpha)]^2} x > 0$, and $\frac{\partial\hat{R}^s(x_i,x_j)}{\partial\alpha} = \frac{-\varepsilon(1-\beta-\varepsilon)}{\beta[\alpha+(1-\alpha)\varepsilon]^2} \frac{x_j}{x_i} < 0$. So again, prices are increasing, and all state-by-state returns are decreasing in the probability the asset can help the agent satisfy his liquidity needs.

4 Bonds

In this section I introduce a government-issued one-period risk-free real bill, which I will refer to as a "bond." Let B_t denote the stock of bonds that are outstanding in period t, to be redeemed before the centralized trading session of period t. (The government sells B_{t+1} in the centralized market at the end of period t.) What I call the "government," is essentially summarized by the budget constraint,

$$B_t = \phi_t^b B_{t+1} + \tau_t, \tag{21}$$

where ϕ_t^b is the price of a bond, and τ_t a lump-sum tax levied on all agents during the centralized trading session, both expressed in terms of fruit. The focus here is not on how the government should select the path $\{B_t, \tau_t\}$, but rather on characterizing the equilibrium, and in particular asset prices and returns, given such a path. Let $\mathbf{a}_t \in \mathbb{R}_+^2$ denote an agent's portfolio. Since they can now hold two assets, $\mathbf{a}_t = \left(a_t^s, a_t^b\right)$, where a_t^s and a_t^b denote the holdings of shares and bonds, respectively. Let ϕ_t^s be the price of a share in terms of fruit, and $\boldsymbol{\phi}_t = \left(\phi_t^s, \phi_t^b\right)$. The value function of an agent who enters the centralized market with portfolio $\mathbf{a}_t$ in a period when dividends are d_t now satisfies:

$$W\left(\mathbf{a}_t, d_t\right) = \max_{c_t, n_t, h_t, \mathbf{a}_{t+1}} \left\{U\left(c_t\right) + v\left(n_t\right) - Ah_t + \beta EV\left(\mathbf{a}_{t+1}, d_{t+1}\right)\right\} \tag{22}$$

$$\text{s.t. } c_t + w_t n_t + \boldsymbol{\phi}_t \mathbf{a}_{t+1} = \left(\phi_t^s + d_t\right) a_t^s + a_t^b + w_t h_t - \tau_t$$

$$0 \leq c_t,\ 0 \leq n_t,\ 0 \leq h_t \leq \bar{n},\ 0 \leq \mathbf{a}_{t+1}.$$

In the decentralized market, the terms at which a buyer with portfolio $\mathbf{a}$ trades with a seller with portfolio $\tilde{\mathbf{a}}$ are $\left[q\left(\mathbf{a}, \tilde{\mathbf{a}}\right), \mathbf{p}\left(\mathbf{a}, \tilde{\mathbf{a}}\right)\right]$, where $q\left(\mathbf{a}, \tilde{\mathbf{a}}\right) \in \mathbb{R}_+$ is the quantity of special good traded, and $\mathbf{p}\left(\mathbf{a}, \tilde{\mathbf{a}}\right) \in \mathbb{R}_+^2$ represents the transfer of assets from the buyer to the seller. The value of an agent who enters the decentralized market holding portfolio $\mathbf{a}$ in a period when the dividend realization is x, satisfies

$$\begin{aligned} V\left(\mathbf{a}, x\right) &= \alpha \int \left\{u\left[q\left(\mathbf{a}, \tilde{\mathbf{a}}\right)\right] + W\left[\mathbf{a} - \mathbf{p}\left(\mathbf{a}, \tilde{\mathbf{a}}\right), x\right]\right\} d\mathbf{G}\left(\tilde{\mathbf{a}}\right) + \\ &\quad \alpha \int \left\{-e\left[q\left(\tilde{\mathbf{a}}, \mathbf{a}\right)\right] + W\left[\mathbf{a} + \mathbf{p}\left(\tilde{\mathbf{a}}, \mathbf{a}\right), x\right]\right\} d\mathbf{G}\left(\tilde{\mathbf{a}}\right) + \left(1 - 2\alpha\right) W\left(\mathbf{a}, x\right), \end{aligned}$$

where $\mathbf{G}$ denotes the distribution of portfolios. The terms of trade $\left[q\left(\mathbf{a}, \tilde{\mathbf{a}}\right), \mathbf{p}\left(\mathbf{a}, \tilde{\mathbf{a}}\right)\right]$ are still determined by Nash bargaining where the buyer has all the bargaining power:

$$\max_{q_t, \mathbf{p}_t \leq \mathbf{a}_t} \left[u\left(q_t\right) + W\left(\mathbf{a}_t - \mathbf{p}_t, d_t\right) - W\left(\mathbf{a}_t, d_t\right)\right] \text{ s.t. } -e\left(q_t\right) + W\left(\tilde{\mathbf{a}}_t + \mathbf{p}_t, d_t\right) \geq W\left(\tilde{\mathbf{a}}_t, d_t\right).$$

Let $\lambda_t^b = \frac{A}{w_t}$, $\lambda_t^s = \frac{A}{w_t}\left(\phi_t^s + d_t\right)$, and note that $W\left(\mathbf{a}_t + \mathbf{p}_t, d_t\right) - W\left(\mathbf{a}_t, d_t\right) = \boldsymbol{\lambda}_t \mathbf{p}_t$, where $\boldsymbol{\lambda}_t = \left(\lambda_t^s, \lambda_t^b\right)$. These observations imply that the bargaining problem reduces to

$$\max_{q_t, \mathbf{p}_t \leq \mathbf{a}_t} \left[u\left(q_t\right) - \boldsymbol{\lambda}_t \mathbf{p}_t\right] \text{ s.t. } -e\left(q_t\right) + \boldsymbol{\lambda}_t \mathbf{p}_t \geq 0.$$

The solution is $q\left(\boldsymbol{\lambda}_t \mathbf{a}_t\right)$, where the function $q\left(\cdot\right)$ is still given by (1). Using this bargaining solution and the linearity of $W\left(\mathbf{a}, d\right)$, the value of search can be written as:

$$V\left(\mathbf{a}, d\right) = \mathcal{S}\left(\boldsymbol{\lambda}\mathbf{a}\right) + W\left(\mathbf{a}, d\right). \tag{23}$$

In the centralized market, the agent's choices of c_t and n_t are still characterized by (3) and (4), respectively. The first-order conditions for asset holdings, $\mathbf{a}_{t+1}$, are:

$$\frac{A}{w_t}\phi_t^i = \beta E_t \frac{\partial V\left(\mathbf{a}_{t+1}, d_{t+1}\right)}{\partial a_{t+1}^i}, \text{ for } i = s, b, \tag{24}$$

where $\frac{\partial V(\mathbf{a}_{t+1},d_{t+1})}{\partial a_{t+1}^s} = \left[1 + \alpha\left(\frac{u'[q(\boldsymbol{\lambda}_{t+1}\mathbf{a}_{t+1})]}{e'[q(\boldsymbol{\lambda}_{t+1}\mathbf{a}_{t+1})]} - 1\right)\right]\lambda_{t+1}^s$, and $\frac{\partial V(\mathbf{a}_{t+1},d_{t+1})}{\partial a_{t+1}^b} = \frac{\partial V(\mathbf{a}_{t+1},d_{t+1})}{\partial a_{t+1}^s}\frac{\lambda_{t+1}^b}{\lambda_{t+1}^s}$, are obtained from (23). Again, the agent's choices do not depend on his individual asset holdings, and the distribution of assets will be degenerate in equilibrium.[9]

Given the dividend process $\{d_t\}_{t=0}^{\infty}$ and a path $\{B_t, \tau_t\}_{t=0}^{\infty}$, an equilibrium is an allocation $\{c_t, n_t, \mathbf{a}_{t+1}\}_{t=0}^{\infty}$, together with a set of prices $\{w_t, \boldsymbol{\phi}_t\}_{t=0}^{\infty}$ and bilateral terms of trade $\{q_t\}_{t=0}^{\infty}$, such that: *(a)* the individual choices $\{c_t, n_t, \mathbf{a}_{t+1}\}_{t=0}^{\infty}$ solve the agent's problem in the decentralized market, given prices; *(b)* the terms of trade are determined by Nash bargaining, i.e., $q_t = \min\left\{q^*, e^{-1}\left[\frac{A}{w_t}\left(\phi_t^s + d_t\right)a_t^s + \frac{A}{w_t}a_t^b\right]\right\}$; and *(c)* prices are such that the centralized market clears, i.e., $c_t = d_t$, $a_{t+1}^s = 1$, and $a_{t+1}^b = B_{t+1}$.

In a *recursive equilibrium*, allocations and prices are time-invariant functions of the aggregate state, x, the current realization of the dividend process. Specifically, $\phi_t^s = \phi^s\left(x_t\right)$, $\phi_t^b = \phi^b\left(x_t\right)$, and using (3), $w_t = w\left(x_t\right) \equiv \frac{A}{U'(x_t)}$. Also, $\lambda_t^s = \lambda^s\left(x_t\right) \equiv U'\left(x_t\right)\left[\phi^s\left(x_t\right) + x_t\right]$, and $\lambda_t^b = U'\left(x_t\right)$. In addition, suppose that the government follows a stationary policy $B_{t+1} = B\left(x_t\right)$, and that the aggregate state follows a Markov process with $\Pr\left(x_{t+1} \le x' | x_t = x\right) = F\left(x', x\right)$. Then, the first-order conditions imply that asset prices satisfy:

$$\lambda^s(x) - U'(x)\,x = \beta\int\left\{1 + \alpha\left(\frac{u'[\lambda^s(x') + U'(x')B']}{e'[\lambda^s(x') + U'(x')B']} - 1\right)\right\}\lambda^s\left(x'\right)dF\left(x', x\right) \tag{25}$$

$$U'(x)\,\phi^b(x) = \beta\int\left\{1 + \alpha\left(\frac{u'[\lambda^s(x') + U'(x')B']}{e'[\lambda^s(x') + U'(x')B']} - 1\right)\right\}U'\left(x'\right)dF\left(x', x\right), \tag{26}$$

where $B' = B(x)$.

If we set $\alpha = 0$ (a world with no liquidity needs), then (25) and (26) reduce to the usual functional equations describing the prices of shares and bonds that one would get from Lucas

[9]Note that the objective function is strictly concave in asset holdings if $\boldsymbol{\lambda}_t\mathbf{a}_t < e\left(q^*\right)$ with positive probability (weakly concave otherwise). The degeneracy of the distribution of asset holds is subject to the same caveats mentioned earlier, and discussed in the appendix, i.e., that the equilibrium path for h_t stays off corners.

(1978) or Mehra and Prescott (1985). For that special case, note that the price function for the share in an economy with bonds is the same as it would be in an economy without the bond: The supply of bonds would be irrelevant in an economy with no liquidity needs. Conversely, if agents are short of liquidity in some states (i.e. if $\alpha > 0$ and $\lambda^s(x') + U'(x') B' < q^*$ with positive probability), then changes in the stock of outstanding bonds affect both, the price of shares and the price of bonds. Note that since shares and bonds can be used freely in decentralized exchange, they both provide the same liquidity return in every state, i.e., $1 + \alpha \left(\frac{u'[\lambda^s(x') + U'(x')B']}{e'[\lambda^s(x') + U'(x')B']} - 1 \right)$. As in the simpler model with a single asset, in applications one will typically have to solve (25) and (26) numerically. (One would first solve (25) for $\lambda^s(x)$, and given this function, then simply read $\phi^b(x)$ from (26).) But it is instructive to first consider some tractable special cases.

4.1 Examples

Suppose that $B_t = B$ for all t, and that $\{d_t\}$ is a sequence of independent random variables, i.e., $F(x_{t+1}, x_t) = F(x_{t+1})$. Condition (25) implies $\lambda^s(x) = \beta\Delta^s + xU'(x)$, where Δ^s solves

$$\Delta = \int \left[1 + \alpha \left(\frac{u'\{q[\beta\Delta + (z+B)U'(z)]\}}{e'\{q[\beta\Delta + (z+B)U'(z)]\}} - 1 \right) \right] [\beta\Delta + zU'(z)]\, dF(z). \tag{27}$$

(The same arguments used in Lemma 1 in the Appendix can be used to establish that a unique Δ^s satisfying this condition exists, and that it is increasing in α.) Given Δ^s, asset prices are given by $\phi^s(x) = \frac{\beta\Delta^s}{U'(x)}$, and $\phi^b(x) = \frac{\beta\Delta^b}{U'(x)}$, where

$$\Delta^b = \int \left\{ 1 + \alpha \left(\frac{u'\{q[\beta\Delta^s + (z+B)U'(z)]\}}{e'\{q[\beta\Delta^s + (z+B)U'(z)]\}} - 1 \right) \right\} U'(z)\, dF(z).$$

The set of realizations of the dividend process for which there is a liquidity premium is:

$$\Omega = \left\{ x \in \Xi : \beta\Delta^s + (x+B)U'(x) < e(q^*) \right\}. \tag{28}$$

With regards to the interaction between the stock of bonds and asset prices, in this simple example it is possible to show that $\frac{\partial\Delta^s}{\partial B} \leq 0$, with strict inequality if $\Omega \neq \varnothing$. That is, in an economy with liquidity needs, equity prices are lower (in every state) the larger is the outstanding stock of bonds. Having bonds in circulation also affects the structure of the set Ω, as the following two examples illustrate.

Example 4 *Suppose $B_t = B$ for all t, $F(x_{t+1}, x_t) = F(x_{t+1})$, and $U(c) = \log c$. Then, (28) specializes to $\Omega = \left\{x \in \Xi : 1 + \frac{B}{x} < e(q^*) - \beta\Delta^s\right\}$, where Δ^s solves (27). If $e(q^*) \leq \frac{1}{1-\beta}$, then $\Omega = \varnothing$. Alternatively, if $e(q^*) > \frac{1}{1-\beta}$, then $\Omega = \{x \in \Xi : x > x^*\}$, with $x^* = \frac{B}{e(q^*)-\beta\Delta^s-1}$. So there can be a liquidity premium only in economies where q^* is large enough, and given this is the case, only for relatively high realizations of the dividend process. Note that as $B \to 0$, in the parametrization with $e(q^*) > \frac{1}{1-\beta}$ things converge to a situation where there is a liquidity premium in every state. Having bonds circulating can qualitatively affect the structure of the equilibrium.*

Example 5 *Suppose $B_t = B$ for all t, $F(x_{t+1}, x_t) = F(x_{t+1})$, and $U(c) = \frac{c^{1-\sigma}}{1-\sigma}$, with $\sigma > 1$. Then, (28) specializes to $\Omega = \left\{x \in \Xi : x^{1-\sigma} + \frac{B}{x^\sigma} < e(q^*) - \beta\Delta^s\right\}$. If $e(q^*) \leq \frac{\beta}{1-\beta}\int z^{1-\sigma}dF(z)$, then $\Omega = \varnothing$. In this case, $\phi^s(x) = \beta\Delta_0^s x^\sigma$, and $\phi^b(x) = \beta\Delta_0^b x^\sigma$, where $\Delta_0^s = \frac{1}{1-\beta}\int z^{1-\sigma}dF(z)$, and $\Delta_0^b = \int z^{-\sigma}dF(z)$. Conversely, if $e(q^*) > \frac{\beta}{1-\beta}\int z^{1-\sigma}dF(z)$, then $\Omega = \{x \in \Xi : x > x^*\}$, where x^* is defined implicitly by $x^{1-\sigma} + \frac{B}{x^\sigma} - e(q^*) + \beta\Delta_1^s = 0$, and Δ_1^s solves*

$$\Delta = \int \left[1 + \alpha\left(\frac{u'\left[q\left(\beta\Delta + z^{1-\sigma} + \frac{B}{z^\sigma}\right)\right]}{e'\left[q\left(\beta\Delta + z^{1-\sigma} + \frac{B}{z^\sigma}\right)\right]} - 1\right)\right]\left(\beta\Delta + z^{1-\sigma}\right)dF(z).$$

In this case asset prices are: $\phi^s(x) = \beta\Delta_1^s x^\sigma$, and $\phi^b(x) = \beta\Delta_1^b x^\sigma$, where

$$\Delta_1^b = \int\left(1 + \alpha\left\{\frac{u'\left[q\left(\beta\Delta_1^s + z^{1-\sigma} + \frac{B}{z^\sigma}\right)\right]}{e'\left[q\left(\beta\Delta_1^s + z^{1-\sigma} + \frac{B}{z^\sigma}\right)\right]} - 1\right\}\right)z^{-\sigma}dF(z).$$

Note that $\Delta_1^i > \Delta_0^i$ for $i = b, s$, so asset prices are uniformly higher in the economy with liquidity needs. (The asset prices corresponding to Example 4 are obtained by setting $\sigma = 1$ in these expressions.)

State-by-state intrinsic returns in the economy of Example 5 are:

$$\hat{R}_k^s(x_i, x_j) = \left(1 + \frac{x_j^{1-\sigma}}{\beta \Delta_k^s}\right)\frac{x_j^\sigma}{x_i^\sigma}$$
$$\hat{R}_k^b(x_i) = \frac{1}{\beta \Delta_k^b x_i^\sigma},$$

where the subscript $k = 0$ indicates the parametrization with $e(q^*) \leq \frac{\beta}{1-\beta}\int z^{1-\sigma} dF(z)$, and $k = 1$ the one with $e(q^*) > \frac{\beta}{1-\beta}\int z^{1-\sigma} dF(z)$. One can think of $\hat{R}_k^s(x_i, x_j) - \hat{R}_k^b(x_i)$ as a state-by-state equity premium. Looking ahead, it is instructive to see how the liquidity of the assets affect this basic notion of equity premium. In this particular example,

$$\hat{R}_1^s(x_i, x_j) - \hat{R}_1^b(x_i) - \left[\hat{R}_0^s(x_i, x_j) - \hat{R}_0^b(x_i)\right] = \left[\left(\frac{1}{\Delta_1^s} - \frac{1}{\Delta_0^s}\right)x_j + \left(\frac{1}{\Delta_0^b} - \frac{1}{\Delta_1^b}\right)\right]\frac{x_i^{-\sigma}}{\beta}. \quad (29)$$

Since $\Delta_1^i > \Delta_0^i$ for $i = s, b$, we have $\frac{1}{\Delta_1^s} - \frac{1}{\Delta_0^s} < 0$ and $\frac{1}{\Delta_0^b} - \frac{1}{\Delta_1^b} > 0$. The state-by-state returns of *both*, shares and bonds are lower in the economy with liquidity needs, so the sign of (29) is ambiguous in general.[10] The key observation is that an asset's intrinsic return, $\hat{R}^i$, is lower when the asset provides agents with liquidity services. So if bonds are more readily accepted in decentralized exchange, then their intrinsic return, $\hat{R}^b$, would be lower relative to the intrinsic return on equity, $\hat{R}^s$.

5 Differential Liquidity

In this section I extend the model of the previous one to situations in which bonds and shares have different (exogenous) liquidity properties. Specifically, suppose an agent can find himself in two types of meetings in the decentralized market: With probability θ_2, he is in a meeting where he can use any of the two assets for payment, while with probability $\theta_1 = 1 - \theta_2$, he can only use bonds.[11] Thus, $\theta_1 \in [0, 1]$ indexes the degree of "illiquidity" of equity shares. (The

[10] Quantitatively, in Section 7 I will find that for the baseline calibration, the net effect of this sort of mechanism on the average equity premium is positive, but modest: about one tenth of one percent. (This calculation is based on a comparison of the last three rows of Tables 2 and 3.)

[11] To interpret the specific modelling choice, one can follow Aiyagari and Wallace (1997), and relate θ_1 to a *government transaction policy* carried out by a small mass of *government agents*. This has become the standard

subscript refers to the number of assets that can be used for payment in that particular type of meeting.)

The value function of an agent who enters the centralized market with portfolio $\mathbf{a}_t$ in a period when dividends are d_t still satisfies (22). As before, the terms of trade in each meeting will be determined by bargaining. Since there are now two types of meetings, there will be two bargaining solutions. For simplicity, I will refer to a meeting where i assets can be used for payment as a "meeting of type i." The terms of trade in a meeting of type i between a buyer who holds portfolio $\mathbf{a}_t = \left(a_t^s, a_t^b\right)$ and a seller with portfolio $\tilde{\mathbf{a}}_t = (\tilde{a}_t^s, \tilde{a}_t^b)$, are $\left[q^i\left(\mathbf{a}_t, \tilde{\mathbf{a}}_t\right), \mathbf{p}^i\left(\mathbf{a}_t, \tilde{\mathbf{a}}_t\right)\right]$, where $q^i\left(\mathbf{a}_t, \tilde{\mathbf{a}}_t\right) \in \mathbb{R}_+$, and $\mathbf{p}^i\left(\mathbf{a}_t, \tilde{\mathbf{a}}_t\right) \in \mathbb{R}_+^2$. In particular, $\mathbf{p}^2\left(\mathbf{a}_t, \tilde{\mathbf{a}}_t\right) = \left[p^s\left(\mathbf{a}_t, \tilde{\mathbf{a}}_t\right), p^b\left(\mathbf{a}_t, \tilde{\mathbf{a}}_t\right)\right]$, where $p^s\left(\mathbf{a}_t, \tilde{\mathbf{a}}_t\right)$ denotes the quantity of shares, and $p^b\left(\mathbf{a}_t, \tilde{\mathbf{a}}_t\right)$ the quantity of bonds that the buyer hands over to the seller. The trading restrictions imply that $\mathbf{p}^1\left(\mathbf{a}_t, \tilde{\mathbf{a}}_t\right) = \left[0, p^b\left(\mathbf{a}_t, \tilde{\mathbf{a}}_t\right)\right]$. The value of an agent who enters the decentralized market holding portfolio $\mathbf{a}$ in a period when the dividend realization is s, satisfies

$$
\begin{aligned}
V\left(\mathbf{a}, s\right) &= \alpha \sum_{i=1,2} \theta_i \int \left\{u\left[q^i\left(\mathbf{a}, \tilde{\mathbf{a}}\right)\right] + W\left[\mathbf{a} - \mathbf{p}^i\left(\mathbf{a}, \tilde{\mathbf{a}}\right), s\right]\right\} d\mathbf{G}\left(\tilde{\mathbf{a}}\right) + \\
&\quad \alpha \sum_{i=1,2} \theta_i \int \left\{-e\left[q^i\left(\tilde{\mathbf{a}}, \mathbf{a}\right)\right] + W\left[\mathbf{a} + \mathbf{p}^i\left(\tilde{\mathbf{a}}, \mathbf{a}\right), s\right]\right\} d\mathbf{G}\left(\tilde{\mathbf{a}}\right) + \left(1 - 2\alpha\right) W\left(\mathbf{a}, s\right),
\end{aligned}
$$

where $\mathbf{G}$ denotes the distribution of portfolios.

