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Collateral Pool Settlement System: A Theoretical Model

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Abstract

This paper investigates a collateral pool settlement (CPS) payment system – a system that provides intra-day liquidity against a collateral pool. First, we show that participants of CPS do not have incentives to delay payments once they have committed to participate. This is in striking contrast to a Real Time Gross Settlement (RTGS) system where banks have strong incentives to free-ride on liquidity provided by incoming payments. Second, we establish conditions under which banks prefer to participate in a collateral pool instead of settling payments in RTGS. Third, we show that a late payment equilibrium may arise in RTGS in the presence of a possible intra-day failure of a participant if the cost of intra-day liquidity is sufficiently high.

Key words: collateral pool settlement, RTGS, payments

JEL classification: E42, E58, G21

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Summary

Over the last two decades large value interbank payment systems across the world have evolved towards real time settlement. By now it is accepted that large value payment systems that cannot ensure real time finality of settlement pose unacceptable settlement risk. Therefore, almost all large value payment systems across the world moved away from settlement of netted payment flows at the end of the day to real time gross settlement (RTGS). In this paper we develop a theoretical model of a payment system with real time finality but net settlement.

In RTGS systems, banks rely on three sources of liquidity to make their payments. The first one is banks' own reserves. The second one is an intra-day liquidity facility made available by the central bank. Holding cash is costly and access to intra-day liquidity facility carries either an explicit fee or a requirement to post collateral with the central bank. The third source of liquidity is virtually free – the payment inflow from other payment system participants. Therefore, by delaying payment outflows banks can minimize their intra-day liquidity costs.

Mitigating the incentives for settlement delay in RTGS systems is a key issue for central bankers and the participants of these systems. A range of solutions has been proposed and implemented. For example, the UK CHAPS payment system imposes throughput guidelines for a percentage of daily payment flows to be settled by a specific time of the day, while the Swiss SIC system levies a time varying payment tariff which is increasing throughout the day. We suggest an alternative mechanism to incentivise early settlement.

In a collateral pool settlement (CPS) system the collateral pool is used to secure the system wide net payment position. Banks are granted access to the intra-day liquidity facility only if in aggregate the net payment position across *all* banks is smaller than the value of collateral in the collateral pool.

In the model, initially, the intra-day liquidity facility is available to all participants on a first-come first-served basis. By submitting payment instructions first, a participant

potentially uses more liquidity than it has posted collateral. For other banks who pay later, it is likely that they have received payments or that some early payments have offset and made the collateral pool available again. Either way, the initial payments made are likely to trigger cascades of further payments in a self reinforcing way.

Payments made throughout the day are immediately final. But the settlement of obligations takes place at the end of the day. Effectively it is an RTGF (F = finality) system, with end of day settlement. If at the end of the day not all intra-day net payment positions are covered, a loss sharing arrangement is triggered where any failed bank's collateral in the shared collateral pool is liquidated first. If insufficient, the shortfall is then covered by liquidating the collateral of the surviving banks. The sum of the net amounts of the payments never exceeds the value of the collateral pool at any point in time, which ensures that the settlement agent can guarantee finality in real time without incurring credit risk. By contrast, in RTGS, banks do not share collateral, thus there is no need for a loss sharing arrangement, but to make a payment each bank must obtain intra-day liquidity for itself or rely on incoming payments. This creates an incentive to delay settlement.

CPS not only eliminates incentives of settlement delay (by making the cost of liquidity sunk), but also provides additional incentives to submit payments early. Such a payment system arrangement provides several other theoretical benefits as well. First, it is less likely to result in a classic gridlock situation as banks can always submit payments to banks in a net payment position. Second, the collateral pool has the potential to reduce system wide liquidity needs, as banks sharing the collateral can use it more efficiently to meet their peak liquidity needs when these occur at different times during the day.

1 Introduction

This paper proposes a collateral pool arrangement to support intra-day liquidity in real time payment systems. We compare a collateralized Real Time Gross Settlement (RTGS) system to a collateral pool settlement (CPS) system – a system that provides intra-day liquidity against a collateral pool and ensures real time finality of settlement. We show that the CPS system provides incentives to submit payments early while RTGS leads to strategic payment delay. We also show that under very general conditions banks prefer to participate in a collateral pool.