The terms of trade $[q^i\left(\mathbf{a}, \tilde{\mathbf{a}}\right), \mathbf{p}^i\left(\mathbf{a}, \tilde{\mathbf{a}}\right)]$ for $i = 1, 2$, are still determined by Nash bargaining

way of introducing legal or other institutional restrictions into environments with decentralized exchange. See, for example, Aiyagari, Wallace, and Wright (1997), or Li and Wright (1998). Also more recently, Shi (2005) has used a similar formulation in a model with fiat money and nominal bonds to study the effects of open market operations. He finds that even an arbitrarily small probability that matured nominal bonds will not be accepted in decentralized exchange is enough for fiat money to drive them out of circulation. Regarding my use of this device, I would like to emphasize that the spirit of this whole exercise is *positive*. That is, here, I want to explore the implications of (small) liquidity differences for the behavior of asset prices in general, and for the equity premium in particular. In fact, given the nature of the findings, below I will argue that understanding the deeper reasons for these differences in the liquidity of these assets—by which I mean the likelihood they will be accepted as means of payment in decentralized exchange—is a necessary next step. In terms of relating these theoretical institutional or legal restrictions to actual features of "real world" trades, consider the following readily-verifiable fact. An investor who places an order to sell shares of an S&P 500 firm on a given day T will typically have to wait until $T + 2$ for settlement, while the settlement for the sale of 90–day U.S. Treasury Bills will usually take place at $T + 1$. These types of considerations seem to suggest that bonds are a more readily available source of funds for agents who must act quickly on some purchase opportunity, which is at least broadly in line with the tradeoffs at work in the theoretical formulation laid out here.

where the buyer has all the bargaining power. Now let $\boldsymbol{\lambda}_t^1 = \left(0, \lambda_t^b\right)$, where $\lambda_t^b = \frac{A}{w_t}$, and $\boldsymbol{\lambda}_t^2 = \boldsymbol{\lambda}_t = \left(\lambda_t^s, \lambda_t^b\right)$, where $\lambda_t^s = \frac{A}{w_t}\left(\phi_t^s + d_t\right)$. Then, the bargaining solution is $q^i\left(\mathbf{a}_t, \tilde{\mathbf{a}}_t\right) = q(\boldsymbol{\lambda}_t^i \mathbf{a}_t)$, where the function $q\left(\cdot\right)$ is still given by (1). Using the bargaining solution and the fact that $W\left[\mathbf{a} + \mathbf{p}^i\left(\mathbf{a}, \tilde{\mathbf{a}}\right), d\right] - W\left(\mathbf{a}, d\right) = \boldsymbol{\lambda}_t \mathbf{p}^i\left(\mathbf{a}, \tilde{\mathbf{a}}\right)$, the value of search can be written as:

$$V\left(\mathbf{a}, d\right) = \sum_{i=1,2} \theta_i \mathcal{S}\left(\boldsymbol{\lambda}^i \mathbf{a}\right) + W\left(\mathbf{a}, d\right). \tag{30}$$

In the centralized market, the agent's choices of c_t and n_t are still characterized by (3) and (4), respectively. The first-order conditions for asset holdings, $\mathbf{a}_{t+1}$, are still those in (24), but now

$$\begin{aligned}
\frac{\partial V\left(\mathbf{a}_{t+1}, d_{t+1}\right)}{\partial a_{t+1}^b} &= \left[1 + \alpha \sum_{i=1,2} \theta_i \left(\frac{u'\left[q\left(\boldsymbol{\lambda}_{t+1}^i \mathbf{a}_{t+1}\right)\right]}{e'\left[q\left(\boldsymbol{\lambda}_{t+1}^i \mathbf{a}_{t+1}\right)\right]} - 1\right)\right] \lambda_{t+1}^b, \\
\frac{\partial V\left(\mathbf{a}_{t+1}, d_{t+1}\right)}{\partial a_{t+1}^s} &= \left[1 + \alpha \theta_2 \left(\frac{u'\left[q\left(\boldsymbol{\lambda}_{t+1}^2 \mathbf{a}_{t+1}\right)\right]}{e'\left[q\left(\boldsymbol{\lambda}_{t+1}^2 \mathbf{a}_{t+1}\right)\right]} - 1\right)\right] \lambda_{t+1}^s,
\end{aligned}$$

are obtained from (30). As usual, the agent's choices do not depend on his individual asset holdings, and the distribution of assets will be degenerate in equilibrium.

Given the dividend process $\{d_t\}_{t=0}^{\infty}$ and a path $\{B_t, \tau_t\}_{t=0}^{\infty}$, an equilibrium is an allocation $\{c_t, n_t, \mathbf{a}_{t+1}\}_{t=0}^{\infty}$, together with a set of prices $\{w_t, \boldsymbol{\phi}_t\}_{t=0}^{\infty}$ and bilateral terms of trade $\{q_t\}_{t=0}^{\infty}$, such that: *(a)* the individual choices $\{c_t, n_t, \mathbf{a}_{t+1}\}_{t=0}^{\infty}$ solve the agent's problem in the decentralized market, given prices; *(b)* the terms of trade are determined by Nash bargaining, i.e., $q_t^1 = \min\left\{q^*, e^{-1}\left(\frac{A}{w_t} a_t^b\right)\right\}$ and $q_t^2 = \min\left\{q^*, e^{-1}\left[\frac{A}{w_t}\left(\phi_t^s + d_t\right) a_t^s + \frac{A}{w_t} a_t^b\right]\right\}$; and *(c)* prices are such that the centralized market clears, i.e., $c_t = d_t$, $a_{t+1}^s = 1$, and $a_{t+1}^b = B_{t+1}$.

In a *recursive equilibrium*, allocations and prices are time-invariant functions of the aggregate state, x, the current dividend realization. Specifically, $\phi_t^s = \phi^s\left(x_t\right)$, $\phi_t^b = \phi^b\left(x_t\right)$, and using (3), $w_t = w\left(x_t\right) \equiv \frac{A}{U'(x_t)}$. In addition, $\lambda_t^s = \lambda^s\left(x_t\right) \equiv U'\left(x_t\right)\left[\phi^s\left(x_t\right) + x_t\right]$, and $\lambda_t^b = U'\left(x_t\right)$. Also, suppose that the government follows the stationary policy $B_{t+1} = B\left(x_t\right)$, and that the aggregate state follows a Markov process with $\Pr\left(x_{t+1} \leq x' | x_t = x\right) = F\left(x', x\right)$. Then, asset prices satisfy:

$$\lambda^s(x) - U'(x)x = \beta \int L^s(x')\lambda^s(x')\,dF(x',x) \tag{31}$$

$$U'(x)\phi^b(x) = \beta \int L^b(x')U'(x')\,dF(x',x), \tag{32}$$

where

$$L^s(x') = 1 + \alpha(1-\theta)\left(\frac{u'\{q[\lambda^s(x') + U'(x')B']\}}{e'\{q[\lambda^s(x') + U'(x')B']\}} - 1\right), \tag{33}$$

$B' = B(x)$, and

$$L^b(x') = L^s(x') + \alpha\theta\left(\frac{u'\{q[U'(x')B']\}}{e'\{q[U'(x')B']\}} - 1\right). \tag{34}$$

I have set $\theta_1 = \theta$ in (33) and (34) to simplify notation. Note that if $\theta = 0$, then (31) and (32) reduce to (25) and (26), respectively.

In this formulation, for different realizations of the aggregate uncertainty, agents may find themselves short of liquidity in all meetings, or perhaps just in those meetings where they can only use bonds, or possibly in no meeting. Formally, $\Omega = \{x' \in \Xi : \lambda^s(x') + U'(x')B' < e(q^*)\}$ is the set of realizations for which agents are short of liquidity in all meetings, while $\Omega_\theta = \{x' \in \Xi : U'(x')B' < e(q^*)\}$ is the set of realizations for which they lack liquidity only in meetings where shares cannot be used as means of payment. For example, if $\Omega_\theta = \varnothing$, then $\Omega = \varnothing$, and agents are never short of liquidity; i.e., $L^s(x) = L^b(x) = 1$ for all x. Alternatively, if $\Omega \neq \varnothing$, then $\Omega_\theta \neq \varnothing$, and there are realizations for which agents are short of liquidity in all meetings; i.e., $L^b(x) > L^s(x) > 1$ with positive probability. Another possibility, is $\Omega = \varnothing$, but $\Omega_\theta \neq \varnothing$, so that agents are short of liquidity for some realizations of the dividend process, but only in trades where just bonds can be used as means of payment; i.e., $L^s(x) = 1$ for all x, and $L^b(x) > 1$ with positive probability.

6 The Equity Premium and Risk-Free Rate Puzzles

Use the *intrinsic returns* $\hat{R}^b(x)$ and $\hat{R}^s(x,x')$, to define the *full returns* $R^b(x,x') = L^b(x')\hat{R}^b(x)$, and $R^s(x,x') = L^s(x')\hat{R}^s(x,x')$, and write (31) and (32), as

$$\int \beta \frac{U'(x')}{U'(x)} R^i(x,x')\, dF(x',x) = 1,$$

for $i = s, b$, and all x. Let $\bar{F}$ be the invariant distribution for the transition function F; then the Euler equations lead to:

$$\begin{aligned} \int\int \beta \frac{U'(x')}{U'(x)} \left[R^s(x,x') - R^b(x,x')\right] dF(x',x)\, d\bar{F}(x) &= 0, \qquad (35)\\ \int\int \beta \frac{U'(x')}{U'(x)} R^b(x,x')\, dF(x',x)\, d\bar{F}(x) - 1 &= 0, \qquad (36)\end{aligned}$$

a pair of statistical restrictions on equilibrium asset returns, $R^i(x,x')$, and the marginal rate of substitution, $\beta\frac{U'(x')}{U'(x)}$.

For the special case of an economy with no liquidity needs (i.e., either $\alpha = 0$, or $\alpha > 0$, but $\Omega_\theta = \Omega = \varnothing$ in equilibrium), the intrinsic returns equal the full returns, i.e., $\hat{R}^s(x,x') = R^s(x,x')$ and $\hat{R}^b(x) = R^b(x,x')$, so (35) and (36) are equivalent to (2a') and (2b') in Kocherlakota (1996).[12] In this case, one can estimate the expectations on the left hand sides of (35) and (36) using the sample means

$$\begin{aligned} \hat{\omega}^e &= \frac{1}{T}\sum_{t=1}^{T} \beta \frac{U'(c_{t+1})}{U'(c_t)} (\hat{R}^s_{t+1} - \hat{R}^b_t), \qquad (37)\\ \hat{\omega}^b &= \frac{1}{T}\sum_{t=1}^{T} \beta \frac{U'(c_{t+1})}{U'(c_t)} \hat{R}^b_t - 1. \qquad (38)\end{aligned}$$

A vast body of work has been devoted to trying to rationalize the finding that for standard parametrizations of preferences, the statistical restrictions $\hat{\omega}^e = \hat{\omega}^b = 0$ are violated by U.S. data. For instance, suppose that $\hat{R}^s_{t+1} = \frac{\phi^s_{t+1} + d_{t+1}}{\phi^s_t}$ is constructed using the Standard and Poor's stock index for ϕ^s_t, and the real dividends for the Standard and Poor's series for d_t, and that $\hat{R}^b_t$

[12] The expressions in Kocherlakota (1996) assume $U(c) = \frac{c^{1-\sigma}}{1-\sigma}$. Also, Kocherlakota's (2a') has been divided through by β, and has a typo (the marginal rate of substitution contains $(C_{t+1}C_t)^{-\sigma}$ instead of $(C_{t+1}/C_t)^{-\sigma}$).

is taken to be the real return on 90–day Treasury bills. Then, if $\beta = 0.99$, and $U(c) = \frac{c^{1-\sigma}}{1-\sigma}$, one finds $\hat{\omega}^e > 0$, and $\hat{\omega}^b < 0$ for "reasonable" values of σ. (E.g., Tables 2 and 3 in Kocherlakota (1996) report $\hat{\omega}^e > 0$ for $\sigma \leq 8.5$, and $\hat{\omega}^b < 0$ for $\sigma \geq 0.5$.) The finding that $\hat{\omega}^e > 0$ constitutes the *equity premium puzzle*, while $\hat{\omega}^b < 0$ is referred to as the *risk-free rate puzzle*.

The statistics $\hat{\omega}^e$ and $\hat{\omega}^b$ that define these puzzles are constructed using only the *intrinsic returns* , $\hat{R}^s_{t+1}$ and $\hat{R}^b_t$. But according to the model developed in the previous sections, agents price assets using the *full returns* R^s_{t+1}, and R^b_{t+1}, that is, the intrinsic returns augmented by their respective liquidity factors. To the extent that the agents experience liquidity needs in some states, the theory does *not* imply that $\hat{\omega}^e = \hat{\omega}^b = 0$. To derive the restrictions that the theory imposes on $\hat{\omega}^e$ and $\hat{\omega}^b$, one can rewrite conditions (35) and (36) as:

$$\int\int \beta \frac{U'(x')}{U'(x)} \left[\hat{R}^s(x,x') - \hat{R}^b(x)\right] dF(x',x)\, d\bar{F}(x) = \omega^e \tag{39}$$

$$\int\int \left[\beta \frac{U'(x')}{U'(x)} \hat{R}^b(x) - 1\right] dF(x',x)\, d\bar{F}(x) = \omega^b, \tag{40}$$

where

$$\begin{aligned} \omega^e &= \int\int \beta \frac{U'(x')}{U'(x)} \left\{\left[L^b(x') - 1\right]\hat{R}^b(x) - \left[L^s(x') - 1\right]\hat{R}^s(x,x')\right\} dF(x',x)\, d\bar{F}(x) \\ \omega^b &= -\int\int \beta \frac{U'(x')}{U'(x)} \left[L^b(x') - 1\right]\hat{R}^b(x)\, dF(x',x)\, d\bar{F}(x). \end{aligned}$$

Note that ω^e and ω^b are the theoretical counterparts to $\hat{\omega}^e$ and $\hat{\omega}^b$, respectively. The left hand side of condition (39) is a weighted average of the excess intrinsic return of equity over bonds, which in equilibrium must equal ω^e. The wedge ω^e will be larger, the larger $\frac{L^b-1}{L^s-1} - \frac{\hat{R}^s}{\hat{R}^b}$ is on average, i.e., the larger is the excess liquidity return of bonds over equity relative to the excess intrinsic return of equity over bonds. The sign of ω^e is ambiguous in general. But to the extent that bonds are more liquid than equity (in the sense of larger θ), the latter will tend to pay an "illiquidity premium," implying $\omega^e > 0$.[13] This is easy to see in the special case $\theta = 1$, which implies

[13] Quantitatively, in Section 7 I will find $\omega^e > 0$ for the baseline calibration, whenever there are liquidity needs in the equilibrium, *even if* $\theta = 0$; i.e, even without assuming any exogenous liquidity differences between equity and bonds. (See the last three rows of Table 3.)

$L^s(x') = 1$ for all x', since in this case $\omega^e = \int\int \beta \frac{U'(x')}{U'(x)} \left[L^b(x') - 1\right] \hat{R}^b(x)\, dF(x', x)\, d\bar{F}(x) \geq 0$, with strict inequality if $\Omega_\theta \neq \varnothing$. Note that even without assuming any exogenous liquidity differences between equity and bonds, we have $\omega^b < 0$, to the extent that there are liquidity needs at least for some realizations of the dividend process. Thus, qualitatively, the model with liquidity needs is always consistent with the risk-free rate puzzle. And in addition, if the liquidity return on bonds is large enough relative to that on equity, then the model may also help rationalize the equity premium puzzle.

6.1 Examples

First consider the special case with $\theta = 1$ (equity is never accepted in decentralized exchange), which implies $\omega^e = -\omega^b$. In this case asset prices satisfy

$$\begin{aligned} U'(x)\,\phi^s(x) &= \beta \int U'(x')\left[\phi^s(x') + x'\right] dF(x', x) \\ U'(x)\,\phi^b(x) &= \beta \int \left[1 + \alpha\left(\frac{u'\{q[U'(x')B']\}}{e'\{q[U'(x')B']\}} - 1\right)\right] U'(x')\, dF(x', x), \end{aligned}$$

and $\omega^e = \int \omega^e(x)\, d\bar{F}(x)$, where

$$\omega^e(x) = 1 - \frac{\int U'(x')\, dF(x', x)}{\int \left[1 + \alpha\left(\frac{u'\{q[U'(x')B']\}}{e'\{q[U'(x')B']\}} - 1\right)\right] U'(x')\, dF(x', x)}.$$

Example 6 *Suppose $\theta = 1$, $U(c) = \frac{c^{1-\sigma}}{1-\sigma}$, $u(q) = \frac{q^{1-\rho}}{1-\rho}$, and $e(q) = q$, with $\sigma > 0$ and $\rho > 0$. Then,*

$$\omega^e(x_t) = 1 - \frac{1}{1 + \alpha \frac{\int \max\left(x_{t+1}^{\sigma\rho} B_{t+1}^{-\rho} - 1, 0\right) x_{t+1}^{-\sigma} dF(x_{t+1}, x_t)}{\int x_{t+1}^{-\sigma} dF(x_{t+1}, x_t)}}. \tag{41}$$

Note that $\omega^e(x) \geq 0$ for all x, and thus $\omega^e \geq 0$ (with strict inequality if $\Omega_\theta \neq \varnothing$). In addition, $\omega^e(x) < 1$ for all x so $\omega^e < 1$. Suppose that $B_{t+1} = B x_t^\gamma$, with $0 \leq \gamma \leq \sigma$. So, for example, the outstanding stock of bonds is constant over time if $\gamma = 0$, and it is a constant proportion of output if $\gamma = 1$. Then (41) specializes to:

$$\omega^e(x_t) = 1 - \frac{1}{1 + \alpha \frac{\int \max\left[\left(\frac{x_{t+1}^\sigma}{B x_t^\gamma}\right)^\rho - 1, 0\right] x_{t+1}^{-\sigma} dF(x_{t+1}, x_t)}{\int x_{t+1}^{-\sigma} dF(x_{t+1}, x_t)}}.$$

Clearly, this wedge (the "equity premium puzzle") will be larger for large α, small B, and large ρ; it is also increasing in σ, under mild conditions (e.g., $\rho \geq 1$, and $\inf \Xi \geq 1$).

Next, consider the model with $\theta \in [0,1]$, $dF(x',x) = dF(x')$, and $B' = B(x) = B$ for all x. For this case, asset prices are $\phi^i(x) = \frac{\beta\Delta^i}{U'(x)}$, for $i = s, b$, where Δ^s solves

$$\Delta^s = \int \left[1 + \alpha(1-\theta)\left(\tfrac{u'\{q[\beta\Delta^s + (x+B)U'(x)]\}}{e'\{q[\beta\Delta^s + (x+B)U'(x)]\}} - 1\right)\right]\left[\beta\Delta^s + xU'(x)\right] dF(x). \tag{42}$$

(The same arguments used in Lemma 1 in the Appendix can be used to establish that a unique Δ^s satisfying this condition exists, and that it is decreasing in θ.) And given Δ^s,

$$\Delta^b = \int \left[1 + \alpha(1-\theta)\left(\tfrac{u'\{q[\beta\Delta^s + (x+B)U'(x)]\}}{e'\{q[\beta\Delta^s + (x+B)U'(x)]\}} - 1\right) + \alpha\theta\left(\tfrac{u'\{q[BU'(x)]\}}{e'\{q[BU'(x)]\}} - 1\right)\right] U'(x)\, dF(x).$$

Also, $\Omega_\theta = \{x \in \Xi : BU'(x) < e(q^*)\}$, and $\Omega = \{x \in \Xi : \beta\Delta^s + (x+B)U'(x) < e(q^*)\}$. In this i.i.d. case, $\omega^b(x)$ is independent of x:

$$\omega^b = -\left(1 - \frac{\int U'(x)\, dF(x)}{\Delta^b}\right).$$

Note that $\Delta^b \geq \int U'(x)\, dF(x)$, so $-1 < \omega^b \leq 0$ (< 0 only if $\Omega_\theta = \varnothing$). Similarly,

$$\omega^e = -\omega^b + \beta - \left(1 - \frac{\int xU'(x)\, dF(x)}{\Delta^s}\right).$$

Note that $\theta = 1$ implies $\Delta^s = \frac{\int xU'(x)dF(x)}{1-\beta}$, and therefore, that $\omega^e = -\omega^b$. Since Δ^s is decreasing in θ, this is an upper bound on how severe the "equity premium puzzle" can be in this simple economy. The following examples illustrate the structure of the sets Ω and Ω_θ.

Example 7 *Suppose $U(c) = \log c$, with $B(x) = B$ for all x. In this case, $\Omega_\theta = \{x \in \Xi : x > x^*_\theta\}$, where $x^*_\theta = B/e(q^*)$. The structure of the set Ω depends on the parametrization: $\Omega = \varnothing$ if $e(q^*) \leq \frac{1}{1-\beta}$, and if $e(q^*) > \frac{1}{1-\beta}$, $\Omega = \{x \in \Xi : x > x^*\}$, where $x^* = \frac{B}{e(q^*) - \beta\Delta^s - 1}$, and Δ^s solves (42). In summary, if $e(q^*) \leq \frac{1}{1-\beta}$, then agents are never liquidity constrained in matches where they can use both shares and bonds as a medium of exchange ($\Omega = \varnothing$), while agents in matches where only bonds can be used, are constrained for high realizations of the dividend process (i.e.,* *realizations above x^*_θ). Alternatively, if $e(q^*) > \frac{1}{1-\beta}$, then: (a) for realizations $x \leq x^*_\theta$, agents are unconstrained in all matches, (b) for realizations $x^*_\theta < x \leq x^*$, the only agents who are liquidity constrained are those in matches where just bonds can be used for payment, and (c) all agents are liquidity constrained for realizations $x^* < x$.*

Example 8 *Suppose $U(c) = \frac{c^{1-\sigma}}{1-\sigma}$, with $\sigma > 1$, and $B(x) = B$ for all x. Then $\Omega_\theta = \{x \in \Xi : x > x^*_\theta\}$, with $x^*_\theta = \left[\frac{B}{e(q^*)}\right]^{\frac{1}{\sigma}}$. To characterize the set Ω, we need to consider two cases: If $e(q^*) \leq \frac{\beta}{1-\beta} \int z^{1-\sigma} dF(z)$, then $\Omega = \varnothing$. If $e(q^*) > \frac{\beta}{1-\beta} \int z^{1-\sigma} dF(z)$, then $\Omega = \{x \in \Xi : x > x^*\}$, where x^* solves $(x + B) x^{-\sigma} = e(q^*) - \beta \Delta^s$, and Δ^s satisfies (42). In summary, if $e(q^*) \leq \frac{\beta}{1-\beta} \int z^{1-\sigma} dF(z)$, then agents are never liquidity constrained in matches where they can use both shares and bonds as a medium of exchange ($\Omega = \varnothing$), while agents in matches where only bonds can be used are constrained for high realizations of the dividend process (i.e., realizations above x^*_θ). Alternatively, if $e(q^*) > \frac{\beta}{1-\beta} \int z^{1-\sigma} dF(z)$, then: (a) for realizations $x \leq x^*_\theta$, agents are unconstrained in all matches, (b) for realizations $x^*_\theta < x \leq x^*$, the only agents who are liquidity constrained are those in matches where just bonds can be used for payment, and (c) all agents are liquidity constrained for realizations $x^* < x$.*

7 Quantitative Analysis

Since the level of per-capita consumption is non-stationary in U.S. data, I will follow Mehra and Prescott (1985) and carry out the quantitative exercise in a version of the model where the *growth rate* of the aggregate endowment follows a Markov process. (In contrast to Lucas' (1978) formulation, where the endowment *level* follows a Markov process.) To this end, let the utilities of a buyer and a seller who trade q in a period when the aggregate dividend is d, be given by $u[(1-\epsilon)\kappa d + q]$, and $u[(1+\epsilon)\kappa d - q] - u[(1+\epsilon)\kappa d]$, respectively, with $\epsilon \in [0,1]$, and $\kappa > 0$. This formulation has the natural feature that as d grows over time, so do the quantities traded in the decentralized market. (In the Appendix I show that with this preference structure, the economy is isomorphic to one where a homogeneous good—instead of the special good—

is traded bilaterally in the decentralized market, and where the double-coincidence-of-wants problem stems from the fact that some agents receive a high, and others a low endowment of the homogeneous good.)

Let γ_{t+1} denote the growth rate of the aggregate dividend, i.e., $d_{t+1} = x_{t+1}d_t$, where $x_{t+1} \in \{\gamma_1, ..., \gamma_n\}$, and $\Pr\left(x_{t+1} = \gamma_j | x_t = \gamma_i\right) = \mu_{ij}$. The conditional expectation, E, now is defined with respect to this transition probability. (As before, the realization of x_t is known when agents enter the decentralized market at the beginning of period t.) In equilibrium, asset prices satisfy:

$$
\begin{aligned}
U'\left(d_t\right)\phi_t^s &= \beta E_t U'\left(d_{t+1}\right) L_{t+1}^s \left[\phi_{t+1}^s + d_{t+1}\right] && (43)\\
U'\left(d_t\right)\phi_t^b &= \beta E_t U'\left(d_{t+1}\right) L_{t+1}^b, && (44)
\end{aligned}
$$

where

$$
\begin{aligned}
L_{t+1}^s &= 1 + \alpha\left(1-\theta\right)\left\{\frac{u'\left[(1-\epsilon)\kappa d_{t+1} + q\left(U'(d_{t+1})\left(\phi_{t+1}^s + d_{t+1} + B_{t+1}\right);\kappa d_{t+1}\right)\right]}{u'\left[(1+\epsilon)\kappa d_{t+1} - q\left(U'(d_{t+1})\left(\phi_{t+1}^s + d_{t+1} + B_{t+1}\right);\kappa d_{t+1}\right)\right]} - 1\right\} && (45)\\
L_{t+1}^b &= L_{t+1}^s + \alpha\theta\left\{\frac{u'[(1-\epsilon)\kappa d_{t+1} + q(U'(d_{t+1})B_{t+1};\kappa d_{t+1})]}{u'[(1+\epsilon)\kappa d_{t+1} - q(U'(d_{t+1})B_{t+1};\kappa d_{t+1})]} - 1\right\}. && (46)
\end{aligned}
$$

(As in Section 5, at this point I have let $\theta = \theta_1$ to simplify notation.)

Now suppose $U\left(c\right) = u\left(c\right) = \frac{c^{1-\sigma}}{1-\sigma}$, with $0 < \sigma < \infty$. For this parametrization, $q\left(z; \kappa d_{t+1}\right)$, namely the quantity of goods traded in a meeting where the buyer can pay from a portfolio of end-of-subperiod real value[14] z, when the aggregate dividend is d_{t+1}, is given by

$$
q\left(z;\kappa d_{t+1}\right) = \begin{cases} \epsilon\kappa d_{t+1} & \text{if } z \geq g\left(\kappa d_{t+1}\right) \\ \left(1+\epsilon\right)\kappa d_{t+1} - \left\{\left[\left(1+\epsilon\right)\kappa d_{t+1}\right]^{1-\sigma} - \left(1-\sigma\right)z\right\}^{\frac{1}{1-\sigma}} & \text{if } z < g\left(\kappa d_{t+1}\right), \end{cases}
$$

where $g\left(\kappa d_{t+1}\right) = \left[\left(1+\epsilon\right)^{1-\sigma} - 1\right]\frac{\left(\kappa d_{t+1}\right)^{1-\sigma}}{1-\sigma}$. (If $\sigma = 1$, then $q\left(z;\kappa d_{t+1}\right) = \epsilon\kappa d_{t+1}$ if $z \geq \ln\left(1+\varepsilon\right)$, and $q\left(z;\kappa d_{t+1}\right) = \left(1 - e^{-z}\right)\left(1+\epsilon\right)\kappa d_{t+1}$ otherwise.)

Hereafter, I specify that the government chooses the stock of bonds according to a policy rule $B_{t+1} = f\left(d_t, x_t\right)$. At any date, the variables $\left(d_t, x_t\right)$ are sufficient for predicting the

[14] By "real value" here I mean *real value in terms of marginal utility of fruit*. For example, in a period when the aggregate dividend is d_t, the "real value" of a portfolio $\mathbf{a}_t = \left(a_t^s, a_t^b\right)$ is $\boldsymbol{\lambda}_t\mathbf{a}_t = \lambda_t^s a_t^s + \lambda_t^b a_t^b$, where $\lambda_t^b = U'\left(d_t\right)$, and $\lambda_t^s = U'\left(d_t\right)\left(\phi_t^s + d_t\right)$.

subsequent evolution of the economy. So in a recursive equilibrium (one where all equilibrium values are invariant functions of the state), one can think of the state of the economy as being the pair (d, i), if $d_t = d$ and $x_t = \gamma_i$. With this convention, asset prices can be written as $\phi^s(d, i)$, and $\phi^b(d, i)$. To guarantee that the ratio of government debt to GDP is stationary, let $f(d_t, x_t) = \hat{B} E\left[d_{t+1} | (d_t, i)\right]$ if $x_t = \gamma_i$, where $\hat{B} \geq 0$. Restricting attention to equilibria that are stationary in growth rates, amounts to focusing on equilibria where share prices are homogeneous of degree one in d, and can therefore be written as $\phi^s(d, j) = \phi_j^s d$, where ϕ_j^s is a constant. In turn, this implies that $\phi^b(d, i)$ is independent of d, so one can write $\phi_i^b = \phi^b(d, i)$. Now (43)–(46) reduce to:

$$\phi_i^s = \beta \sum_j \mu_{ij} \gamma_j^{1-\sigma} L_{ij}^s(\phi_j^s)(1 + \phi_j^s) \tag{47}$$

$$\phi_i^b = \beta \sum_j \mu_{ij} \gamma_j^{-\sigma} L_{ij}^b(\phi_j^s), \tag{48}$$

for $i = 1, ..., n$, with

$$L_{ij}^s(\phi_j^s) = 1 + \alpha(1-\theta)\max\left\{\left[\frac{2}{\left[(1+\epsilon)^{1-\sigma} + \frac{(\sigma-1)}{\kappa^{1-\sigma}}\left(1 + \phi_j^s + \frac{B_i}{\gamma_j}\right)\right]^{\frac{1}{1-\sigma}}} - 1\right]^{-\sigma} - 1, 0\right\} \tag{49}$$

$$L_{ij}^b(\phi_j^s) = L_{ij}^s(\phi_j^s) + \alpha\theta\max\left\{\left[\frac{2}{\left[(1+\epsilon)^{1-\sigma} + \frac{(\sigma-1)}{\kappa^{1-\sigma}}\frac{B_i}{\gamma_j}\right]^{\frac{1}{1-\sigma}}} - 1\right]^{-\sigma} - 1, 0\right\}, \tag{50}$$

and where $B_i = \hat{B} \sum_{j=1}^n \mu_{ij} \gamma_j$.

The expressions analogous to (39) and (40) are:

$$\sum_i \sum_j \bar{\mu}_i \mu_{ij} \beta \gamma_j^{-\sigma} (\hat{R}_{ij}^s - \hat{R}_i^b) = \omega^e$$

$$\sum_i \sum_j \bar{\mu}_i \mu_{ij} (\beta \gamma_j^{-\sigma} \hat{R}_i^b - 1) = \omega^b,$$

where $\hat{R}^s_{ij} = \frac{(1+\phi^s_j)\gamma_j}{\phi^s_i}$, $\hat{R}^b_i = 1/\phi^b_i$, and where

$$\omega^e = \sum_i \sum_j \bar{\mu}_i \mu_{ij} \beta \gamma_j^{-\sigma} \left\{ \left[L^b_{ij}(\phi^s_j) - 1 \right] \hat{R}^b_i - \left[L^s_{ij}(\phi^s_j) - 1 \right] \hat{R}^s_{ij} \right\}, \tag{51}$$

$$\omega^b = -\sum_i \sum_j \bar{\mu}_i \mu_{ij} \beta \gamma_j^{-\sigma} \left[L^b_{ij}(\phi^s_j) - 1 \right] \hat{R}^b_i. \tag{52}$$

Here, $(\bar{\mu}_i)_{i=1}^n$ denotes the vector of stationary probabilities on i, i.e., it solves $\bar{\mu} = \mu' \bar{\mu}$, with $\sum_i \bar{\mu}_i = 1$, and $\mu' = [\mu_{ji}]$. The average intrinsic returns on equity and the bond are, respectively,

$$\hat{R}^s = \sum_i \sum_j \bar{\mu}_i \mu_{ij} \hat{R}^s_{ij} \qquad \text{and} \qquad \hat{R}^b = \sum_i \bar{\mu}_i \hat{R}^b_i.$$

The average equity premium is $\pi = \hat{R}^s - \hat{R}^b$.

Note that given that last period's growth rate was γ_i, the sets of next-period states in which the liquidity constraints bind are

$$\Omega_\theta = \left\{ j \in \{1, \dots, n\} : \frac{B_i}{\gamma_j} < \frac{\kappa^{1-\sigma}}{\sigma - 1} \left[1 - (1+\epsilon)^{1-\sigma} \right] \right\}, \text{ and}$$

$$\Omega = \left\{ j \in \{1, \dots, n\} : 1 + \phi^s_j + \frac{B_i}{\gamma_j} < \frac{\kappa^{1-\sigma}}{\sigma - 1} \left[1 - (1+\epsilon)^{1-\sigma} \right] \right\}.$$

Thus, as usual, $\Omega \subseteq \Omega_\theta$.[15] If $\Omega_\theta = \varnothing$ for all i, then $L^s_{ij}\left(\phi^s_j\right) = L^b_{ij}\left(\phi^s_j\right) = 1$ for all i and j, $\omega^e = \omega^b = 0$, and the model reduces to Mehra and Prescott (1985). The following example can help build some intuition.

Example 9 *Suppose* $\mu_{ij} = \mu_j$ *for all* i*, let* $\Delta \equiv \beta \sum_j \mu_j \gamma_j^{1-\sigma} < 1$*,* $\overline{\gamma} = \max_i \gamma_i$*, and* $\underline{\gamma} = \min_i \gamma_i$*. Also let* $\overline{B} = \left\{ \frac{\kappa^{1-\sigma}}{\sigma-1} \left[1 - (1+\epsilon)^{1-\sigma} \right] - \frac{1}{1-\Delta} \right\} \overline{\gamma}$*. Then,* $\phi^s_i = \phi^s$*,* $\phi^b_i = \phi^b$*, and* $B_i = B$*,*

[15] For $\sigma = 1$, $\Omega = \left\{ j \in \{1, \dots, n\} : 1 + \phi^s_j + \frac{B_i}{\gamma_j} < \ln(1+\epsilon) \right\}$, $\Omega^c_\theta = \left\{ j \in \{1, \dots, n\} : \frac{B_i}{\gamma_j} < \ln(1+\epsilon) \right\}$, and (49) and (50) become:

$$L^s_{ij}\left(\phi^s_j\right) = 1 + \alpha(1-\theta) \max\left\{ \left(\frac{2e^{1+\phi^s_j + B_i/\gamma_j}}{1+\epsilon} - 1 \right)^{-1} - 1, 0 \right\}$$

$$L^b_{ij}\left(\phi^s_j\right) = L^s_{ij}\left(\phi^s_j\right) + \alpha\theta \max\left\{ \left(\frac{2e^{B_i/\gamma_j}}{1+\epsilon} - 1 \right)^{-1} - 1, 0 \right\}.$$

for all i, and: (a) For $B \geq \overline{B}$, we have $\phi^s = \frac{\Delta}{1-\Delta}$, $\phi^b = \beta \sum_j \mu_j \gamma_j^{-\sigma}$, and $\Omega_\theta = \Omega = \varnothing$; i.e., $L_j^s(\phi^s) = L_j^b(\phi^s) = 1$ for all j. (b) For $B < \overline{B}$, $\phi^s = \phi^*$, where ϕ^* is the unique solution to $\frac{\phi}{1+\phi} = \beta \sum_j \mu_j \gamma_j^{1-\sigma} L_j^s(\phi)$, and given ϕ^*, $\phi^b = \beta \sum_j \mu_j \gamma_j^{-\sigma} L_j^b(\phi^*)$. In this case, $\Omega_\theta \neq \varnothing$, and $\Omega \neq \varnothing$, so $L_j^b(\phi^*) > L_j^s(\phi^*)$ for (at least) some j, and $L_i^s(\phi^*) > 1$ for (at least) some i. To provide a full characterization, let $\overline{B}_\theta = \frac{\kappa^{1-\sigma}}{\sigma-1}\left[1-(1+\epsilon)^{1-\sigma}\right]\overline{\gamma}$, $\underline{B}_\theta = \left(\underline{\gamma}/\overline{\gamma}\right)\overline{B}_\theta$, and $\underline{B} = \left\{\frac{\kappa^{1-\sigma}}{\sigma-1}\left[1-(1+\epsilon)^{1-\sigma}\right]-(1+\phi^*)\right\}\underline{\gamma}$. (It is easy to see that in general, $\underline{B}_\theta < \overline{B}_\theta$, $\underline{B} < \overline{B}$, $\overline{B} < \overline{B}_\theta$, and $\underline{B} < \underline{B}_\theta$.) Then it can be shown, that (i) $\Omega_\theta^c = \varnothing$ iff $B < \underline{B}_\theta$, and $\Omega_\theta = \varnothing$ iff $\overline{B}_\theta \leq B$; and (ii) that $\Omega^c = \varnothing$ iff $B < \underline{B}$, and $\Omega = \varnothing$ iff $\overline{B} \leq B$.

7.1 Calibration

There are three basic types of parameters: those that define preferences (β and σ), those that define technology (α, ϵ, κ, and the elements of $[\mu_{ij}]$ and $[\gamma_i]$), and $\hat{B}$, which defines government policy. In the formulations that assume differential liquidity, there is also θ, the probability that a randomly drawn trading partner will not accept shares as a means of payment in a decentralized exchange.[16] I follow Mehra and Prescott (1985) and assume that the growth rate of the endowment follows a two-state Markov chain with $\gamma_1 = \bar{\gamma} + \delta$, $\gamma_2 = \bar{\gamma} - \delta$, $\mu_{11} = \mu_{22} = \mu$, and $\mu_{12} = \mu_{21} = 1 - \mu$, and set $\bar{\gamma} = 1.018$, $\delta = 0.036$, and $\mu = 0.43$.[17]

The strategy for calibrating κ consists on selecting the value for which the *fraction of GDP that corresponds to production units ("trees") that have outstanding tradeable equity shares*, in the model matches the sample value for the U.S. economy.[18] Since this number is not readily

[16] There are two additional parameters in the detailed formulation laid out in the appendix: the preference parameter A, and the technology parameter Z. Since the values of these parameters are irrelevant for asset prices within the class of equilibria studied, I set $A = Z = 1$.

[17] Mehra and Prescott selected these parameter values so that the average growth rate of per capita consumption, the standard deviation of the growth rate of per capita consumption, and the first-order serial correlation of this growth rate (all with respect to the model's invariant distribution) matched the sample values for the U.S. economy between 1889–1978.

[18] At this point, it may be useful to think about the model in the following way. There is a measure 2 of trees. Half of them only give fruit in the second subperiod and have outstanding shares. The other half gives fruit in the first subperiod: each of these trees is sitting in its single owner's backyard, and they have no equity shares outstanding (these trees themselves are not traded, but their fruit may be traded in the decentralized market). Each tree, each period, gives out a dividend $(1+\epsilon)$ with probability $1/2$, or $(1-\epsilon)$ with probabilty $1/2$. Agents choose to each hold the same portfolio of the traded trees, e.g., they buy the "index," namely each holds an equal

available, I have instead guided the choice of κ by using the *gross value added of the nonfinancial corporate business sector as share of GDP*. In the U.S. National Income and Product Accounts (NIPA), this number is slightly above 0.5 for the fifty-seven-year period 1947–2004.[19] But clearly not all businesses have outstanding tradeable equity, so this is just an upper bound. With this in mind, I target 0.4 in the benchmark, which results in $\kappa = 0.5$.[20] Below I will report results for different targets, both larger and smaller.

The policy parameter $\hat{B}$ determines the size of the stock of government-issued assets that can be used as a medium of exchange, and hence the extent to which agents may experience liquidity needs in decentralized trades. Government bonds are the only such asset in the model, but in the U.S., there are other government-issued assets (and claims to these assets) that agents may use to satisfy their liquidity needs. With this in mind, I constructed a new empirical monetary aggregate, M1*, which augments and adjusts the standard M1 measure. The measure M1* differs from M1 in two ways. First, it includes Treasury bills in circulation. And second, it makes an adjustment for the amounts of M1 and Treasury bills outstanding that are held outside the United States.[21] The ratio of M1* to annual GDP was about 0.3 in 1959, declined through

share of each of the individual trees. Therefore, each gets dividend d_t for sure each period. On the other hand, since the there are no shares (or claims to the dividends) for the trees sitting in the individual agent's backyard, the only way they can hope to smooth their consumption of fruit in the first subperiod is by trading fruit itself, and this requires a medium of exchange. So—from the point of view of the theory—the key distinction between the two types of trees is whether agents can trade and hold claims to their dividend flow. With this in mind, one would like to set κ in order to match the *fraction of GDP that corresponds to production units (i.e., "trees," or "corporations") that have outstanding tradeable equity shares.*

[19] The data for Gross Value Added of Domestic Corporate Business and GDP are from NIPA, Table 1.1.4 and Table 1.15, respectively.

[20] In the model, the fraction of GDP produced by trees that have outstanding, tradeable equity shares is $\frac{1}{2+\kappa}$. To see this, recall that there are three commodities in this formulation: fruit, general goods, and labor. In period t, there is an endowment κd_t of fruit in the first subperiod, and an endowment d_t in the second, so the total endowment of fruit in period t is $(1+\kappa)\, d_t$. The output of general goods in the second subperiod is $Zn^* d_t$, where $n^* = \left(AZ^{\sigma-1}\right)^{-1/\sigma}$. Multiplying this quantity by the relative price of general goods in terms of fruit, $\frac{A}{Z}$, yields the value of production of general good expressed in terms of fruit, $An^* d_t$. So real GDP (in terms of fruit) is $(1+\kappa+An^*)\, d_t$, or just $(2+\kappa)\, d_t$, given $A = Z = 1$.

[21] Data for M1 is from the Board of Governors of the Federal Reserve System (*Money Stock Measures, Release H.6*). The amount of Treasury Bills outstanding for 1959–2004 was obtained from Bureau of the Public Debt (*Monthly Statement of the Public Debt, Table I: Summary of Treasury Securities Outstanding*). The *Flow of Funds Accounts of the United States* compiled by the Board of Governors of the Federal Reserve System provides estimates of the portion of checkable deposits and currency and of Treasury securities that is held by the rest of the world (Table L.204, line 20, and Table L209, line 11, respectively). The series for M1 was adjusted by

1979, hovered around 0.2 until 1995, and has been about 0.15 since then. So summing up, this ratio has been in the interval $[0.15, 0.3]$ over the last 40 years. I use 0.3 as a benchmark target—a conservative choice, since large values of this ratio mitigate the relevance of the liquidity channel I am exploring. This choice, together with the benchmark value for κ, implies $\hat{B} = 0.75$. Below, I will also report results for different $\hat{B}$.