In our theoretical CPS model discussed below, participating banks post collateral into a "collateral pool" at the beginning of the day. During the day banks can obtain intra-day liquidity against the collateral on a first-come first-served basis up to the total value of collateral.[1] By submitting payment instructions first, a participant potentially uses more liquidity than it has posted collateral. For these banks who pay later, it is likely that they have received payments or that some early payments have offset and have made the collateral pool available again. Either way, the initial payments made are likely to trigger cascades of further payments in a self reinforcing way.

Since all payments are backed by collateral, payments made throughout the day are immediately final in CPS, even though the settlement of obligations takes place at the end of the day. Effectively it is an RTGF (F = finality) system, with end of day settlement. If at the end of the day not all intra-day net payment positions are covered, a loss sharing arrangement is triggered where any failed bank's collateral in the shared collateral pool is liquidated first. If insufficient, the shortfall is then covered by liquidating the collateral of the surviving banks. The sum of the net amounts of the payments never exceeds the value of the collateral pool at any point in time, which ensures that the settlement agent can guarantee finality in real time without incurring credit risk.[2]

[1]Individual participant limits can be put in place to ensure that not all of the pooled liquidity is used by just one (or a few) participants. For simplicity, our model does not include such bilateral limits. The main implication of introducing individual participant limits into our model would be to reduce its liquidity efficiency; but they would not alter the incentives that we observe in a CPS system.

[2]The shared collateral arrangements in Canada are different in that the Bank of Canada is potentially exposed in the unlikely event of multiple bank failures - but there is no need for this to be a feature of a shared collateral pool.

By contrast, in RTGS, banks do not share collateral, thus there is no need for a loss sharing arrangement. In RTGS systems, banks rely on three sources of liquidity to make their payments. The first one is banks' own reserves. The second one is an intra-day liquidity facility made available by the central bank. Holding cash is costly and access to intra-day liquidity facility carries either an explicit fee or a requirement to post collateral with the central bank. The third source of liquidity is virtually free – the payment inflow from other payment system participants. If the intra-day liquidity cost is relatively large, banks find it optimal to delay payments and rely on payment inflows for their source of liquidity. However, this approach to liquidity saving does not work well if all banks pursue the same delay strategy.

Mitigating the problem of settlement delay, or the trade-off between intra-day liquidity cost and intra-day delay cost, is a key issue for central bankers and the participants of RTGS systems. A range of solutions has been proposed and implemented. For example, The UK CHAPS payment system imposes throughput guidelines for a percentage of daily payment flows to be settled by a specific time of the day, while the Swiss SIC system levies a time varying payment tariff which is increasing throughout the day.

In this paper we set up a general game theoretic structure that we use to compare RTGS and CPS payment systems. In the model the main trade-offs for the banks are (i) in deciding if they want to commit collateral to a collateral pool (CPS) or if they would rather rely on their own liquidity and the liquidity of incoming payments (RTGS); (ii) in choosing the timing of payments. We solve this dual stage problem backwards and demonstrate that (i) banks settle their payments as soon as possible in CPS while delaying them to the end of the day in RTGS, (ii) in anticipation of such equilibrium behavior banks choose to participate in a collateral pool, unless default probability is high.

In CPS, banks pledge collateral at the beginning of the day to the collateral pool. This is a similar arrangement to Canadian Large Value Transfer System (LVTS). While in LVTS banks pledge collateral that goes some way to secure their bilateral net payment position, in CPS the collateral pool is used to secure the aggregate net payment position. Thus banks can utilize collateralized intra-day credit facility for payments on a "first come, first served" basis – banks obtain intra-day credit only if the aggregate net payment position across *all* banks is equal to or smaller than the margined value of collateral in a collateral pool. Therefore compared to LVTS CPS has an "exhaustibility constraint".