The parameters α and ϵ index the agent's ability and desire to engage in bilateral exchange, respectively. If $\alpha = 0$, then the economy is just a sequence of Walrasian markets, and assets are only valued for their intrinsic payoffs, but not for their role as media of exchange: In this case, the economy reduces to the one studied by Mehra and Prescott. If $\epsilon = 0$, then each agent has the same endowment in every first subperiod, so there are no benefits from trade in any bilateral meeting, and therefore, again, no role for a medium of exchange. In this case asset prices and returns will again be just as in the economy studied by Mehra and Prescott.

The choice of ϵ has implications for consumption inequality. In the benchmark calibration, I set $\epsilon = 0.8$, which implies the variance of the log of total (i.e., including general goods and fruit in both subperiods) per capita consumption is 0.013.[22] Below, I will also explore the implications of higher and lower values for ϵ. Note that $\alpha = \hat{\alpha}/4$, where $\hat{\alpha}$ is the probability that the agent finds an opportunity to trade in the decentralized market (see the Appendix for details.) In the benchmark, I set $\hat{\alpha} = 1$, which basically says that every agent encounters one trading opportunity in the decentralized market over the course of the year. However, only the product $\alpha\theta$ matters for asset prices and returns, and I will be experimenting extensively with

assuming that the proportion that is held domestically is the same as the proportion of currency and checkable deposits that is held domestically. Similarly, the stock of outstanding Treasury Bills held domestically was estimated by assuming that the fraction of Treasury Bills held by the rest of the world is the same as the fraction of Treasury Securities held by the rest of the world. The series for M1* was constructed by adding the estimated domestic holdings of Treasury Bills to the estimated stock of M1 that is held domestically.

[22] This number is rather low relative to the values of about 0.15 that Krueger and Perri (2005) report for the within-group (i.e., conditioning on education, age, sex, experience, occupation, region of residence) variance of the log of per capita consumption in the U.S. The cross-sectional variance of the log of total consumption is low even for values of ϵ close to 1. This is so because in the model, there is only inequality in consumption of fruit in the first subperiod, and this is only a fraction of total consumption (in addition, all agents also consume the same amounts of fruit and general goods in the second subperiod). For the various parametrizations, I have also computed the variance of the log of per capita consumption *in the first subperiod only*—and found it to be in line with the estimates of Krueger and Perri (2005).

θ. In the Appendix, I verify the robustness of the benchmark results to changes in the values of the parameters $\hat{B}$, κ, and ε, the key parameters that were not in Mehra and Prescott (1985).

The nature of Mehra and Prescott's "test" of the theory was to experiment with different values of β and σ to find out for which combinations the model's average risk-free rate and equity premium matched those observed for the U.S. economy. (Table 1, taken from Mehra and Prescott (2003), reports these average returns based on various datasets.) Here, I carry out a similar exercise, except that I will not experiment with β, and instead set $\beta = 0.99$ throughout. I first consider the economy with no exogenous liquidity differences between equity shares and bonds, and assess the model's ability to produce bond returns and equity premia that resemble those observed for the U.S. economy. I then go on to consider the specification with exogenous liquidity differences, where shares are not accepted as a means of payment in a fraction θ of decentralized exchanges. For this specification, the question I pose is: how large does θ (the relative illiquidity of equity) need to be for the model to generate an average risk-free rate of 1% and an average equity premium that matches the one observed in the U.S. economy?

Insert Table 1 here

7.2 Results

Table 2 reports the average percentage return on equity, the average percentage return on the bond, their difference (the equity premium), the percentage standard deviation of equity returns, and the ratio of the equity premium to the standard deviation of returns (the Sharpe Ratio) in the Mehra–Prescott economy, for values of σ ranging from 1 to 10. For these "reasonable" values of σ, the risk-free rate is too high and the equity premium too low. Table 3 corresponds to the benchmark calibration of the basic model, augmented to allow for a liquidity motive, as outlined in the previous sections. (This table assumes $\theta = 0$, i.e., it assumes no exogenous liquidity differences between equity and bonds.) The last column reports the pair of wedges (ω^e, ω^b) to the agent's Euler Equations (see (51) and (52)). Recall that these wedges are zero if agents have enough liquidity to conduct decentralized transactions in all states of the world, or if the liquidity channel is shut off, as in the Mehra–Prescott economy.

Insert Tables 2 and 3 here

The first thing to note, is that the first seven rows in Table 3 are identical to the corresponding rows in Table 2: The economy behaves exactly as the Mehra–Prescott economy up to $\sigma = 7$. By the time σ reaches 8, equilibrium asset prices and returns begin to differ across both economies. For example, in Table 2, when the curvature parameter goes from 7 to 8, the return on equity rises from 12.24% to 13.52%, and the bond return rises from 10.67% to 11.6%. In contrast, in the economy of Table 3, the equity and bond returns *fall* from 12.24% to 8.8%, and from 10.67% to 6.76%, respectively. The reason why the two economies behave differently for higher values of σ is that agents face binding liquidity constraints in bilateral trades for high but not for low values of σ. To see why this is the case, let $\Omega\left(B_i\right)$ denote the set of states for which the liquidity constraints bind in the economy with $\theta = 0$, when the outstanding stock of bonds is $\frac{B_i}{\gamma_j}d$. Then, from the analysis in the previous sections, we know that:

$$\Omega\left(B_i\right) = \left\{ j \in \{1,2\} : 1 + \phi_j^s + \frac{B_i}{\gamma_j} < \frac{\kappa^{1-\sigma}}{\sigma - 1}\left[1 - \left(1+\epsilon\right)^{1-\sigma}\right] \right\}. \tag{53}$$

Recall that $(1 + \phi_j^s + \frac{B_i}{\gamma_j})d$ is the real (value in terms of fruit of the) equilibrium portfolio at the beginning of the first subperiod when the state is (d, j) and the outstanding stock of bonds is $\frac{B_i}{\gamma_j}d$, and hence $(1 + \phi_j^s + \frac{B_i}{\gamma_j})d^{1-\sigma}$ is the "price" of this portfolio in terms of marginal utility of fruit. The definition of the set $\Omega\left(B_i\right)$ states that the buyer's liquidity constraint binds if this value of her portfolio falls short of $u\left[\left(1+\epsilon\right)\kappa d\right] - u\left(\kappa d\right) = \frac{\kappa^{1-\sigma}}{\sigma-1}\left[1 - \left(1+\epsilon\right)^{1-\sigma}\right]d^{1-\sigma}$, namely, the amount of utility she has to transfer to the seller for him to be willing to consume κd instead of $\left(1+\epsilon\right)\kappa d$ in the first subperiod. In the benchmark parametrization, it can be shown that $\frac{\kappa^{1-\sigma}}{\sigma-1}\left[1 - \left(1+\epsilon\right)^{1-\sigma}\right]$ is strictly increasing in σ. This means that, keeping ϕ_j^s constant, the set Ω nondecreasing in σ. Of course, in the general equilibrium σ also affects equity prices ϕ_j^s, but the direct effect that tends to tighten the liquidity constraint, quantitatively dominates the indirect effect through the price of equity (which in general tends to loosen the liquidity constraint, since equity prices typically rise with σ when the liquidity constraints are not always slack). To be more precise, in the benchmark, for all $i \in \{1,2\}$, we have $\Omega(B_i) = \varnothing$ for $\sigma \leq 7$, and $\Omega(B_i) = \{1,2\}$ for $\sigma \geq 8$. That is, for $\sigma \geq 8$ buyers in the decentralized market face binding liquidity constraints for all realizations of the aggregate endowment process and every outstanding stock of bonds implied by the government's policy rule.

When liquidity constraints bind, assets help relax those constraints, and this additional "liquidity service" they provide is reflected in asset prices and in their intrinsic returns—the asset returns as they are conventionally measured. When the liquidity constraints bind, asset prices *rise*, and their measured, intrinsic returns *fall* as we increase σ. The reason is that the liquidity constraint becomes progressively tighter with larger values of σ, so agents have an additional incentive to hold assets: they help relax liquidity constraints. This means that—relative to a world with no liquidity—they no longer require such large expected intrinsic returns to be willing to hold those assets. (Conversely, in the Mehra–Prescott economy of Table 2, asset prices are strictly decreasing and returns strictly increasing in σ, at least for reasonable values, e.g., $\sigma \leq 20$.)

Comparing the third columns of Tables 2 and 3, it is clear that, for $\sigma \geq 8$, the equity premium is only slightly higher in the model with liquidity and $\theta = 0$, relative to the model in Mehra–Prescott. Take the row with $\sigma = 10$, for instance: the standard model generates a premium of 2.69% and the model with liquidity a premium of 2.85%. The big difference is in the actual returns. In the standard model the 2.69% is the difference between an equity return of 15.79% and a risk-free rate of 13.10%—both too high. While the 2.85% premium in the model with liquidity is the difference between an equity return of 4.35% and a risk-free rate of 1.5%. The former is rather low, but the latter is within the range of estimates reported in Table 1.

The last column of Table 3 reports the "wedges" to the Euler Equations defined in (51) and (52). This column can be used as an indicator of when the liquidity constraints bind and when they do not. When these constraints are slack in all states, both wedges equal zero, and the equilibrium asset prices and returns coincide with those in the Mehra–Prescott economy. As discussed earlier, for $\sigma = 8$ and higher, the liquidity constraints bind, so these wedges become nonzero. Since Table 3 assumes $\theta = 0$, we have $L^b_{ij}(\phi^s_j) = L^s_{ij}(\phi^s_j)$, and therefore (see (51)), the wedge $-\omega^e$, is a weighted average of $(\hat{R}^s_{ij} - \hat{R}^b_i)$, the state-by-state equity premium. Since the average equity premium is modest, and the "weight" $\beta\gamma_j^{-\sigma} L_{ij}$ is large in the low state, when $(\hat{R}^s_{ij} - \hat{R}^b_i) < 0$ and small in the high state, when $(\hat{R}^s_{ij} - \hat{R}^b_i) > 0$, the model delivers $\omega^e > 0$.[23] The wedge $-\omega^b$ is a weighted average of the risk free rate, $\hat{R}^b_i$, so we have $\omega^b < 0$ (see (52)).

As mentioned in Section 6, a large body of empirical work in asset pricing specifies and tests the sample counterpart to the moment restriction implied by the Euler Equation of candidate structural models (e.g., Hansen and Singleton (1982)). The wedges ω^e and ω^b in the last column of Table 3 can be though of as the theoretical counterparts of the sample means reported in Kocherlakota (1996) (Tables 2 and 3, p. 50–51.) Interestingly, even with $\theta = 0$, the model is able to replicate the signs of these wedges. Quantitatively, ω^e is too small relative to Kocherlakota's sample means in his Table 2[24] The value of ω^b for σ ranging from 7 to 10, is roughly in line with magnitudes of the sample means that Kocherlakota reports in his Table 3—an indication that, even with $\theta = 0$, the model is able to rationalize the risk-free rate puzzle.

The risk-free rate remains low for high risk aversion because assets, and in particular bonds, also yield a liquidity return which agents take into account when choosing their portfolios, but financial analysts don't include in their calculations of actual, "intrinsic" returns. In the usual

[23] For $\sigma = 10$, for example, $\hat{R}^b_1 = 0.96$, $\hat{R}^b_2 = 1.07$, $\hat{R}^s_{11} = 1.076$, $\hat{R}^s_{21} = 1.169$, $\hat{R}^s_{12} = 0.92$, and $\hat{R}^s_{22} = 1$. Note that $\hat{R}^s_{i1} - \hat{R}^b_i > 0$, and $\hat{R}^s_{i2} - \hat{R}^b_i < 0$ for $i = 1, 2$. The liquidity factors are $L_{11} = 1.0726$, $L_{21} = 1.0725$, $L_{12} = 1.1355$, and $L_{22} = 1.1353$.

[24] For σ ranging from 7 to 10, the magnitude of the ω^e wedge generated by the model is relatively close to some of the mean values one obtains from other data sets than the one used by Kocherlakota (1996). See Table 3 in İmrohoroğlu (2003), for example. There, he compares estimates of the wedges for two measures of the "risk-free" rate, the usual one based on 90–day Treasury Bills, and another based on high-grade long-term corporate bonds; and for two measures of equity returns, the standard one based on S&P 500, and another based on S&P 500 but subject to the adjustments suggested by McGrattan and Prescott (2001, 2003). Their adjustments are intended to account for taxes, regulatory constraints, and diversification costs. McGrattan and Prescott's emphasis on defining the equity premium relative to long-term bonds—as opposed to relative to 90–day Treasury Bills—seems to be an attempt to control for possible liquidity premia: "One problem [with Mehra and Prescott's analysis] is interpreting the return on a 90–day T-bill as the rate at which households intertemporally substitute consumption. We do not interpret it as such. Treasury bills provide considerable liquidity services and are a negligible part of individual's long-term debt holdings." (McGrattan and Prescott (2003), p. 393.)

Mehra–Prescott economy the risk-free rate puzzle arises because at relatively high levels of σ, the equilibrium must offer agents in a growing economy a high return on bonds for them to be willing to hold them. From the optic of the model with liquidity needs, at higher levels of σ the equilibrium compensates agents for holding bonds (and since $\theta = 0$, equity) with a high overall return, composed of a relatively high liquidity return, and a relatively low "intrinsic" return, of about 1.5%.

Insert Table 4 here

Now consider the more general formulation with $\theta \in [0, 1]$. Table 4 was generated with the benchmark parametrization used to generate Table 3, but the difference is that in Table 4, for each value of σ, the parameter θ was chosen so that the risk free rate is equal to 1%, whenever possible. The resulting value of θ is reported in the seventh column. The key difference between an economy with $\theta > 0$ relative to one with $\theta = 0$ is that in the former there are two types of decentralized trades: some where agents can pay using bonds and equity, and a fraction θ where they can only use bonds. This means that in any given state of the world, agents may face binding liquidity constraints in all decentralized trades, just in those where only bonds can be used as a medium of exchange, or in none of the decentralized trades. Formally, in addition to the set $\Omega(\cdot)$ defined in (53), there is now another set,

$$\Omega_\theta(B_i) = \left\{ j \in \{1, 2\} : \frac{B_i}{\gamma_j} < \frac{\kappa^{1-\sigma}}{\sigma - 1} \left[1 - (1 + \epsilon)^{1-\sigma} \right] \right\}, \tag{54}$$

that contains the set of states for which the liquidity constraints are binding in meetings where only bonds can be used in exchange, when the outstanding stock of bonds is $\frac{B_i}{\gamma_j} d$.

The first row of Table 4, is identical to the first row of Table 3 (and of Table 2). That is, if $\sigma = 1$, the agents' liquidity needs are so modest that the value of their equilibrium bond holdings is large enough to allow them to buy the first-best quantity $\epsilon\kappa d_t$ in every round of decentralized trade, for all realizations of the dividend process. Formally, $\Omega_\theta(B_i) = \varnothing$ for all i, and $\Omega(B_i) = \varnothing$, since $\Omega(B_i) \subseteq \Omega_\theta(B_i)$. Thus, the equilibrium is independent of the value of θ. For $\sigma = 2$ agent's liquidity needs are already strong enough so that the liquidity constraint is no longer slack in all decentralized meetings. (The nonzero wedges indicate the agent is sometimes short of liquidity.) In fact, for the benchmark parametrization, the equilibrium has $\Omega_\theta(B_i) = \{1,2\}$ for all i. That is, for all realizations of the aggregate endowment process and every outstanding stock of bonds implied by the government's policy rule, the liquidity constraint binds in decentralized trades where the buyer can only use bonds. With $\sigma = 2$ liquidity constraints do not bind in decentralized trades where both equity shares and bonds can be used as means of payment, i.e., $\Omega(B_i) = \varnothing$ for all i. The equilibrium remains qualitatively the same (in the sense that $\Omega(B_i) = \varnothing$ and $\Omega_\theta(B_i) = \{1,2\}$ for all i) up to $\sigma = 7$. For $\sigma \geq 8$ the liquidity motive becomes so strong that the equilibrium has $\Omega_\theta(B_i) = \Omega(B_i) = \{1,2\}$ for all i, that is, liquidity constraints bind at all dates, in *all* decentralized trades—even when agents can pay with bond and equity shares.[25]

Comparing the equity returns across Tables 2, 3, and 4, we see that they are identical up to $\sigma = 7$. The reason is that, although equity shares can in principle provide liquidity in the economies of Tables 3 and 4, they don't for relatively low levels of σ: For $\sigma \leq 7$, equity prices and returns in the economy with liquidity needs (for any θ) are just as in the Mehra–Prescott economy that abstracts from liquidity considerations. Accordingly, for this range of σ, equity returns are increasing in σ.[26] And in particular, the equilibrium equity returns and the equity premium implied by $\sigma = 3$, $\sigma = 4$, and $\sigma = 5$, are within the range of estimates reported in Table 1.[27] The corresponding values of θ that imply a risk-free rate of 1% are 0.0742, 0.0222, and 0.0062, respectively. So, for example, if $\sigma = 4$, this says that in 2 out of 100 decentralized trading opportunities, agents find themselves in a situation in which they cannot trade away

[25] Again, notice that the column that reports the Euler Equation "wedges" ω^e and ω^b, can be used as and indicator of the binding patterns of the liquidity constraints. To see this, refer back to (51) and (52), and note that: *(i)* $\omega^e = \omega^b = 0$ if $\Omega_\theta(B_i) = \Omega(B_i) = \varnothing$, *(ii)* $\omega^b < 0$ iff $\Omega_\theta(B_i) \neq \varnothing$, and *(iii)* $\omega^e = |\omega^b|$ iff $\Omega(B_i) = \varnothing$.

[26] Just as in Table 3, equity returns start falling for $\sigma \geq 8$, once equity shares can be used to relax liquidity constraints at the margin, i.e., once liquidity constraints start binding in decentralized trades where agents can pay with any combination of shares and bonds.

[27] Note that although the equity return (and consequently also the equity premium) is too high for $\sigma = 6$ and 7, the equity return and the equity premium are also in line with the data for $\sigma = 8$. The reason is that liquidity constraints start binding in all trades for $\sigma = 8$, so equity starts yielding a liquidity return, and this depresses the measured intrinsic return back to a level consistent with some of the estimates in Table 1.

the shares in their portfolios. At face value, this seems rather small, especially given that the actual institutional arrangements for trading away equity and bonds are rather different in actual economies.[28] But this view may leave room for dissent, so in the following section, I propose a more systematic way to gauge the absolute size of the difference in the relative ability of shares and bonds to serve as a medium of exchange which are needed for the liquidity motive to be able to rationalize the full observed equity premium.[29]

The standard deviation of equity returns in the model with liquidity (for any value of θ) do not differ much from the Mehra–Prescott model. (They are identical up to $\sigma = 7$, and vary only slightly for higher values of σ.) Even though this standard deviation is rather low (it is roughly 16% in the data), the Mehra–Prescott model still generates a Sharpe Ratio which is too low, relative to the standard estimates (about 0.5). Interestingly, the Sharpe Ratio is too high in the model with liquidity for the parametrizations that generate the right premium. Of course, this is just another way of saying that the liquidity mechanism as modeled here, does not induce additional volatility in equity returns.

To conclude, note that the signs and magnitudes of the Euler Equation wedges in Table 4 are in line with those estimated by Kocherlakota (1996) using Mehra and Prescott's data. For example, if $\sigma = 5$, $(\omega^e, \omega^b) = (.0687, -.0687)$, compared to $(.0433, -.0675)$ in Kochelakota's Tables 2 and 3. For $\sigma = 4$, the model implies $(\omega^e, \omega^b) = (.0576, -.0576)$, and Kocherlakota's estimates are $(.0464, -.0569)$, and for $\sigma = 3$, $(\omega^e, \omega^b) = (.0453, -.0453)$, compared to $(.0496, -.0448)$.[30]

[28] Here I have in mind the fact that from an investor's standpoint, executing and settling a sale of shares routinely takes longer than executing and settling a sale of bonds (see footnote 11). These differences may imply that once in a while, the proceeds of a sale of equity shares may not be available in time to take advantage of a fleeting trade opportunity.

[29] Lest there be any confusion, I want to stress that these exercises are not meant to suggest that *only* liquidity considerations are behind the observed equity premium. The spirit is rather to put this liquidity mechanism to an extreme test by asking whether it could conceivably account for the whole premium. To the extent that these sort of liquidity considerations are only one of several ingredients of a broader theory of the equity premium, the implied values of θ will be even smaller.

[30] For $\sigma = 8$, the other parametrization that can generate reasonable average equity return and premium, the model has $(\omega^e, \omega^b) = (.0526, -.0950)$, and Kocherlakota reports $(.0341, -.0910)$.

7.3 Bond–Equity Trade-Volume Bounds

A natural way to assess whether the values of θ that are needed for the model to rationalize the equity premium as a liquidity premium are "reasonable," follows from the realization that θ determines the frequency with which bonds get traded vis-a-vis shares. For example, in the extreme case with $\theta = 1$, equity shares never change hands in this economy: Each agent holds a single equity share at the end of the period, and since these shares are never used in decentralized trades, each agent also enters every second subperiod with the same share, so no shares are traded along the equilibrium. Conversely, in this case the volume of bonds traded will be positive along the equilibrium path, since at least some bonds will be exchanged in the first subperiod, and again in the second subperiod. The idea is then to construct the ratio of the value of bonds traded in a given period (as a proportion of the total value of outstanding stock of bonds), relative to the value of shares traded (as a proportion of the total value of outstanding equity).

Let v_{ij}^b and v_{ij}^s denote the quantities or *volumes* of bonds and equity shares that are traded in a period when the aggregate state is (d, j) and the outstanding stock of bonds is $\frac{B_i}{\gamma_j}d$. Then v_{ij}^b and $\phi_j^s d v_{ij}^s$ are respectively the value of bonds and shares that change hands during the period, expressed in terms of fruit, the numeraire. The total value of outstanding bonds and shares in the same period are $\frac{B_i}{\gamma_j}d$ and $\phi_j^s d$. The proportion of the outstanding value of the stock of bonds traded during the period—the *bond turnover ratio*—is $v_{ij}^b \gamma_j / (B_i d)$. Similarly, the proportion of the outstanding value of equity that was traded during the period—the *equity turnover ratio*—is v_{ij}^s. Therefore,

$$v_{ij} = \frac{v_{ij}^b \gamma_j}{v_{ij}^s B_i d}$$

can be used as a measure of the turnover of bonds relative to the turnover of equity.

The total volumes traded are $v_{ij}^b = \frac{\hat{\alpha}}{2}[\theta p_{ij}^b + (1-\theta)\,\hat{p}_{ij}^b]$, and $v_{ij}^s = \hat{\alpha}\,(1-\theta)\,\hat{p}_{ij}^s$, where p_{ij}^b is the quantity of bonds traded in a match where shares cannot be used, and $(\hat{p}_{ij}^s, \hat{p}_{ij}^b)$ is the portfolio that changes hands in a match where both shares and bonds can be used for payment, in a period when the aggregate state is (d, j) and the outstanding stock of bonds is $\frac{B_i}{\gamma_j}d$. (Clearly, as mentioned earlier, $v_{ij}^s \to 0$ as $\theta \to 1$, so values of θ close to 1 are unreasonable in that they would imply an unrealistically large volume of trade for bonds relative to shares.)

To get a sense for how the quantities p_{ij}^b, $\hat{p}_{ij}^s$, and $\hat{p}_{ij}^b$, of assets traded in the decentralized market are determined, it is convenient to consider the four possible types of double-coincidence trades a buyer may find himself in when $\theta \in (0, 1)$. The buyer may be in a meeting where he can use shares as means of payment, or where he cannot, and in each case, the relevant liquidity constraint may be binding or slack. In the two cases where the relevant constraint is binding, the agent simply spends all his portfolio if in a meeting where both can be used in exchange, or just all his bond holdings, if in a meeting where shares cannot be traded. If the liquidity constraint is slack in a meeting where only bonds can be used in exchange (for example as is the case when $\sigma = 1$), he will spend just enough bonds to afford $\epsilon\kappa d$. However, if the liquidity constraint is slack in trades where either asset can be used as a medium of exchange, then only the real value of the portfolio that changes hands is pinned down by the equilibrium—the precise composition of the portfolio that the buyer gives the seller in exchange for the $\epsilon\kappa d$ quantity of fruit is indeterminate. (See the discussion around (64) in the appendix for more details.)

This means that in those cases where the liquidity constraint is slack in trades where either asset can be used in exchange (as is the case for $\sigma = 2$ through 7 in the benchmark parametrization of the model with $\theta \in (0, 1)$), the precise trading volumes of bonds and shares are indeterminate in the equilibrium. Nonetheless, for each value of θ, it is possible to derive implied upper, and lower bounds for υ_{ij}, denoted $\overline{\upsilon}_{ij}$ and $\underline{\upsilon}_{ij}$, respectively. From these, I derive the average (with respect to the model's invariant distribution) upper and lower bounds, $\overline{\upsilon}$ and $\underline{\upsilon}$ for the ratio of the value of traded bonds (as a proportion of the value of the outstanding stock of bonds) to the value of traded shares (as a proportion of the value of outstanding equity). (See the appendix for details.) Values of θ that imply bounds $\overline{\upsilon}$ and $\underline{\upsilon}$ such that the ratio of bond to equity trade volumes computed from actual data is smaller than $\underline{\upsilon}$ would be deemed too large. Similarly, values of θ for which the value of the ratio observed in the data is larger than the implied $\overline{\upsilon}$ would be too low.

From Table 4, the values of σ for which the model is able to generate a risk-free rate of 1% and an equity premium that is in line with the data are $\sigma = 3$, 4, and 5. For $\sigma = 5$, $\theta = 0.0062$, and this value of θ implies $[\underline{\upsilon}, \overline{\upsilon}] = [.01, 2.55]$. This means that the model is consistent with the value of traded bonds (relative to the value of outstanding bonds) being at most 2 and a half times and at least 1% of the value of traded equity (relative to the value of outstanding equity). Similarly, the volume bounds implied by $\theta = 0.0222$ (the value corresponding to $\sigma = 4$) are $[.09, 6.29]$, and those implied by $\theta = 0.0742$ (the value corresponding to $\sigma = 3$) are $[0.7, 20]$. The bound for the pair $(\sigma, \theta) = (6, .0015)$ is $[.0015, 1.17]$. Naturally, $\overline{\upsilon}$ is increasing in θ (decreasing in σ).

8 Discussion

The bounds of Hansen and Jagannathan (1991) provide a way of assessing the magnitude of asset-pricing puzzles, and are often used as diagnostics tests for asset pricing models. I begin by asking how the Mehra–Prescott economy and the benchmark economy with liquidity fare against these bounds for different values of σ. This is a useful way to understand how the two models differ at the core; i.e., in terms of the first and second moments of their respective stochastic discount factors.

Let m denote a stochastic discount factor that satisfies, $E(mR) = \mathbf{1}$, the unconditional version of the agent's Euler Equations, where $R = [R_1, R_2]'$, and $\mathbf{1}$ is a vector of ones. Let Σ_m denote the standard deviation of m. The cup-shaped line in Figure 1 is the Hansen–Jagannathan bound for returns; i.e., $\Sigma_m \geq [b' cov(R, R)\, b]^{1/2}$, with $b = [cov(R, R)]^{-1} [\mathbf{1} - E(m)\, E(R)]$. The straight line is the Hansen–Jagannathan bound on excess returns; i.e., $\Sigma_m \geq [b' cov(z, z)\, b]^{1/2}$, with $b = -[cov(z, z)]^{-1} E(m)\, E(z)$, and $z = R_1 - R_2$.[31] The lines are drawn for the original Mehra–Prescott data: $R' = [1.07, 1.01]$, $var(R_1) = 0.0274$, $var(R_2) = 0.00308$, and

[31] See Ljungqvist and Sargent (2000) for a textbook treatment. The bounds in Figure 1 are *identical* to those in Ljungqvist and Sargent's Figure 10.4, p. 287. McGrattan and Prescott (2003) suggest some adjustments to the basic data that result in a different—lower—pair of bounds. See İmrohoroğlu (2003) for details.

$cov(R_1, R_2) = 0.00104$. According to these data, the minimum standard deviation an admissible stochastic discount factor must have is about 0.3509 (the minimum height of the cup-shaped curve). This value of Σ_m corresponds to an $E(m)$ of about 0.9855. Since the sides of the cup are rather steep, a model that satisfies this bound must have a stochastic discount factor with a mean that is close to 0.9855, unless one is willing to accept dramatically higher standard deviation for the stochastic discount factor. (If agents have access to a risk-free return, then this value of $E(m)$ implies a risk-free rate close to 1.0147.)

Consider the basic model with no exogenous liquidity differences between assets. The Euler Equations (47) and (48) imply: $\sum_j \mu_{ij} m_{ij} \hat{R}^s_{ij} = \sum_j \mu_{ij} m_{ij} \hat{R}^b_i = 1$, where $L_{ij} = L^s_{ij}$ (as given in (49), with $\theta = 0$), and $m_{ij} = \beta \gamma_j^{-\sigma} L_{ij}$ is the stochastic discount factor between states i and j. The circles in Figure 1 give the mean–standard deviation pairs of this discount factor implied by the benchmark calibration of Table 3, for values of σ ranging from 1 to 10. The crosses are the analogous mean–standard deviation pairs obtained by setting $L_{ij} = 1$ for all i and j. In this case the stochastic discount factor for state j is just $m_j = \beta \gamma_j^{-\sigma}$: this is the basic Mehra–Prescott economy of Table 2. For example, if $\sigma = 1$, the mean–standard deviation pair is $(0.9737, 0.0346)$ for both models. This is the lowest point in the figure; higher points correspond to higher values of σ. In fact, for $\sigma = 1$ through 7, the liquidity constraints are slack (recall that the Euler Equation wedges are zero in Table 3), so both models share the same stochastic discount factor and hence they generate the same mean–standard deviation pair for each σ. (These seven points appear marked with a circle *and* a cross in Figure 1.) The stochastic discount factors of the two models diverge for $\sigma = 8$ and higher, i.e., once the liquidity constraints start to bind. The liquidity mechanism increases the standard deviation, but especially the mean of the stochastic discount factor. For instance, at $\sigma = 10$, the model with liquidity has a stochastic discount factor with mean and standard deviation $(0.9878, 0.3602)$, just inside the admissible cup-shaped area.[32]

[32] For $\sigma = 10$, the Mehra–Prescott economy implies $(0.8861, 0.3010)$, which does not even satisfy the bound for excess returns. It takes a value of σ of about 20 in the Mehra–Prescott model to bring the mean and standard deviation of their stochastic discount factor within the bounds implied by their data.

I would like to stress that this discussion is not intended to suggest that the model developed here solves the equity premium puzzle "because the point corresponding to $\sigma = 10$ lies inside the cup-shaped area." Instead, the spirit is that, by understanding how the new ingredients I have added to the standard Lucas-type asset pricing model (e.g., decentralized exchange, anonymity, and the resulting need for a medium of exchange) affect the moments of the stochastic discount factor that prices returns, one can get a better sense for why it is that these ingredients can enhance the ability of the model to explain asset returns.

Next, consider the more general formulation with $\theta \in [0, 1]$. The (unconditional versions of the) corresponding Euler Equations (43) and (44) imply the following expression for the average risk-premium:

$$\begin{aligned} E(\hat{R}^s - \hat{R}^b) &= \frac{-cov(R^s, MRS)}{E\,(MRS)} + \frac{cov(R^b, MRS)}{E\,(MRS)} - cov(\hat{R}^s, L^s) + cov(\hat{R}^b, L^b) \\ &\quad + [E(L^b) - 1]E(\hat{R}^b) - [E(L^s) - 1]E(\hat{R}^s). \end{aligned} \tag{55}$$

where MRS_{t+1} denotes the marginal rate of substitution, $\beta U'\left(c_{t+1}\right)/U'\left(c_t\right)$. (Recall that $\hat{R}^i_{t+1}$ is the *intrinsic return* of asset i, and $R^i_{t+1} = L^i_{t+1}\hat{R}^i_{t+1}$ is the *full return,* as perceived by the agents in the model.) The first two terms are standard: excess returns are partly due to the fact that the full return on equity covaries with the growth rate of consumption more than does the average bond return. The third term is an adjustment for the degree to which equity returns covary with the liquidity constraints; i.e., it reflects the extent to which equity shares are a good hedge against binding liquidity constraints. For example, if the liquidity constraints that can be relaxed with shares are looser in periods when the equity return is relatively high, then $cov(L^s, \hat{R}^s) < 0$, and the second term tends to magnify the measured equity premium. The fourth term is an analogous adjustment for bonds. The last two terms reflect the liquidity return differential between the assets: their sum will be positive if $\ln E(L^b) - \ln E(L^s) > \ln E(\hat{R}^s) - \ln E(\hat{R}^b)$, namely if the average (geometric) excess liquidity return of bonds over shares is larger than the average (geometric) excess intrinsic return of shares over bonds. When this is the case, the combination of the last two terms adds on to the equity premium as it is typically measured.[33]

Insert Tables 13 and 14 here

To get some intuition about the signs and relative magnitudes of the various components of the equity premium, Tables 13 and 14 report the first and second moments of the asset returns, liquidity premia, and marginal rate of substitution implied by the model. Table 13 corresponds to the baseline economy, but with $\theta = 0$, and $\sigma = 10$ (the tenth line of Table 3). In this case we know that there is no liquidity differential, i.e., $L^b = L^s = L$ always, so the premium can be written as

$$\underbrace{E(\hat{R}^s - \hat{R}^b)}_{.028477} = \underbrace{\frac{-cov(R^s, MRS)}{E\,(MRS)\,E\,(L)}}_{.018904} + \underbrace{\frac{cov(R^b, MRS)}{E\,(MRS)\,E\,(L)}}_{.00735} - \underbrace{\frac{cov(\hat{R}^s, L)}{E\,(L)}}_{.002433} + \underbrace{\frac{cov(\hat{R}^b, L)}{E\,(L)}}_{-.000210}$$

The numbers below each term are for the decomposition implied by the model. The first two terms account for about 92% of the premium, and the third for 8.5%. In this case, since $cov(L, MRS) > 0$, the liquidity mechanism is causing the total return on equity, R^s, to covary more with the MRS than the intrinsic return $\hat{R}^s$, and this tends to bring the equity premium down from what it would have been in an economy with $L = 1$.[34] However, the fact that $cov(L, MRS) > 0$ turns $cov\left(R^b, MRS\right)$ positive even though $cov(\hat{R}^b, MRS) < 0$ (see Table 13). Intuitively, the intrinsic asset returns $\hat{R}^s$ and $\hat{R}^b$ tend to be high in times when the growth rate of consumption is high (the MRS low), but these are also times when the value of the average portfolio is high, which means that each agent will face looser liquidity constraints, or equivalently, that L will tend to be low in those periods.

[33] If $L_t^s = L_t^b = 1$ with probability 1 at all t, as is the case for the benchmark parametrization with $\theta = 0$ for $\sigma = 7$ or lower, and for any parametrization of the Mehra–Prescott model, then

$$E\,(R^s) - R^b = \frac{-cov\,(R^s, MRS) + cov\,\left(R^b, MRS\right)}{E\,(MRS)},$$

and the model relies only on the first two terms disussed above to account for the whole equity premium.

[34] For Mehra–Prescott, in this parametrization the premium is 0.02689, and it is accounted for fully by the negative covariance of equity returns with the MRS.

Table 14 corresponds to the baseline economy, with $\sigma = 4$, and $\theta = 0.0222$ (this is the fourth row of Table 4). In this case $L^s = 1$ always, so the third and fifth terms of (55) are identically zero, and the premium can be decomposed as:

$$\underbrace{E(\hat{R}^s - \hat{R}^b)}_{.068871} \approx \underbrace{[-cov\,(R^s, MRS) + cov(R^b, MRS)]E\,(MRS)^{-1}}_{.0063} + \underbrace{[E(L^b) - 1]E(\hat{R}^b)}_{.062571}.$$

The liquidity differential between bonds and equity accounts for about 90% of the equity premium.[35] Yet another way to try to assess which fraction of the equity premium is due to risk and which to liquidity considerations, I have set δ (the standard deviation of the growth rate of the endowment) to zero in the baseline. For example, with $\sigma = 4$ and $\theta = 0.0222$ (Tables 4 and 14). The equity premium goes down from 6.8871% to 6.3146%. So in this case about 90% of the equity premium is due to the fact that equity pays an *illiquidity premium*.

9 Conclusion

I have presented an asset-pricing model in which financial assets are valued for their liquidity—the degree to which they are valued as a medium of exchange—as well as for being claims to streams of consumption. The key implications of the model for average asset returns, the equity-premium and risk-free rate puzzles, were explored analytically and quantitatively.

Explicitly modeling the exchange process, and allowing for the possibility that the value of equity shares and bonds may partly depend on the role that each plays in exchange, rationalizes the two most commonly addressed asset-pricing anomalies. Quantitatively, the model performs better than the Mehra–Prescott frictionless benchmark, even if shares are just as useful as bonds for exchange purposes. But with standard constant relative risk aversion preferences, it still takes a coefficient of relative risk aversion of about 10 for the model to be consistent with asset return data.

If, in addition, one allows for the fact that bonds may be (slightly) better suited than equity shares to play the medium-of-exchange role, then the model is able to match the historical

[35] The fourth term in (55), $cov(\hat{R}^b, L^b)$, is negative, but zero to the fifth decimal, as a percentage.

average return to equity and the risk-free rate for the U.S. with values of the risk aversion coefficient between 3 and 5. These results indicate that prying deeper into the microeconomics of the decentralized exchange process may add to our understanding of how asset prices are determined in actual economies.

Kocherlakota (1996) ended his survey on the equity premium puzzle by drawing a parallel between the pure theory of money—much of which seeks to understand issues such as the coexistence of interest-bearing risk-free nominal bonds and fiat money—and the branch of financial economics that deals with the equity-premium puzzle. He argued that "we must seek to identify what fundamental features of goods and asset markets lead to large risk-adjusted price differences between stocks and bonds." And he concluded with, "While I have no idea what these 'fundamental features' are, it is my belief that any true resolution to the equity premium puzzle lies in finding them."

In this paper I have tried to pursue this line of reasoning a step further. I have advanced some candidate "fundamental features." These features are those that go into making an asset a medium of exchange, which—aside from the intrinsic properties of the asset—are bound to be related to the frequency of trade, the determination of the terms of trade, and the nature of the information structure. I have also asked whether these features stand a chance quantitatively. They do.

Having identified these features, at a deeper level the key issue becomes, why is asset X more generally accepted or better suited than asset Y to function as a medium of exchange? In terms of the equity premium, the next step is to explain precisely how these particular features can lead to differences in acceptability or, more generally, in the readiness for exchange between equity and bonds. In this regard, Kocherlakota (2003) and Zhu and Wallace (2005) may provide some valuable hints.

References

[1] Aiyagari, S. Rao, and Mark Gertler. "Asset Returns with Transaction Costs and Uninsured Individual Risk." *Journal of Monetary Economics* 27(3) (June 1991): 311–331.

[2] Aiyagari, S. Rao, and Neil Wallace. "Government Transaction Policy, the Medium of Exchange, and Welfare." *Journal of Economic Theory* 74(1) (May 1997): 1–18.

[3] Aiyagari, S. Rao, Neil Wallace, and Randall Wright. "Coexistence of Money and Interest-bearing Securities." *Journal of Monetary Economics* 37(3) (June 1996): 397–419.

[4] Bansal, Ravi, and John Wilbur Coleman II. "A Monetary Explanation of the Equity Premium, Term Premium, and Risk-free Rate Puzzles." *Journal of Political Economy* 104(6) (December 1996): 1135–1171.

[5] Campbell, John Y. "Asset Pricing at the Millennium." *The Journal of Finance* 55(4) (August 2000): 1515–1567.

[6] Cochrane, John H., and Lars Peter Hansen. "Asset Pricing Explorations for Macroeconomics." *NBER Macroeconomics Annual* (1992): 115–165.

[7] Diamond, Peter A. "Aggregate Demand Management in Search Equilibrium." *Journal of Political Economy* 90(5) (October 1982): 881–94.

[8] Duffie, Darrell, Nicolae Gârleanu, and Lasse Heje Pedersen. "Over-the-Counter Markets." *Econometrica*, forthcoming, 2005a.

[9] Duffie, Darrell, Nicolae Gârleanu, and Lasse Heje Pedersen. "Valuation in Over-the-Counter Markets." NYU Stanford University Graduate School of Business mimeo, 2005b.

[10] Greenwood, Jeremy, Richard Rogerson, and Randall Wright. "Household Production in Real Business Cycle Theory." *Frontiers of Business Cycle Research*, Ed. Thomas J. Cooley, Princeton University Press, Princeton, 1995.

[11] Hansen, Lars Peter, and Ravi Jagannathan. "Implications of Security Market Data for Models of Dynamic Economies." *Journal of Political Economy* 99(2) (April 1991): 225–62.

[12] Hansen, Lars Peter, and Kenneth J. Singleton. "Generalized Instrumental Variables Estimation of Nonlinear Rational Expectations Models." *Econometrica* 50(5) (September 1982): 1269–1286.

[13] He, Hua, and David M. Modest. "Market Frictions and Consumption-Based Asset Pricing." *Journal of Political Economy* 103(1) (February 1995): 94–117.

[14] Heaton, John, and Deborah J. Lucas. "The Importance of Investor Heterogeneity and Financial Market Imperfections for the Behavior of Asset Prices." Carnegie–Rochester Conference Series on Public Policy 42 (June 1995): 1–32.

[15] Heaton, John, and Deborah J. Lucas. "Evaluating the Effects of Incomplete Markets on Risk Sharing and Asset Pricing." *Journal of Political Economy* 104(3) (June 1996): 443–87.

[16] İmrohoroğlu, Selahattin. "A Note on the McGrattan and Prescott (2003) Adjustments and the Equity Premium Puzzle." University of Southern California mimeo, 2003.

[17] Kiyotaki, Nobuhiro, and John Moore. "Liquidity and Asset Prices." *International Economic Review* 46(2) (May 2005): 317–350.

[18] Kiyotaki, Nobuhiro, and Randall Wright. "On Money as a Medium of Exchange." *Journal of Political Economy* 97(4) (August 1989): 927–954.

[19] Kocherlakota, Narayana. "The Equity Premium: It's Still a Puzzle." *Journal of Economic Literature* 34 (March 1996): 42–71.

[20] Kocherlakota, Narayana R. "Societal Benefits of Illiquid Bonds." *Journal of Economic Theory* 108(2) (February 2003): 179–193.

[21] Krueger, Dirk, and Fabrizio Perri. "Does Income Inequality Lead to Consumption Inequality?: Evidence and Theory." Federal Reserve Bank of Minneapolis mimeo, 2005.

[22] Lagos, Ricardo. "Asset Prices, Liquidity, and Monetary Policy." Federal Reserve Bank of Minneapolis mimeo (in progress), 2005.

[23] Lagos, Ricardo, and Guillaume Rocheteau. "Money and Capital as Competing Media of Exchange." Federal Reserve Bank of Minneapolis Staff Report 341, August 2004.

[24] Lagos, Ricardo, and Randall Wright. "A Unified Framework for Monetary Theory and Policy Analysis." *Journal of Political Economy* 113(3) (June 2005): 463–84.

[25] Li, Yiting, and Randall Wright. "Government Transaction Policy, Media of Exchange, and Prices." *Journal of Economic Theory* 81(2) (August 1998): 290–313.

[26] Ljungqvist, Lars, and Thomas J. Sargent. *Recursive Macroeconomic Theory.* MIT Press, 2000.

[27] Lucas, Deborah J. "Asset Pricing with Undiversifiable Income Risk and Short Sales Constraints: Deepening the Equity Premium Puzzle." *Journal of Monetary Economics* 34(3) (December 1994): 325–341.

[28] Lucas, Robert E., Jr. "Asset Prices in an Exchange Economy." *Econometrica* 46(6) (November 1978): 1426–1445.

[29] Luttmer, Erzo G. J. "Asset Pricing in Economies with Frictions." *Econometrica* 64(6) (November 1996): 1439–1467.

[30] McGrattan, Ellen R., and Edward C. Prescott. "Taxes, Regulations, and Asset Prices." Federal Reserve Bank of Minneapolis Working Paper 610, 2001.