CPS not only eliminates incentives of settlement delay (by making the cost of liquidity sunk), but also provides additional incentives to submit payments early, as by paying early a bank is less likely to be exposed to the loss sharing in case of intra-day default. Counterparty risk in deferred settlement systems emerges only when a risky bank has a net payment position. Thus a bank can reduce the counterparty risk (= net payment position) by making a payment to the risky bank.) This is a critical difference from payment behaviour in RTGS, where making early payments is costly.

But even in RTGS banks may find it optimal to pay early, as late payments increase the risk to be a net payer at the end of the day. If a bank fails to make a payment, its recipient is left with unanticipated cash shortfall and is forced to obtain overnight funding at a probably higher cost. Probability of being in such a situation can be minimized by making early payments which trigger cascades of incoming early payments. Banks in RTGS thus face a trade-off between the cost of liquidity and the intra-day risk coming from the counterparty's default: early payments reduce the risk of own payment failure and the risk to be a net payer by the end of the day, but early payments raise the cost of liquidity. Thus banks pay late, unless the expected intra-day default cost dominates the cost of liquidity. In CPS such a trade-off does not arise and therefore banks pay early.

CPS arrangement provides several other theoretical benefits as well. First, it is less likely to result in a classic gridlock situation as banks can always submit payments to banks in a net payment position. This is possible as making a payment to the bank in a net payment position does not increase the utilization of a collateral pool. Second, the collateral pool has the potential to reduce system wide liquidity needs, as banks sharing the collateral can use it more efficiently to meet their peak liquidity needs when these occur at different times during the day. Third, the CPS system also ensures that at any point in time the collateral pledged fully covers the aggregate net debit positions of the banks, thereby ensuring that the guarantor of the settlement (usually the central bank) does not face additional credit risk (as is the case with a traditional RTGS arrangement).

The theoretical model used in this paper has several new features not present in the literature. First we model credit risk explicitly, as the incentives created by sharing collateral management is a core feature of this paper. An optimal collateral arrangement cannot be studied unless we explicitly model payment systems in which participants can default. Second, CPS and pure RTGS are modeled in the same analytical framework. The difference between CPS and pure RTGS is described as the difference in parameters. To our knowledge, this is the first paper that compares the behavior and the performance of these different payment systems directly. Third, the choice between the payment systems is modeled explicitly, while the literature usually studies banks' behavior in a given payment system.

The rest of the paper is structured as follows. We overview relevant literature in the next section. We set up a stylized payments model that nests the features of the pure RTGS and CPS in section 3. Payment timing equilibria for both pure RTGS and CPS are provided in Section 4. In Section 5 we show the conditions under which banks would prefer to participate in a collateral pool instead of settling payments in pure RTGS. We discuss the main results in Section 6 while section 7 concludes.

2 Related Literature

Literature on bank payment behavior in RTGS payment system is well established. As shown by Angelini (1998, 2000) and Bech and Garratt (2003) banks may find it optimal to delay payments in an RTGS payment system. This unfortunately appears to be not only a theoretical possibility, but a practical feature of some payment systems. Armantier, Arnold and McAndrews (2008) show that a large proportion of payments in Fedwire are settled late in the day with the peak activity on average at around 5:11pm in 2006[3]. Significant intra-day payment delay carry a non-pecuniary cost of "delay" (e.g. customer satisfaction), but most importantly it can exacerbate the costs of an operational failure or costs due to the default of a payment system participant.

Literature on shared collateral payment systems is scarce and exclusively concentrates on studying the Canadian LVTS. None of the studies addresses the incentives to submit

[3]Fedwire is open for payments until 6:30pm.

payments early.

The risk sharing nature of LVTS T2 is believed to pose limited systemic risk, but it is not clear if individual participants are robust to other participants' defaults. McVanel (2005) evaluates this aspect of LVTS conducting a simulation of unanticipated defaults of one or several LVTS participants and evaluating the impact of such an event on the surviving participants. It is found that even taking the maximum possible shortfalls, the surviving members of LVTS will not incur significant losses measured as a proportion of their capital. Although resulting losses are small compared to capital, in some cases it leads to a significant increase in leverage. The major limitation of the study is that the payment system simulator used at a time did not account for credit limits. Thus some of the payments in their simulation exercise settle prematurely instead of being delayed. In an extension of the original paper, Ball and Engert (2007) confirm the results of McVanel (2005) using an updated payment system simulator that accounts for credit limits and LVTS queuing arrangements.