[31] McGrattan, Ellen R., and Edward C. Prescott. "Average Debt and Equity Returns: Puzzling?" *American Economic Review* 93(2) (May 2003): 392–397.

[32] Mehra, Rahnish. "The Equity Premium: Why Is It a Puzzle?" *Financial Analysis Journal* 59(1) (January/February 2003): 54–69

[33] Mehra, Rajnish, and Edward C. Prescott. "The Equity Premium: A Puzzle." *Journal of Monetary Economics* 15(2) (March 1985): 145–161.

[34] Mehra, Rajnish, and Edward C Prescott. "The Equity Premium in Retrospect." *Handbook of the Economics of Finance*, Edited by G. M. Constantinides, M. Harris, and R. Stulz, 2003.

[35] Neumeyer, Pablo A., and Fabrizio Perri. "Business Cycles in Emerging Economies: The Role of Interest Rates." *Journal of Monetary Economics* 52(2) (March 2005): 345–380.

[36] Ravikumar, B., and Enchuan Shao. "Search Frictions and Asset Price Volatility." University of Iowa mimeo, 2005.

[37] Reed, Robert, and Christopher Waller. "Money and Risk Sharing." University of Kentucky and University of Notre Dame mimeo, 2005.

[38] Sargent, Thomas J. *Dynamic Macroeconomic Theory*. Harvard University Press, 1987.

[39] Shi, Shouyong. "A Divisible Search Model of Fiat Money." *Econometrica* 65(1) (January 1997): 75–102.

[40] Shi, Shouyong. "Nominal Bonds and Interest Rates." *International Economic Review* 46(2) (May 2005): 579–612.

[41] Stokey, Nancy, and Lucas, Robert E., Jr. *Recursive Methods in Economic Dynamics*. Harvard University Press, 1989.

[42] Telmer, Chris, I. "Asset-pricing Puzzles and Incomplete Markets." *Journal of Finance* 48(5) (December 1993): 1803–1832.

[43] Vayanos, Dimitri, and Tan Wang. "Search and Endogenous Concentration of Liquidity in Asset Markets." London School of Economics mimeo, 2005.

[44] Vayanos, Dimitri, and Pierre Olivier Weill. "A Search-Based Theory of the On-the-Run Phenomenon." NYU Stern School of Business mimeo, 2005.

[45] Weil, Philippe. "The Equity Premium Puzzle and the Risk-Free Rate Puzzle." Journal of Monetary Economics 24(3) (November 1989): 401–421.

[46] Weill, Pierre Olivier. "Leaning Against the Wind." NYU Stern School of Business mimeo, 2005a.

[47] Weill, Pierre Olivier. "Liquidity Premia in Dynamic Bargaining Markets." NYU Stern School of Business mimeo, 2005b.

[48] Zhu, Tao, and Neil Wallace. "Pair-wise Trade and the Coexistence of Money and Higher-Return Assets." Pennsylvania State University mimeo, 2005.

A Appendix

Lemma 1 *There exists a unique Δ that solves*

$$\Delta = \int \left[1 + \alpha \left(\frac{u' \{ q [\beta\Delta + zU'(z)] \}}{e' \{ q [\beta\Delta + zU'(z)] \}} - 1 \right) \right] [\beta\Delta + zU'(z)] \, dF(z),$$

and it is strictly increasing in α.

Proof. Define $\Upsilon(\Delta) \equiv \int \left[1 + \alpha \left(\frac{u'\{q[\beta\Delta + zU'(z)]\}}{e'\{q[\beta\Delta + zU'(z)]\}} - 1 \right) \right] [\beta\Delta + zU'(z)] \, dF(z) - \Delta$. Note that $\Upsilon(0) > 0$. Also, $\Upsilon(\Delta) = \int zU'(z) \, dF(z) - (1 - \beta)\Delta$ for all $\Delta \geq e(q^*)/\beta$, so $\lim_{\Delta \to \infty} \Upsilon(\Delta) = -\infty$ and since Υ is continuous, there exists a Δ that satisfies $\Upsilon(\Delta) = 0$. Differentiating,

$$\Upsilon'(\Delta) = -[1 - \beta(1 - \alpha)] + \alpha\beta \int_\Omega \tfrac{u''e' - e''u'}{e'^3} \lambda(z) \, dF(z) + \alpha\beta \left[\int_\Omega \tfrac{u'}{e'} dF(z) + \int_{\Omega^c} dF(z) \right],$$

where $\lambda(z) = \beta\Delta + zU'(z)$, $\Omega = \{z \in \Xi : \lambda(z) < e(q^*)\}$, and u'', e'', u' and e' are functions of $\{q[\lambda(z)]\}$. Note that $\Upsilon(\Delta) = 0$ implies

$$\alpha\beta \left[\int_\Omega \tfrac{u'}{e'} dF(z) + \int_{\Omega^c} dF(z) \right] = 1 - \beta(1 - \alpha) - \varsigma,$$

where $\varsigma \equiv \frac{1}{\Delta} \left[\int zU'(z) \, dF(z) + \alpha \int_\Omega \left(\frac{u'}{e'} - 1 \right) zU'(z) \, dF(z) \right] > 0$. Therefore

$$\Upsilon'(\Delta)\big|_{\Upsilon(\Delta)=0} = \alpha\beta \int_\Omega \tfrac{u''e' - e''u'}{e'^3} \lambda(z) \, dF(z) - \varsigma < 0,$$

so $\Upsilon(\Delta) = 0$ has a unique solution. Finally,

$$\frac{\partial \Upsilon(\Delta)}{\partial \alpha} = \int_\Omega \left(\frac{u' \{ q [\beta\Delta + zU'(z)] \}}{e' \{ q [\beta\Delta + zU'(z)] \}} - 1 \right) [\beta\Delta + zU'(z)] \, dF(z) \geq 0, \text{ “ > ” if } \Omega \neq \varnothing,$$

so

$$\left. \frac{d\Delta}{d\alpha} \right|_{\Upsilon(\Delta)=0} = \frac{\partial \Upsilon(\Delta) / \partial \alpha}{- \Upsilon'(\Delta)|_{\Upsilon(\Delta)=0}} \geq 0, \text{ “ > ” if } \Omega \neq \varnothing.$$

This concludes the proof. ■

Lemma 2 *Suppose $xU'(x)$ is continuous and bounded, and that for any continuous function g, $\int g(x') \, dF(x', x)$ is a continuous function of x. There is a unique continuous and bounded function λ that solves (15).*

Proof. Let $\mathcal{C}$ denote the space of continuous and bounded real-valued functions. Let T be the mapping defined by the right-hand side of (15), namely

$$(Tg)(x) = \beta \int \left\{ (1-\alpha) g(x') + \alpha \max \left[g(x'), 1 \right] \right\} dF(x', x) + xU'(x).$$

The conditions in the statement of the lemma imply that $T : \mathcal{C} \to \mathcal{C}$. In addition, T satisfies Blackwell's sufficient conditions, so it is a contraction on $\mathcal{C}$. ■

A.1 Sufficient conditions for interior solutions in the centralized market

Throughout, I have focused on equilibria with an interior solution to the agent's problem in the centralized market. In particular, the constraints $0 \leq h_t \leq \bar{n}$ have so far simply presumed to have been slack. Here I show how to derive sufficient conditions so that this will indeed be the case along the equilibrium path.[36] In the economy with only equity shares, for instance,

$$h_t = \frac{1}{w_t} \left[c_t + w_t n_t + \phi_t a^s_{t+1} - (\phi_t + x_t) a^s_t \right],$$

where x_t denotes the dividend realized in period t. In the equilibrium I constructed, $c_t = x_t$, $w_t = w(x_t)$, $n_t = y^*$, $\phi_t = \phi(x_t)$, and $a^s_{t+1} = 1$, so $0 \leq h_t \leq \bar{n}$ for $t \geq 1$ is equivalent to

$$0 \leq N(a^s_t, x_t) \leq \bar{n} \tag{56}$$

where $N(a^s_t, x_t) \equiv y^* + \frac{\lambda(x_t)}{A}(1 - a^s_t)$. Notice that $N_1(a^s, x) < 0$. Also, in equilibrium, $0 \leq a^s_t \leq 2$ for all $t \geq 1$. Thus $N(2, x_t) \leq N(a^s_t, x_t) \leq N(0, x_t)$ holds for all x_t along the equilibrium path. Therefore the conditions

$$\lambda(x) \leq Ay^* \quad \text{and} \quad \lambda(x) \leq A(\bar{n} - y^*) \quad \text{for all } x \tag{57}$$

imply (56). These sufficient conditions are still rather "implicit" because λ is the solution to a functional equation (i.e. (7)). But since in the equilibrium $\lambda(x)$ is bounded, say by $\bar{\lambda}$, then

$$\bar{\lambda} \leq An(x) \leq A\bar{n} - \bar{\lambda} \quad \text{for all } x, \tag{58}$$

[36] In general one also has to specify restrictions on the initial distribution of assets so that the constraints $0 \leq h_0 \leq \bar{n}$ are slack. These conditions essentially amount to requiring the initial distribution not to be too disperse.

is a simple parametric restriction on $\bar{\lambda}$, A, $\bar{n}$, and $v(\cdot)$ that implies (57). For instance, consider the model in Example 3. There, $\lambda(x) = \frac{\alpha\beta+\varepsilon}{1-\beta(1-\alpha)}$ for all x, so (57) reduces to $\frac{\alpha\beta+\varepsilon}{[1-\beta(1-\alpha)]A} \leq y^* \leq \bar{n} - \frac{\alpha\beta+\varepsilon}{[1-\beta(1-\alpha)]A}$.

A.2 Alternative Formulation

In the first part of the paper, the double-coincidence problem that generates the role for a medium of exchange arises from bilateral trade in the first subperiod, combined with the fact that each agent has the ability to produce a subset, but wishes to consume a different subset of the special goods traded in that subperiod. Since the work of Kiyotaki and Wright (1989), this has become the standard formalization of the double-coincidence-of-wants problem in monetary theory. For the theoretical derivations in the preliminary sections, this standard formulation was useful because it underscored the connections between the cannonical asset pricing model, and the pure theory of money based on search theory that emphasizes the role of assets as media of exchange. However, since the level of per-capita consumption is non-stationary in U.S. data, for the empirical implementation in Section 7, I will want to follow Mehra and Prescott (1985) in formulating an economy where the *growth rate* of the aggregate endowment follows a Markov process. Here, I show how to reformulate the model developed in the first part of the paper so that it is stationary in growth rates.

Consider the following alternative formulation. There are now only two nonstorable and perfectly divisible consumption goods at each date: fruit, and general goods. (There are no special goods in this formulation.) As before, there is a set of trees that yield quantity of fruit d_t—the "dividend"—in the second subperiod of every period. Let γ_{t+1} denote the growth rate of this dividend, i.e., $d_{t+1} = x_{t+1} d_t$, where $x_{t+1} \in \{\gamma_1, ..., \gamma_n\}$, and $\Pr\left(x_{t+1} = \gamma_j | x_t = \gamma_i\right) = \mu_{ij}$. The conditional expectation, E, used below, is defined with respect to this transition probability. The realization of x_t is known when agents enter the decentralized market at the beginning of period t. I will assume the Markov chain is ergodic. In the second subperiod every agent is also endowed with $\bar{n}$ units of time that can be employed as labor services, and each agent has access to a production technology that allows him to transform a unit of labor services into $Z(d_t)$ general goods. (Think of d_t as indexing the economy-wide "state of technology.") In the first subperiod, half of the agents are endowed with $(1+\epsilon)\,\bar{q}_t$ units of the general good, and the remaining half are endowed with $(1-\epsilon)\,\bar{q}_t$ units, with $\epsilon \in [0,1]$. In what follows, I will assume that the aggregate endowment of the general good in the first subperiod ($\bar{q}_t$), is proportional to the aggregate endowment of fruit (d_t); in particular, $\bar{q}_t = \kappa d_t$, with $\kappa > 0$.[37] The agents who receive the high endowment are selected at the beginning of each period at random from the population, so from the perspective of an individual agent, the endowment process is independent across periods.[38]

Let $u(Q)$ now be the utility of consuming quantity Q of general goods in the first subperiod. As before, $U(c)$ is the utility from consuming quantity c of fruit, and $v(y)$ is the utility from consuming quantity y of general goods in the second superiod. Each agent now wishes to maximize

$$E\left\{\sum_{t=0}^{\infty}\beta^t\left[u(Q_t^i)+U(c_t^i)+v(y_t^i)-A_t h_t^i\right]\right\},$$

where $\beta \in (0,1)$, Q_t^i is agent i's consumption of general goods in the decentralized market, y_t^i his consumption of general goods in the centralized market, c_t^i his consumption of fruit, and h_t^i his labor supply in the second subperiod. The marginal disutility from working is $A_t > 0$. (The "t" subscript is meant to allow for specifications where A depends on the state of technology, d_t. The usefulness of such specifications has to do with balanced growth considerations, as discussed below.) I still assume that $u' > 0$, $v' > 0$, $U' > 0$, $u'' < 0$, $v'' \leq 0$, $U'' < 0$. It is also convenient to have $u'(0) = v'(0) = U'(0) = +\infty$.[39]

Let $\hat{\alpha}$ denote the probability of a meeting in the decentralized market. Bilateral trade, and

[37] One can also interpret the endowment in the first subperiod as being an endowment of fruit, e.g., stemming from a set of trees for which there are no equity shares that represent ownership.

[38] It would not be difficult to allow the individual endowment process to be correlated over time. Reed and Waller (2005) use a similar formulation in their study of monetary policy and risk sharing.

[39] Note that I no longer need to assume $u(0) = 0$, which is useful, because—in contrast to the formulation in Lagos and Wright (2005)—I can now use standard CRRA preferences over the good traded in the decentralized market.

the fact that some agents receive a high endowment while others receive a low one, give rise to a double-coincidence-of-wants problem in the first subperiod. There are two relevant types of meetings: $\frac{\hat{\alpha}}{2}$ of them involve an agent with a high endowment and an agent with a low endowment (naturally, in these "single-coincidence meetings," the agent with the low endowment will be the "buyer" and the other one the "seller"); and $\frac{\hat{\alpha}}{2}$ of them are "no-coincidence meetings" that involve either two agents with high, or two agents with low endowment.

Next, I turn to the Bellman equations that summarize the agents' optimization problem, in the context of the model with differential liquidity laid out in Section 5. Let $V_l(\mathbf{a}_{t+1}, \mathbf{s}_{t+1})$ be the value of search for an agent who enters the decentralized market holding portfolio $\mathbf{a}_{t+1}$ and receives the low endowment of general good, in a period where the aggregate state of the economy is $\mathbf{s}_{t+1} = (d_{t+1}, x_{t+1}, B_{t+1})$, i.e., in a period when the realization of the endowment process is $d_{t+1} = x_{t+1}d_t$, and there are B_{t+1} bonds in circulation. Then,

$$\begin{aligned} V_l(\mathbf{a}, \mathbf{s}) &= \frac{\hat{\alpha}}{2} \sum_{i=1,2} \theta_i \int \left\{ u\left[(1-\epsilon)\kappa d + q^i(\mathbf{a}, \tilde{\mathbf{a}})\right] + W\left[\mathbf{a} - \mathbf{p}^i(\mathbf{a}, \tilde{\mathbf{a}}), \mathbf{s}\right] \right\} d\mathbf{G}(\tilde{\mathbf{a}}) \\ &\quad + (1 - \frac{\hat{\alpha}}{2}) \left\{ u\left[(1-\epsilon)\kappa d\right] + W(\mathbf{a}, \mathbf{s}) \right\}. \end{aligned}$$

The analogous value function for an agent who gets the high endowment is

$$\begin{aligned} V_h(\mathbf{a}, \mathbf{s}) &= \frac{\hat{\alpha}}{2} \sum_{i=1,2} \theta_i \int \left\{ u\left[(1+\epsilon)\kappa d - q^i(\tilde{\mathbf{a}}, \mathbf{a})\right] + W\left[\mathbf{a} + \mathbf{p}^i(\tilde{\mathbf{a}}, \mathbf{a}), \mathbf{s}\right] \right\} d\mathbf{G}(\tilde{\mathbf{a}}) \\ &\quad + (1 - \frac{\hat{\alpha}}{2}) \left\{ u\left[(1+\epsilon)\kappa d\right] + W(\mathbf{a}, \mathbf{s}) \right\}. \end{aligned}$$

The value to an agent of entering the decentralized market before knowing his endowment of the general good in the first subperiod, is $V(\mathbf{a}, \mathbf{s}) = \frac{1}{2}\left[V_l(\mathbf{a}, \mathbf{s}) + V_h(\mathbf{a}, \mathbf{s})\right]$. Explicitly,

$$\begin{aligned} V(\mathbf{a}, \mathbf{s}) &= \alpha \sum_{i=1,2} \theta_i \int \left\{ u\left[(1-\epsilon)\kappa d + q^i(\mathbf{a}, \tilde{\mathbf{a}})\right] + W\left[\mathbf{a} - \mathbf{p}^i(\mathbf{a}, \tilde{\mathbf{a}}), \mathbf{s}\right] \right\} d\mathbf{G}(\tilde{\mathbf{a}}) \\ &\quad + \alpha \sum_{i=1,2} \theta_i \int \left\{ u\left[(1+\epsilon)\kappa d - q^i(\tilde{\mathbf{a}}, \mathbf{a})\right] + W\left[\mathbf{a} + \mathbf{p}^i(\tilde{\mathbf{a}}, \mathbf{a}), \mathbf{s}\right] \right\} d\mathbf{G}(\tilde{\mathbf{a}}) \\ &\quad + (1 - 2\alpha)\left[\bar{u}(d) + W(\mathbf{a}, \mathbf{s})\right], \end{aligned}$$

where $\alpha \equiv \hat{\alpha}/4$, and $\bar{u}(d) \equiv \frac{1}{2}\{u[(1+\epsilon)\kappa d] + u[(1-\epsilon)\kappa d]\}$. (Note that this expression for the value of search is essentially the same as the one in Section 5, except that now, the utility of a buyer is $u\left[(1-\epsilon)\kappa d + q^i(\mathbf{a}, \tilde{\mathbf{a}})\right]$ instead of $u\left[q^i(\mathbf{a}, \tilde{\mathbf{a}})\right]$, the utility cost of a seller is $u[(1+\epsilon)\kappa d] - u\left[(1+\epsilon)\kappa d - q^i(\tilde{\mathbf{a}}, \mathbf{a})\right]$ instead of $e\left[q^i(\tilde{\mathbf{a}}, \mathbf{a})\right]$, and agents without trade opportunity consume $(1+\epsilon)\kappa d$ or $(1-\epsilon)\kappa d$, each with probability a half, instead of 0 with certainty.)

The value function of an agent who enters the centralized market with portfolio $\mathbf{a}_t$ in a period when the state is $\mathbf{s}_t$, namely $W(\mathbf{a}_t, \mathbf{s}_t)$, satisfies:

$$W(\mathbf{a}_t, \mathbf{s}_t) = \max_{c_t, y_t, n_t, h_t, \mathbf{a}_{t+1}} \{U(c_t) + v(y_t) - A_t h_t + \beta E V(\mathbf{a}_{t+1}, \mathbf{s}_{t+1})\}$$

$$\text{s.t. } c_t + w_t n_t + \boldsymbol{\phi}_t \mathbf{a}_{t+1} = (\phi_t^s + d_t) a_t^s + a_t^b + w_t h_t - \tau_t$$

$$0 \le c_t,\ 0 \le n_t,\ 0 \le h_t \le \bar{n},\ 0 \le \mathbf{a}_{t+1},\ y_t = Z(d_t) n_t.$$

The agent chooses consumption of fruit (c_t) and of the general good (y_t), how many hours of work to demand (n_t) and supply (h_t), and an end-of-period portfolio ($\mathbf{a}_{t+1}$). Let $\lambda_t^b = \frac{A_t}{w_t}$, $\lambda_t^s = (\phi_t^s + d_t)\lambda_t^b$, and $\boldsymbol{\lambda}_t = \left(\lambda_t^s, \lambda_t^b\right)$. Then this problem can be rewritten as

$$\begin{aligned} W(\mathbf{a}_t, \mathbf{s}_t) \quad &= \quad \boldsymbol{\lambda}_t \mathbf{a}_t - \lambda_t^b \tau_t + \max_{c_t, n_t} \left\{ U(c_t) + v[Z(d_t) n_t] - \frac{A_t}{w_t}(c_t + w_t n_t) \right\} \\ &\quad + \max_{\mathbf{a}_{t+1}} \left\{ -\frac{A_t}{w_t} \boldsymbol{\phi}_t \mathbf{a}_{t+1} + \beta E V(\mathbf{a}_{t+1}, \mathbf{s}_{t+1}) \right\}, \end{aligned}$$

with $y_t = Z(d_t) n_t$,

$$h_t = \frac{1}{w_t}\left[c_t + w_t n_t + \boldsymbol{\phi}_t \mathbf{a}_{t+1} - (\phi_t^s + d_t) a_t^s - a_t^b + \tau_t\right], \tag{59}$$

and subject to $0 \le c_t$, $0 \le n_t$, $0 \le \mathbf{a}_{t+1}$, and $0 \le h_t \le \bar{n}$.