When granting Bilateral Credit Limits (BCLs) payment system participants are expected to evaluate the creditworthiness of the counterparties they are dealing with. Thus it seems obvious that when a participant is expected to default on a particular day, nobody will grant a BCL to it. Controversially O'Connor and Caldwell (2008) show that this might not be the case if the expected credit loss is relatively small. The key argument is that the network externalities of granting a BCL to a bank that is subject to an imminent closure may outweigh the losses and may be Pareto improving. The authors also point out that complex payment systems, like LVTS T2, may have complicated incentive structures and network effects that need to be better understood before appropriate policy decisions may be prescribed.

In a related study, O'Connor, Chapman and Millar (2008) evaluate possible network equilibria of the LVTS T2. The authors assert that virtually all time-critical payments are settled in LVTS T1. Since non-time critical payments provide more options for strategic interaction, they focus on LVTS T2. In a highly stylized analysis they find that LVTS T2 is a network of three related subgroups of direct member banks that are quite asymmetric in terms of their activity, liquidity efficiency, and degree of network centrality. The asymmetry that is identified could lead to important non-neutral effects of a system wide shock. Therefore the asymmetric nature of the different subgroups must be taken into consideration when evaluating different policy choices.

More generally, this paper is relevant to the studies of payment behavior under risky circumstances. The first paper studying that issue is Rochet and Tirole (1996). They argue that if a bank transfers cash to a counterparty that then must transfer cash back to the bank, the first payer is exposed to its counterparty's credit risk until the counterparty has paid back. They show that the exposure is smaller in a deferred net settlement (DNS) system. Kahn, McAndrews and Roberds (2003) also reach a similar conclusion, by studying strategic default in a payment system. Mills Jr. and Nesmith (2008) argue that operational risk incentivises settlement delay in FedWire. All these papers study bilateral payment relationships.

Mills Jr. and Nesmith (2008) is the closest to our paper. In their model, delay is costless because no intra-day delay cost is assumed and operational failures are costless for the stricken bank. A bank falls short of cash if an operational failure hinders a counterparty's payment to the bank after the bank has made a payment to the counterparty. A bank has no reason to be the first payer, because once they observe their counterparty get stricken, they experience no cost if payments are not made.

This paper also studies credit risk, but the situation analyzed is different. The payment obligations are multilaterally offsetting, not bilaterally, meaning that making a payment does not directly expose the payer to the recipient's counterparty risk. Instead, a member's failure (insolvency, or an operational failure) creates a liquidity sink, which is costly for the other members. In addition the surviving solvent members have to share the loss of the insolvent member to ensure the finality of payments. We show that the member banks of CPS are exposed to the other members' credit risk ex-ante, but the exposure incentivises early payments: as a result, the ex-post exposure of CPS is contained. In this sense, this paper is close to Fujiki, Green and Yamazaki (2008) who argues that payment "systems should be designed to encourage its participants to take optimal degrees of risk in accordance with their attitude toward risk".

3 Payment Model

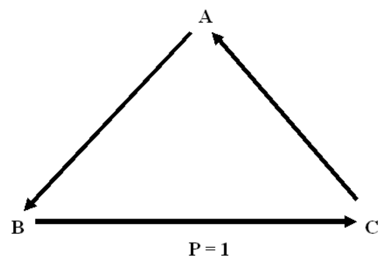

Chart 1: Payment flows are normalized to 1 and are assumed to be perfectly offsetting.

There are three ex-ante identical banks (indexed A, B, and C) that have multilaterally offsetting payment obligations of equal size, p, as shown in Chart 1. For simplicity p is normalized to 1. It is the most parsimonious structure within which we can capture important features of both payment system arrangements. Clearly, in reality banks do not know with certainty that their payment flows are perfectly offsetting, but that seems to be broadly the case at least on average.