As before, the terms of trade between a buyer who holds portfolio $\mathbf{a}$ and is in a match of type i with a seller who holds portfolio $\tilde{\mathbf{a}}$, i.e., $[q^i(\mathbf{a}, \tilde{\mathbf{a}}), \mathbf{p}^i(\mathbf{a}, \tilde{\mathbf{a}})]$, for $i = 1, 2$, are determined by a take-it-or-leave-it offer by the buyer. That is, $\left(q^i, \mathbf{p}^i\right)$ solves

$$\max_{q^i, \mathbf{p}^i} \left\{ u\left[(1-\epsilon)\kappa d + q^i\right] + W\left(\mathbf{a} - \mathbf{p}^i, \mathbf{s}\right) - u[(1-\epsilon)\kappa d] - W(\mathbf{a}, \mathbf{s}) \right\}$$

$$\text{s.t. } u\left[(1+\epsilon)\kappa d - q^i\right] + W\left(\tilde{\mathbf{a}} + \mathbf{p}^i, \mathbf{s}\right) - u\left[(1+\epsilon)\kappa d\right] - W\left(\tilde{\mathbf{a}}, \mathbf{s}\right) \geq 0,$$

and subject to $\mathbf{p}^1 = \left(0, p^b\right)$, and $p^b \leq a^b$, if $i = 1$ (matches where only bonds can be used as means of payment), or subject to $\mathbf{p}^2 = \left(p^s, p^b\right) \leq \mathbf{a}$, if $i = 2$ (matches where both shares and bonds can be used for payment). Define the function $q(z; y)$ as

$$q(z; y) = \begin{cases} \epsilon y & \text{if } z \geq u\left[(1+\epsilon) y\right] - u(y) \\ \hat{q}(z; y) & \text{if } z < u\left[(1+\epsilon) y\right] - u(y), \end{cases}$$

where $\hat{q}(z; y)$ denotes the q that solves $u\left[(1+\epsilon) y\right] - u\left[(1+\epsilon) y - q\right] = z$. Then, the bargaining solution is $q^i(\mathbf{a}_t, \tilde{\mathbf{a}}_t) = q\left(\boldsymbol{\lambda}_t^i \mathbf{a}_t; \kappa d_t\right)$, for $i = 1, 2$, where $\boldsymbol{\lambda}_t^1 \equiv \left(0, \lambda_t^b\right)$, and $\boldsymbol{\lambda}_t^2 \equiv \boldsymbol{\lambda}_t = \left(\lambda_t^s, \lambda_t^b\right)$. This, and the fact that $W\left[\mathbf{a} + \mathbf{p}^i(\mathbf{a}, \tilde{\mathbf{a}}), \mathbf{s}\right] = W(\mathbf{a}, \mathbf{s}) + \boldsymbol{\lambda}\mathbf{p}^i(\mathbf{a}, \tilde{\mathbf{a}})$, imply that the value of search can be written as

$$V(\mathbf{a}, \mathbf{s}) = \alpha \sum_{i=1,2} \theta_i \widehat{\mathcal{S}}\left(\boldsymbol{\lambda}^i \mathbf{a}; d\right) + W(\mathbf{a}, \mathbf{s}) + (1 - 2\alpha)\bar{u}(d), \tag{60}$$

where $\widehat{\mathcal{S}}\left(\boldsymbol{\lambda}_t^i \mathbf{a}_t; d_t\right) = u\left[(1-\epsilon)\kappa d_t + q\left(\boldsymbol{\lambda}_t^i \mathbf{a}_t; \kappa d_t\right)\right] + u\left[(1+\epsilon)\kappa d_t - q\left(\boldsymbol{\lambda}_t^i \mathbf{a}_t; \kappa d_t\right)\right]$.

The agent's optimal choices of c_t and n_t are characterized by

$$U'(c_t) = \frac{A_t}{w_t} \tag{61}$$

$$v'\left[Z(d_t) n_t\right] = \frac{A_t}{Z(d_t)}. \tag{62}$$

For $i = s, b$, the portfolio choice, $\mathbf{a}_{t+1}$, satisfies

$$U'(c_t)\phi_t^i = \beta E_t \frac{\partial V(\mathbf{a}_{t+1}, \mathbf{s}_{t+1})}{\partial a_{t+1}^i}, \tag{63}$$

where

$$\frac{\partial V(\mathbf{a}_{t+1}, \mathbf{s}_{t+1})}{\partial a_{t+1}^b} = \left[1 + \alpha \sum_{i=1,2} \theta_i \left(\frac{u'\left[(1-\epsilon)\kappa d_{t+1} + q\left(\boldsymbol{\lambda}_{t+1}^i \mathbf{a}_{t+1}; \kappa d_{t+1}\right)\right]}{u'\left[(1+\epsilon)\kappa d_{t+1} - q\left(\boldsymbol{\lambda}_{t+1}^i \mathbf{a}_{t+1}; \kappa d_{t+1}\right)\right]} - 1\right)\right] \lambda_{t+1}^b,$$

$$\frac{\partial V(\mathbf{a}_{t+1}, \mathbf{s}_{t+1})}{\partial a_{t+1}^s} = \left[1 + \alpha\theta_2 \left(\frac{u'\left[(1-\epsilon)\kappa d_{t+1} + q\left(\boldsymbol{\lambda}_{t+1}^2 \mathbf{a}_{t+1}; \kappa d_{t+1}\right)\right]}{u'\left[(1+\epsilon)\kappa d_{t+1} - q\left(\boldsymbol{\lambda}_{t+1}^2 \mathbf{a}_{t+1}; \kappa d_{t+1}\right)\right]} - 1\right)\right] \lambda_{t+1}^s,$$

are obtained from (60). As usual, the agent's choices do not depend on his individual asset holdings, and the distribution of assets is degenerate in the equilibrium.

Given the dividend process $\{d_t\}_{t=0}^{\infty}$, and a path $\{B_t, \tau_t\}_{t=0}^{\infty}$, an equilibrium is an allocation $\{c_t, y_t, n_t, h_t, \mathbf{a}_{t+1}\}_{t=0}^{\infty}$, together with a set of prices $\{w_t, \boldsymbol{\phi}_t\}_{t=0}^{\infty}$, and bilateral terms of trade $\{q_t\}_{t=0}^{\infty}$, such that: *(i)* the individual choices $\{c_t, y_t, n_t, h_t, \mathbf{a}_{t+1}\}_{t=0}^{\infty}$ solve the agent's problem in the decentralized market, given prices; *(ii)* the terms of trade are determined by Nash bargaining, i.e., $q_t^1 = q\left(\lambda_t^b a_t^b; \kappa d_t\right)$, and $q_t^2 = q\left(\boldsymbol{\lambda}_t \mathbf{a}_t; \kappa d_t\right)$; and *(iii)* prices are such that the centralized market clears, i.e., $c_t = d_t$, $a_{t+1}^s = 1$, $a_{t+1}^b = B_{t+1}$, and (21) is satisfied.

To solve for the equilibruim, first notice that (63) imply that asset prices satisfy (43)–(46). Parametrizing preferences by assuming $U(c) = u(c) = \frac{c^{1-\sigma}}{1-\sigma}$, posing that the supply of bonds is set according to the rule $B_{t+1} = f(d_t, x_t) = B_i d_t$ if $x_t = \gamma_i$, and focusing on recursive equilibria where share prices are homogeneous of degree one in d, leads to (47)–(50). The equilibrium asset prices are: $\phi^s(d,i) = \phi_i^s d$, and $\phi^b(d,i) = \phi_i^b$, where $\left(\phi_i^s, \phi_i^b\right)_{i=1}^n$ solve (47) and (48) for $i = 1, ..., n$. The decision rules for bond holdings, shares, and consumption, are: $a^b(d,i) = B_i d$, $a^s(d,i) = 1$, and $c(d,i) = d$, respectively.

A buyer who enters a bilateral meeting holding $\mathbf{a} = \left(a^s, a^b\right)$ in a period where the state is (d,i), has a portfolio that is worth $(d + \phi_i^s d)\, a^s + a^b$ in terms of fruit in the round of centralized trade that follows.[40] Such a buyer purchases

$$q(a^b d^{-\sigma}; \kappa d) = \min\left\{\epsilon\kappa, (1+\epsilon)\kappa - \left(\frac{1}{[(1+\epsilon)\kappa]^{1-\sigma} + (\sigma-1)\frac{a^b}{d}}\right)^{\frac{1}{\sigma-1}}\right\} d$$

in a decentralized meeting in which he can only pay with bonds, or

$$q[(d + \phi_i^s d)\, a^s d^{-\sigma} + a^b d^{-\sigma}, \kappa d] = \min\left\{\epsilon\kappa, (1+\epsilon)\kappa - \left(\frac{1}{[(1+\epsilon)\kappa]^{1-\sigma} + (\sigma-1)\left[(1+\phi_i^s)a^s + \frac{a^b}{d}\right]}\right)^{\frac{1}{\sigma-1}}\right\} d$$

in a meeting in which he can pay with bonds or shares. Along the equilibrium path, if the state today is (d_t, i), all agents bring $a^s(d_t, i) = 1$ and $a^b(d_t, i) = B_i d_t$ into the following period's round of decentralized trade. Note that if in the following period the state is (d_{t+1}, j), then the bond holdings of agents in that period, $a^b(d_t, i)$, can be written as $\frac{B_i}{\gamma_j} d_{t+1}$. Thus, along the

[40] In terms of marginal utility, the value of this portfolio is $\boldsymbol{\lambda}(d,i)\,\mathbf{a}$, where $\boldsymbol{\lambda}(d,i) = \left[\lambda^s(d,i), \lambda^b(d)\right]$, with $\lambda^b(d) = U'(d)$, and $\lambda^s(d,i) = U'(d)(1+\phi_i^s)\, d$. Intuitively, $\lambda^b(d,i)$ and $\lambda^s(d,i)$ are the end-of-period values (in terms of marginal utility of fruit) of a bond and a share, respectively, in a period where the state is (d,i).

equilibrium path, in a period where the state is (d, j), and the outstanding stock of bonds is $\frac{B_i}{\gamma_j} d$, a buyer purchases a quantity

$$\min\left\{\epsilon\kappa, (1+\epsilon)\,\kappa - \left(\frac{1}{[(1+\epsilon)\kappa]^{1-\sigma}+(\sigma-1)\frac{B_i}{\gamma_j}}\right)^{\frac{1}{\sigma-1}}\right\} d$$

in a decentralized meeting in which he can only pay with bonds, or a quantity

$$\min\left\{\epsilon\kappa, (1+\epsilon)\,\kappa - \left(\frac{1}{[(1+\epsilon)\kappa]^{1-\sigma}+(\sigma-1)\left(1+\phi_j^s+\frac{B_i}{\gamma_j}\right)}\right)^{\frac{1}{\sigma-1}}\right\} d$$

in a meeting in which he can pay with bonds or shares. In the former, the buyer pays

$$\min\left\{\frac{B_i}{\gamma_j}, \left[(1+\epsilon)^{1-\sigma}-1\right]\frac{\kappa^{1-\sigma}}{1-\sigma}\right\} d$$

bonds, and in the latter, he pays with a combination of bonds and shares, $\mathbf{p}^2 = \left(p^s, p^b\right) \in \mathbb{R}$, with real value (in terms of fruit)

$$(1+\phi_j^s)p^s d + p^b = \min\left\{1+\phi_j^s+\frac{B_i}{\gamma_j}, \left[(1+\epsilon)^{1-\sigma}-1\right]\frac{\kappa^{1-\sigma}}{1-\sigma}\right\} d, \tag{64}$$

where $0 \leq p^s \leq 1$, and $0 \leq p^b \leq \frac{B_i}{\gamma_j} d$. The real value of the portfolio that changes hands is pinned down, but the precise composition is indeterminate.[41]

To derive equilibrium real wage (w_t), and the remaining individual choices, namely labor demand (n_t), consumption of general good in the second subperiod (y_t), and labor supply (h_t), we need to parametrize v, $Z(d_t)$, and A_t. Let $v(y) = \frac{y^{1-\sigma}}{1-\sigma}$, $Z(d_t) = Zd_t$, and $A_t = Ad_t^{1-\sigma}$, where σ, Z, and A, are positive constants.[42] Then, (61) and (62) imply $w(d) = Ad$, $n(d) = n^*$, and $y(d) = Zn^* d$, where $n^* = \left(AZ^{\sigma-1}\right)^{-1/\sigma}$.

[41] This is also a feature that can be found in Lagos and Rocheteau (2004), a version of Lagos and Wright (2005) where capital goods and fiat money compete as media of exchange.

[42] Setting $A_t = Ad_t^{1-\sigma}$, means that the disutility from work depends on the level of technology of the economy. This specification is useful for two reasons. First, it makes the agent's preferences consistent with balanced growth. And second, it allows one to derive relatively simple conditions to ensure that the constraints $0 \leq h_t \leq \bar{n}$ are slack at all dates and for all states of the world—which must be the case for the distribution of asset holdings to be degenerate in the equilibrium. Alternatively, one could assume $A_t = \overline{\gamma}^{(1-\sigma)t}$, where $\overline{\gamma}$ is the mean growth rate of the economy, as done in many RBC models (see Greewood, Rogerson, and Wright (1995), or Neumeyer and Perri (2004)). This formulation is enough to guarantee that h_t has no secular trend, but in the present context, one would still have to verify that the structure of the shocks and other primitives of the model are

Finally, along the equilibrium path, (59) implies

$$h_t = n^* + \frac{1}{w_t}\left[\left(\phi_t^s + d_t\right)\left(1 - a_t^s\right) + B_t - a_t^b\right]. \tag{65}$$

Intuitively, agents supply labor to finance their consumption of the general good (the first term), and possibly also to rebalance their portfolios.[43]

Recall that the whole analysis so far has been predicated on the premise that the distribution of assets is degenerate at the beginning of each decentralized round of trade—which will indeed be the case, as in Lagos and Wright (2005)—provided the constraints $0 \leq h_t \leq \bar{n}$ are always slack. To conclude, I provide sufficient conditions on parameters, such that this is indeed the case along the equilibrium path. In a period where the state is (d, i), and the outstanding stock of bonds is $\frac{B_j}{\gamma_i} d$, the right-hand side of (65) can be written as

$$h\left(d, i, B_j\right) = n^* + \frac{1}{A}\left[\left(1 + \phi_i^s\right)\left(1 - a^s\right) + \frac{B_j}{\gamma_i} - \frac{a^b}{d}\right].$$

Along the conjectured equilibrium path, $0 \leq a^s \leq 2$, and $0 \leq a^b \leq 2\frac{B_j}{\gamma_i} d$. That is, after a round of decentralized trade, an agent can enter the centralized market neither with a negative asset position, nor holding more than twice his beginning-of-period portfolio. Thus, $\underline{h}\left(i, j\right) \leq h\left(d, i, B_j\right) \leq \overline{h}\left(i, j\right)$, where $\underline{h}\left(i, j\right) \equiv n^* - \frac{1}{A}(1 + \phi_i^s - \frac{B_j}{\gamma_i})$, and $\overline{h}\left(i, j\right) \equiv n^* + \frac{1}{A}(1 + \phi_i^s - \frac{B_j}{\gamma_i})$. Let $\Delta = \max_{i,j}(1 + \phi_i^s - \frac{B_j}{\gamma_i})$, and note that Δ is independent of Z and A. Then, it follows that $\frac{\Delta}{A} \leq n^* \leq \bar{n} - \frac{\Delta}{A}$ implies $0 \leq h\left(d, i, B_j\right) \leq \bar{n}$, for all i and all j. Equivalently, if $Z^{\frac{1-\sigma}{\sigma}} \in \left[\Delta, \bar{n}A - \Delta\right] A^{\frac{1-\sigma}{\sigma}}$, then the constraints $0 \leq h_t \leq \bar{n}$ are slack at all dates, with probability one.

such that the choice of h_t is interior for all realizations of the uncertainty. In fact, this should not be too difficult, for instance, if the variance of the shocks is small, since both formulations coincide if the economy grows deterministically. To conclude this technical digression, I would like to stress that although the particular specification for A_t (and $Z\left(d_t\right)$, and v) matters "globally," i.e., to ensure that the equilibrium is indeed within the class of those with a degenerate distribution of asset holdings—as was conjectured for the derivations—it does not matter "locally," e.g., for the behavior of asset prices. That is, given that the equilibrium distribution of asset holdings is degenerate, asset prices are independent of A_t, $Z\left(d_t\right)$, and v. Observe that (47)–(50) were derived *before* parametrizing A_t, $Z\left(d_t\right)$, and v.

[43] For example, an agent who neither bought nor sold in the decentralized market enters the centralized market holding $a_t^s = 1$, and $a^b = B_t$, chooses $h_t = n^*$, since he has no need to rebalance his portfolio.

A.3 Derivation of the Bond–Equity Trade-Volume Bounds

In equilibrium, in a period where the state is (d, j) and the outstanding stock of bonds is $\frac{B_i}{\gamma_j} d$, the portfolio that is traded in meetings where only bonds can be used to pay, i.e., $\mathbf{p}^1_{ij} = (0, p^b_{ij})$, where $p^b_{ij} = \min\left\{\frac{B_i}{\gamma_j}, \left[(1+\epsilon)^{1-\sigma} - 1\right] \frac{\kappa^{1-\sigma}}{1-\sigma}\right\} d$. In meetings where both assets can be used in exchange, the portfolio that is traded is $\mathbf{p}^2_{ij} = (1, \frac{B_i}{\gamma_j} d)$ if the liquidity constraint binds, and any vector $(0,0) \leq (p^s_{ij}, p^b_{ij}) \leq (1, \frac{B_i}{\gamma_j} d)$ that satisfies (64), if it is slack. Given all this, it is possible to derive an upper and a lower bound for the quantities of bonds and shares traded in each state, v^b_{ij} and v^s_{ij}. To derive the upper bound, resolve the (potential) indetarminacy in $(\hat{p}^s_{ij}, \hat{p}^b_{ij})$, namely in the quantities of shares and bonds that get traded in a meeting where either can be used for payment, by assuming that buyers always follow a *spend-bonds-first rule*. This rule means that $(\hat{p}^s_{ij}, \hat{p}^b_{ij}) = (\underline{p}^s_{ij}, \overline{p}^b_{ij})$, where

$$
\begin{aligned}
\overline{p}^b_{ij} &= \min\left\{\frac{B_i}{\gamma_j}, \left[(1+\epsilon)^{1-\sigma} - 1\right] \frac{\kappa^{1-\sigma}}{1-\sigma}\right\} d \\
\underline{p}^s_{ij} &= \min\left\{1, \max\left\{\left[(1+\epsilon)^{1-\sigma} - 1\right] \frac{\kappa^{1-\sigma}}{1-\sigma} - \frac{B_i}{\gamma_j}, 0\right\} \frac{1}{1+\phi^s_j}\right\}.
\end{aligned}
$$

To derive the lower bound, resolve the indetarminacy in $(\hat{p}^s_{ij}, \hat{p}^b_{ij})$ by assuming that buyers follow a *spend-shares-first rule*. This implies $(\hat{p}^s_{ij}, \hat{p}^b_{ij}) = (\overline{p}^s_{ij}, \underline{p}^b_{ij})$, where

$$
\begin{aligned}
\overline{p}^s_{ij} &= \min\left\{1, \left[(1+\epsilon)^{1-\sigma} - 1\right] \frac{\kappa^{1-\sigma}}{1-\sigma} \frac{1}{1+\phi^s_j}\right\} \\
\underline{p}^b_{ij} &= \min\left\{\frac{B_i}{\gamma_j}, \max\left\{\left[(1+\epsilon)^{1-\sigma} - 1\right] \frac{\kappa^{1-\sigma}}{1-\sigma} - \left(1+\phi^s_j\right), 0\right\}\right\} d.
\end{aligned}
$$

Thus, $\overline{p}^b_{ij}$ and $\underline{p}^s_{ij}$ are the quantities of assets traded in matches of type 2 when the buyer uses the *spend-bonds-first rule*, and $\overline{p}^s_{ij}$ and $\underline{p}^b_{ij}$ are the assets traded in matches of type 2 when the buyer uses the rule spend-shares-first rule. (Note that the formulas for $\overline{p}^b_{ij}$, $\underline{p}^s_{ij}$, $\underline{p}^b_{ij}$, and $\overline{p}^s_{ij}$ are general, in that they apply both, when the liquidity constraint binds and when it is slack.) Recall that the total quantities traded in the period are $v^b_{ij} = \frac{\hat{\alpha}}{2}\left[\theta p^b_{ij} + (1-\theta)\, \hat{p}^b_{ij}\right]$, and

$v^s_{ij} = \hat{\alpha}\,(1-\theta)\,\hat{p}^s_{ij}$, and define the following bounds on the total quantities traded

$$\begin{aligned}
\overline{v}^b_{ij} &= \frac{\hat{\alpha}}{2}\left[\theta p^b_{ij} + (1-\theta)\,\overline{p}^b_{ij}\right] \\
\underline{v}^b_{ij} &= \frac{\hat{\alpha}}{2}\left[\theta p^b_{ij} + (1-\theta)\,\underline{p}^b_{ij}\right] \\
\overline{v}^s_{ij} &= \hat{\alpha}\,(1-\theta)\,\overline{p}^s_{ij} \\
\underline{v}^s_{ij} &= \hat{\alpha}\,(1-\theta)\,\underline{p}^s_{ij}.
\end{aligned}$$

So $\overline{v}^b_{ij}$ and $\underline{v}^s_{ij}$ are the total quantities of bonds and shares traded under the spend-bonds-first regime, while $\underline{v}^b_{ij}$ and $\overline{v}^s_{ij}$ are the total quantities of bonds and shares traded under the spend-shares-first regime. In turn, these bounds can be used to derive upper and lower bounds for the ratio v_{ij}, namely

$$\overline{v}_{ij} = \frac{\gamma_j \overline{v}^b_{ij}}{\underline{v}^s_{ij} B_i d} \qquad \text{and} \qquad \underline{v}_{ij} = \frac{\gamma_j \underline{v}^b_{ij}}{\overline{v}^s_{ij} B_i d}.$$

Finally, the average upper and lower bounds for the ratio of the value of traded bonds (as a proportion of the value of the outstanding stock of bonds) to the value of traded shares (as a proportion of the value of outstanding equity) are $\overline{v} = \sum_i \sum_j \bar{\mu}_i \mu_{ij} \overline{v}_{ij}$, and $\underline{v} = \sum_i \sum_j \bar{\mu}_i \mu_{ij} \underline{v}_{ij}$. These bounds can be compared with actual trade volume data to assess the plausibility of the underlying value of θ.

A.4 Robustness

In this section I verify the robustness of the benchmark results to changes in the values of the parameters $\hat{B}$, κ, and ε, the key parameters that were not in Mehra and Prescott (1985).

The benchmark parametrization is perhaps too conservative, in the sense that it biases the case against the liquidity mechansim by using 0.3 as the target ratio of M1* to annual GDP, which implied $\hat{B} = 0.75$. Tables 5 and 6 are analogous to Tables 3 and 4, but with $\hat{B} = 0.5$, which is consistent with a ratio of M1* to annual GDP of 0.2 (this is roughly the average for the 1975–1995 period). All the other parameters are kept at their benchmark levels. All entries corresponding to $\sigma = 1$ through 7 in Table 5 are identical to those in Table 3. In both, the liquidity constraints bind for $\sigma \geq 8$. When the constraints bind, the asset returns are only slightly lower in Table 5, as is natural, since the relative scarcity of outside bonds means that both assets yield larger liquidity returns. The equity premium is essentially unchanged.

The first six columns of Table 6 are identical to the corresponding columns of Table 4 through $\sigma = 7$. Again, equity returns and premia are slightly lower in the economy with lower $\hat{B}$ for higher values of σ. Since the bond to GDP ratio is smaller in this economy, the liquidity constraint is even more binding in trades where only bonds can circulate, and therefore the implied values for θ are uniformly lower in Table 6 relative to Table 4. In Table 6, the volume bounds implied by the values of θ corresponding to $\sigma = 5$, 4, and 3, i.e., those that imply returns consistent with the data, are: $[.004, 2]$, $[.03, 5]$, and $[.23, 14]$ respectively.

Insert Tables 5 and 6 here

In the benchmark calibration, I set $\kappa = 0.5$ so that the model implies that the fraction of GDP that corresponds to trees that have outstanding tradeable equity shares equals 0.4. Next, I report how the results change if the share of GDP produced by traded "trees" is in fact larger, say 0.45. (Recall that the gross value added of the nonfinancial corporate business sector as a share of GDP, i.e., the upper bound on the relevant target, is about 0.5.) Table 7 corresponds to the economy with $\kappa = 2/9$ and $\hat{B} = 0.6658$, which imply that tradeable trees produce 45% of GDP, and that the average bond to GDP ratio is 0.3 (as in the benchmark), and $\theta = 0$. For this parametrization, the liquidity constraints start binding for $\sigma = 4$. The equilibrium for lower values of σ (and hence all entries in the table), are just as in Mehra–Prescott. Comparing Table 7 with Table 3, it is apparent that liquidity constraints are tighter in the economy with lower κ.[44] First, they start binding for lower values of σ, i.e., 4 as opposed to 8), and second, they are tighter when they both bind. (This is evident from the comparing the magnitudes of the Euler Equation wedges in Table 7 to their counterparts in Table 3.) Tighter liquidity constraints imply that equity and bond returns are uniformly lower in this economy than in the

[44]The reason is that with CRRA preferences, the difference $u\left[(1+\epsilon)\kappa\right] - u(\kappa)$ is decreasing in κ for $\sigma > 1$. This difference, is essentially what determines the size of the right-hand side of the inequalities that appear in the definitions of the sets Ω and Ω_θ in (53) and (54). Thus, the liquidity mechanism is stronger for smaller κ.

one with higher κ. For $\sigma = 9$ or larger, agent's are so eager for liquidity, that they are willing to hold bonds even though their intrinsic return is negative on average (bonds still sell at a discount in the high state).

Table 8 reports the results for the model with exogenous liquidity differences. For $\sigma = 3$ and lower, the first six columns are the same as those of Table 4. But for each value of σ, the implied values of θ are uniformly lower in Table 8, the case with smaller κ. For $\sigma = 6$ the liquidity needs are so severe that there is no positive θ that can induce a risk-free rate as high as 1%. Summarizing, the liquidity mechanism is magnified in parametrizations with smaller values of κ. This implies that somewhat smaller values of θ are needed to account for the equity premium.[45]

Higher values of κ tend to relax the liquidity constraints, and in fact, they will bind in at least some state of the world iff $\kappa < \overline{\kappa}$, where

$$\overline{\kappa} = \left[\frac{1 - (1+\epsilon)^{1-\sigma}}{(\sigma - 1) \min_{i,j} \left(B_i / \gamma_j \right)} \right]^{\frac{1}{\sigma - 1}}.$$

Given the rest of the parameters and data targets are as in the benchmark, I found that for targets of the share of GDP produced by trees with tradeable equity that are less than 35%, there are no liquidity needs in any state.[46]

Insert Tables 7 and 8 here

In the baseline calibration, I use $\epsilon = 0.8$. Now suppose $\epsilon = 0.5$, and keep all other parameters as in the benchmark. Tables 9 and 10 are very similar to Tables 3 and 4, so the key results appear robust to lowering ε. The volume bounds implied by the values of θ corresponding to

[45] The volume bounds do not appear too sensitive, e.g., those corresponding to $\sigma = 3$, are now $[.69, 20]$.

[46] If this target is set slighthly higher, at 36% for example, then the liquidity constraints are slack in all trades, even those where only bonds can be used, up to $\sigma = 7$. For $\sigma = 8$ and $\sigma = 9$, setting $\theta = 1$ is not enough of a liquidity advantage on bonds to bring their return down to 1%. With $\sigma = 8$ and $\theta = 1$ the risk-free rate is 11.27%, virually as high as in the Mehra–Prescott economy. With $\sigma = 9$ and $\theta = 1$ the risk-free rate is 6%, half of what it would be for Mehra–Prescott, but still high. With $\sigma = 10$, setting $\theta = 0.7$ induces a risk-free rate of 1%, and the return on equity is essentially the same as in Mehra–Prescott, 15.79%. (With $\sigma = 10$, $\theta = 0.7$ implies $[\underline{v}, \overline{v}] = [9, 62]$.)

$\sigma = 5$, 4, and 3 in Table 10 are: $[.05, 3]$, $[.3, 8]$, and $[3, 43]$, respectively.

Insert Tables 9 and 10 here

Next, consider the parametrization with $\epsilon = 1$, and all other parameters kept at their benchmark values. The results for the case with $\theta = 0$ and for the one where θ is chosen so that the risk-free rate is 1% are reported in Tables 11 and 12 respectively. The results do not seem to be altered much by increasing ϵ away from its benchmark value. For example in Table 12, the volume bounds implied by the values of θ corresponding to $\sigma = 5$, 4, and 3 are: $[.007, 2.4]$, $[.05, 5.75]$, and $[.39, 16]$ respectively.

Insert Tables 11 and 12 here

The choice of ϵ has implications for consumption inequality. In the benchmark calibration with $\epsilon = 0.8$, the variance of the log of total (i.e., including general goods and fruit in both subperiods) per capita consumption is 0.013, and the variance of log per capita consumption of in the first subperiod (which is the only source of consumption inequality in the model) is 0.67. The parametrization with $\epsilon = 0.5$ used to generate Tables 9 and 10 implies that the variance of the log of total consumption equals 0.005, while the variance of the log of consumption in the first subperiod equals 0.156. (See footnote 22.)

Data set	% real return on a market index (mean)	% real return on a relatively riskless security (mean)	% equity premium (mean)
1889–1978 (Mehra–Prescott)	6.98	0.8	6.18
1889–2000 (Mehra–Prescott)	8.06	1.14	6.92
1926–2000 (Ibbotson)	8.8	0.4	8.4
1871–1999 (Shiller)	6.99	1.74	5.25
1802–1998 (Siegel)	7.0	2.9	4.1

Table 1: U.S. equity premium from different data sets

σ	Equity Return	Bond Return	Equity Premium	SD Equity Return	Sharpe Ratio
1	2.83	2.70	0.13	3.6	.035
2	4.58	4.30	0.28	4.2	.068
3	6.27	5.79	0.48	4.8	.099
4	7.89	7.18	0.71	5.5	.128
5	9.42	8.45	0.97	6.2	.156
6	10.88	9.62	1.26	6.9	.182
7	12.24	10.67	1.57	7.6	.207
8	13.52	11.60	1.92	8.3	.231
9	14.70	12.41	2.29	9.0	.254
10	15.79	13.10	2.69	9.7	.276

Table 2: The Mehra-Prescott economy

σ	Equity Return	Bond Return	Equity Premium	SD Equity Return	Sharpe Ratio	EE wedges ω^e, ω^b
1	2.83	2.70	0.13	3.6	.035	0,0
2	4.58	4.30	0.28	4.2	.068	0,0
3	6.27	5.79	0.48	4.8	.099	0,0
4	7.89	7.18	0.71	5.5	.128	0,0
5	9.42	8.45	0.97	6.2	.156	0,0
6	10.88	9.62	1.26	6.9	.182	0,0
7	12.24	10.67	1.57	7.6	.207	0,0
8	8.80	6.76	2.04	8.4	.244	.0010,-.0435
9	5.97	3.54	2.43	9.0	.272	.0014,-.0790
10	4.35	1.50	2.85	9.6	.298	.0017,-.1027

Table 3: Benchmark economy with $\theta = 0$

σ	Equity Return	Bond Return	Equity Premium	SD Equity Return	Sharpe Ratio	EE Wedges ω^e, ω^b	θ
1	2.83	2.70	0.13	3.6	0.035	0,0	[0,1]
2	4.58	1	3.58	4.2	0.847	.0316,-.0316	.3799
3	6.27	1	5.27	4.8	1.084	.0453,-.0453	.0742
4	7.89	1	6.89	5.5	1.247	.0576,-.0576	.0222
5	9.42	1	8.42	6.2	1.357	.0687,-.0687	$.062\times10^{-1}$
6	10.88	1	9.88	6.9	1.431	.0786,-.0786	$.015\times10^{-1}$
7	12.24	1	11.24	7.6	1.478	.0873,-.0873	0.03×10^{-2}
8	8.80	1	7.80	8.4	0.929	.0526,-.0950	3.40×10^{-5}
9	5.97	1	4.97	9.0	0.555	.0240,-.1016	2.40×10^{-6}
10	4.35	1	3.35	9.6	0.350	.0062,-.1072	7.26×10^{-8}

Table 4: Benchmark economy with exogenous liquidity differences

σ	Equity Return	Bond Return	Equity Premium	SD Equity Return	Sharpe Ratio	EE Wedges ω^e, ω^b
1	2.83	2.70	0.13	3.6	.035	0,0
2	4.58	4.30	0.28	4.2	.068	0,0
3	6.27	5.79	0.48	4.8	.099	0,0
4	7.89	7.18	0.71	5.5	.128	0,0
5	9.42	8.45	0.97	6.2	.156	0,0
6	10.88	9.62	1.26	6.9	.182	0,0
7	12.24	10.67	1.57	7.6	.207	0,0
8	8.70	6.65	2.05	8.4	.244	.0010,-.0444
9	5.94	3.50	2.44	9.0	.272	.0014,-.0794
10	4.34	1.49	2.85	9.6	.298	.0017,-.1028

Table 5: Robustness, lower $\hat{B}$ with $\theta = 0$

σ	Equity Return	Bond Return	Equity Premium	SD Equity Return	Sharpe Ratio	EE Wedges ω^e, ω^b	θ
1	2.83	2.70	0.13	3.6	0.035	0,0	[0,1]
2	4.58	1	3.58	4.2	0.847	.0316,-.0316	.078
3	6.27	1	5.27	4.8	1.084	.0453,-.0453	.0263
4	7.89	1	6.89	5.5	1.247	.0576,-.0576	.00802
5	9.42	1	8.42	6.2	1.357	.0687,-.0687	.00208
6	10.88	1	9.88	6.9	1.431	.0786,-.0786	.000464
7	12.24	1	11.24	7.6	1.478	.0873,-.0873	9.0×10^{-5}
8	8.70	1	7.70	8.4	0.917	.0517,-.0950	8.7×10^{-6}
9	5.94	1	4.94	9.0	0.551	.0237,-.1016	5.8×10^{-7}
10	4.34	1	3.34	9.6	0.349	.0061,-.1072	1.6×10^{-8}

Table 6: Robustness, lower $\hat{B}$ with exogenous liquidity differences

σ	Equity Return	Bond Return	Equity Premium	SD Equity Return	Sharpe Ratio	EE Wedges ω^e, ω^b
1	2.83	2.70	0.13	3.6	.035	0,0
2	4.58	4.30	0.28	4.2	.068	0,0
3	6.27	5.79	0.48	4.8	.099	0,0
4	6.33	5.58	0.75	5.6	.134	.0003,-.0149
5	3.15	2.13	1.02	6.1	.166	.0006,-.0584
6	2.29	0.96	1.33	6.8	.196	.0008,-.0790
7	2.10	0.43	1.67	7.5	.224	.0011,-.0926
8	2.08	0.04	2.04	8.1	.250	.0014,-.1036
9	2.12	-0.31	2.43	8.8	.276	.0016,-.1133
10	2.18	-0.67	2.85	9.5	.301	.0019,-.1219

Table 7: Robustness, lower κ with $\theta = 0$

σ	Equity Return	Bond Return	Equity Premium	SD Equity Return	Sharpe Ratio	EE Wedges ω^e, ω^b	θ
1	2.83	2.70	0.13	3.6	0.035	0,0	[0,1]
2	4.58	1	3.58	4.2	0.848	.0316,-.0316	.0260
3	6.27	1	5.27	4.8	1.084	.0453,-.0453	$.022\times10^{-1}$
4	6.33	1	5.33	5.6	0.954	.0430,-.0576	$.011\times10^{-2}$
5	3.15	1	2.15	6.1	0.350	.0110,-.0687	17×10^{-7}
6	2.29	0.96	1.33	6.8	0.196	.0008,-.0790	0

Table 8: Robustness, lower κ with exogenous liquidity differences

σ	Equity Return	Bond Return	Equity Premium	SD Equity Return	Sharpe Ratio	EE Wedges ω^e, ω^b
1	2.83	2.70	0.13	3.6	.035	0,0
2	4.58	4.30	0.28	4.2	.068	0,0
3	6.27	5.79	0.48	4.8	.099	0,0
4	7.89	7.18	0.71	5.5	.128	0,0
5	9.42	8.45	0.97	6.2	.156	0,0
6	10.88	9.62	1.26	6.9	.182	0,0
7	12.24	10.67	1.57	7.6	.207	0,0
8	9.13	7.09	2.04	8.4	.243	.0009,-.0405
9	6.11	3.69	2.42	8.9	.271	.0013,-.0777
10	4.41	1.57	2.84	9.5	.297	.0017,-.1021

Table 9: Robustness, $\epsilon = 0.5$ with $\theta = 0$

σ	Equity Return	Bond Return	Equity Premium	SD Equity Return	Sharpe Ratio	EE Wedges ω^e, ω^b	θ
1	2.83	2.70	0.13	3.6	0.035	0,0	[0,1]
2	4.58	4.30	0.28	4.2	0.068	0,0	[0,1]
3	6.27	1	5.27	4.8	1.084	.0453,-.0453	.218
4	7.89	1	6.89	5.5	1.247	.0576,-.0576	.06
5	9.42	1	8.42	6.2	1.357	.0687,-.0687	.02
6	10.88	1	9.88	6.9	1.431	.0786,-.0786	0.06×10^{-1}
7	12.24	1	11.24	7.6	1.478	.0873,-.0873	0.02×10^{-1}
8	9.13	1	8.13	8.4	0.968	.0555,-.0950	3.49×10^{-4}
9	6.11	1	5.11	8.9	0.571	.0252,-.1016	4.20×10^{-5}
10	4.41	1	3.41	9.5	0.357	.0067,-.1072	2.38×10^{-6}

Table 10: Robustness, $\epsilon = 0.5$ with exogenous liquidity differences

σ	Equity Return	Bond Return	Equity Premium	SD Equity Return	Sharpe Ratio	EE Wedges ω^e, ω^b
1	2.8301	2.7038	.1262	3.6	.0347	0,0
2	4.5842	4.2973	.2868	4.2	.0678	0,0
3	6.2722	5.7905	.4817	4.8	.0991	0,0
4	7.8871	7.1774	.7097	5.5	.1285	0,0
5	9.4238	8.4542	.9696	6.2	.1562	0,0
6	10.8780	9.6180	1.260	6.9	.1825	0,0
7	12.2454	10.6663	1.579	7.6	.2075	0,0
8	8.7428	6.6923	2.0506	8.4	.2441	.0010,-.0440
9	5.9525	3.5163	2.4362	9.0	.2718	.00138,-.07924
10	4.3440	1.4950	2.8489	9.56	.2978	.0017,-.1028

Table 11: Robustness, $\epsilon = 1$ with $\theta = 0$

σ	Equity Return	Bond Return	Equity Premium	SD Equity Return	Sharpe Ratio	EE Wedges ω^e, ω^b	θ
1	2.83	2.70	0.13	3.6	0.035	0,0	[0,1]
2	4.58	1	3.58	4.2	0.847	.0316,-.0316	.1693
3	6.27	1	5.27	4.8	1.084	.0453,-.0453	.0467
4	7.89	1	6.89	5.5	1.247	.0576,-.0576	.0140
5	9.42	1	8.42	6.2	1.357	.0687,-.0687	$.037\times10^{-1}$
6	10.88	1	9.88	6.9	1.431	.0786,-.0786	$.083\times10^{-2}$
7	12.24	1	11.24	7.6	1.478	.0873,-.0873	1.62×10^{-4}
8	8.74	1	7.74	8.4	0.922	.0520,-.0950	1.57×10^{-5}
9	5.95	1	4.95	9.0	0.562	.0238,-.1016	1.02×10^{-6}
10	4.34	1	3.34	9.6	0.350	.0061,-.1072	2.71×10^{-8}

Table 12: Robustness, $\epsilon = 1$ with exogenous liquidity differences

	Mean	Variance-Covariance						
		$\hat{R}^s$	$\hat{R}^b$	L^s	L^b	R^s	R^b	MRS
$\hat{R}^s$	1.043518	.00914	.00282	-.00268	-.00268	.00728	.00039	-.02569
$\hat{R}^b$	1.015041		.00269	-.00023	-.00023	.00287	.00274	-.00219
L^s	1.103996			.00099	.00099	-.00193	.00075	.00946
L^b	1.103996				.00099	-.00193	.00075	.00946
R^s	1.149354					.00601	.00121	-.01849
R^b	1.120370						.00378	.00719
MRS	0.886144							.09063

Table 13: Moments of returns, baseline with $\sigma = 10$ and $\theta = 0$

	Mean	Variance-Covariance						
		$\hat{R}^s$	$\hat{R}^b$	L^s	L^b	R^s	R^b	MRS
$\hat{R}^s$	1.078871	.00305	.00042	0	.00028	.00305	.00074	-.00701
$\hat{R}^b$	1.01		.00039	0	-2.8×10^{-8}	.00042	.00042	-.00036
L^s	1.00			0	0	0	0	0
L^b	1.061942				.00003	.00028	.00003	-.00073
R^s	1.078871					.00305	.00074	-.00701
R^b	1.072561						.00048	-.00112
MRS	0.933394							.01722

Table 14: Moments of returns, baseline with $\sigma = 4$ and $\theta = 0.0222$

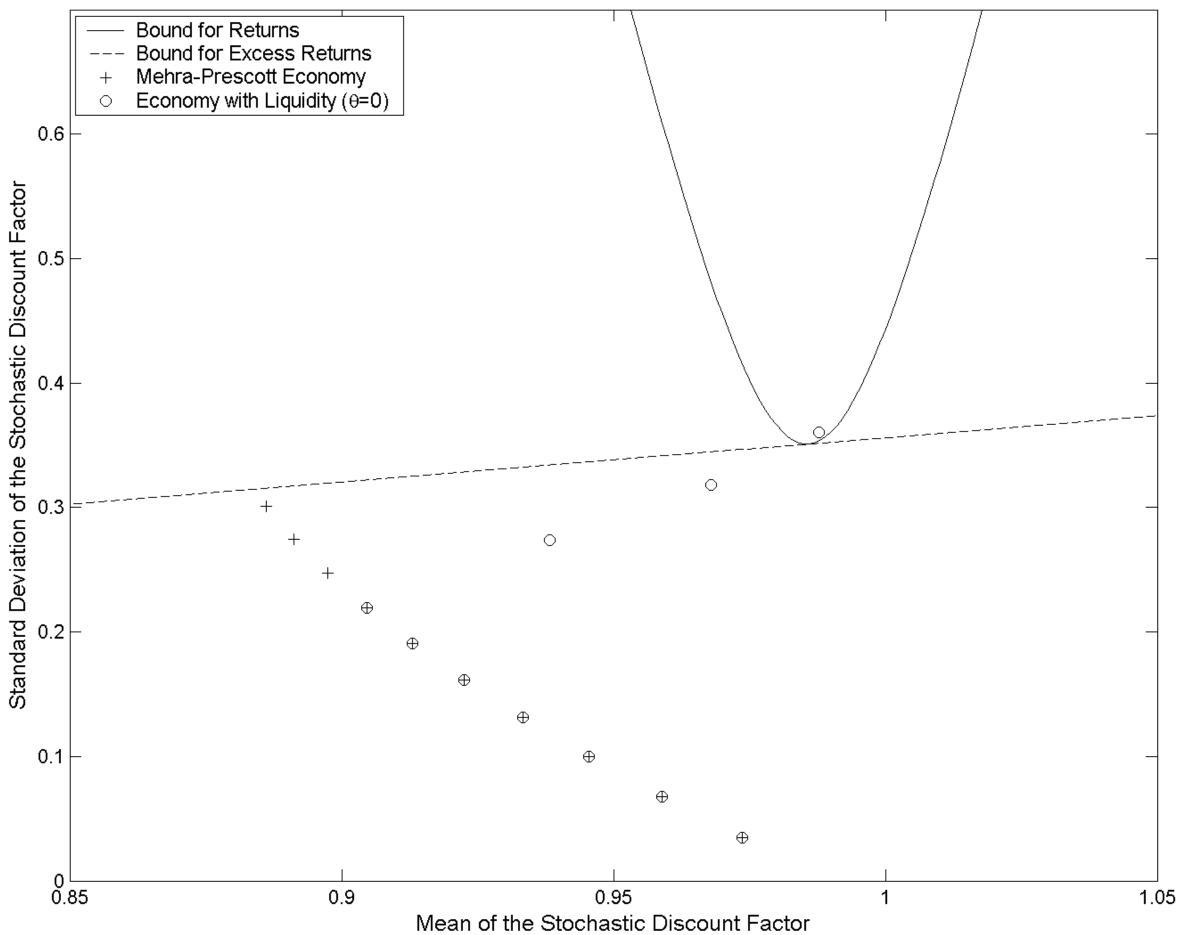

Figure 1: Hansen–Jagannathan bounds