The settlement game consists of two main components. First, banks decide whether they want to settle payments in RTGS or would rather pledge collateral ex-ante in order to participate in the CPS system. The choice of the payment system takes place at $t = 0$, before the system opens. If at least one bank determines that participating in CPS is not in its interest, the CPS system fails to open[4] and all banks have to settle their payment obligations in RTGS. Therefore we can limit our attention to the two simple cases in which all banks take part in CPS or RTGS.

The second component of the settlement game is choosing optimal payment timing, once the payment system is chosen. We initially consider the second component (payment game) assuming that one of the payment systems has been chosen, and then proceed to study the choice between the systems. Each bank chooses the timing of the payment to be made within a continuous time interval: $t_i \in [s_i, 1]$, for $i = \{A, B, C\}$, where $0 \leq s_i \leq 1$ is the payment order arrival time for bank i and can be known ex ante or uncertain.

[4]This is a modeling simplification and not a necessary feature of a collateral pool arrangement.

At any point in time a bank may experience an exogenous solvency shock. We assume that an insolvency of a bank becomes public information after the close of the payment system. An insolvent bank cannot send payments while it can receive payments (because the bank is legally still solvent). We assume that the probability that a bank experiences a solvency shock within a given time period follows a Poisson process with the parameter q_i. Therefore a bank i fails within a time period t with probability $D_i(t)$:

$$D_i(t) = 1 - e^{-q_i t}. \qquad \textbf{(1)}$$

3.1 Payment timing strategies

Based on the timing choices made each bank may find itself in 4 different situations:

(i) it becomes the first payer,
(ii) it becomes the second payer after receiving a payment,
(iii) it becomes the third payer after receiving a payment,
(iv) it becomes the second payer without receiving a payment.

Thus each bank has a strategy profile $\{t_1, \Delta_2, \Delta_3, \widetilde{\Delta}_3\}_i$. Here to avoid confusion we drop the subscripts referring to the individual bank. t_1 is the time at which a bank would become the first payer if nobody else has payed. Δ_2 is the time that the second payer delays its payment after receiving a payment from the first payer. Δ_3 is the time that a bank would delay a payment after receiving a payment from the second payer. $\widetilde{\Delta}_3$ is the time a payer waits to pay after observing the first payment, but not receiving any. If banks don't know their payment obligations at the beginning of the day, the delay strategy is how much to wait, given that a payment order is received.

Without loss of generality we denote the first payer as **A** and the recipient of the **A**'s payment as **B**, and the payer to bank **A** as **C**.

We assume that banks can observe the payments made but cannot observe the solvency status of the other banks. Note, that in a three banks case it is equivalent to the banks observing the aggregate collateral pool usage. For tractability we assume that all banks have the same probability of default process during the day, $q_i = q \ \forall i$. The cost structure of making payments depends on the payment system in place, but the structure of the game does not change.

3.2 Cost structure

Similar to Mills Jr. and Nesmith (2008) we abstract from an intrinsic 'delay' cost introduced by Bech and Garratt (2003). Instead, banks take into consideration a possibility of an intra-day default. This turns out to be a significant factor explaining bank payment timing behavior.

In RTGS if a bank becomes the first payer it has to obtain funds at an intra-day cost of R, where R is the per unit cost. As the decision to participate in the collateral pool is made at the beginning of the day in CPS the cost of intra-day liquidity is sunk at the time the payment is made. But to be able to participate in CPS, each bank i has to post collateral of L_i at a unit cost of R. Note that we assume that the intra-day cost of liquidity in RTGS is equivalent to the cost of posting collateral in CPS at the beginning of the day.

If by the end of the day a bank fails to make a payment, it incurs a cost of technical default β. This is introduced to make sure that banks do not end the day without submitting payments voluntarily.

There is a possibility that a bank may end up the day with a negative net payment balance. If the bank is solvent it has to cover the net payment position by borrowing funds overnight at a cost of r. We assume that $\beta > r + R$. That is the cost of not settling a payment is higher than the cost of funding the payment intra-day and/or overnight, which ensures banks have sufficient incentives to settle payments.

In a collateral pool, a situation unique to this particular payment arrangement may arise. If a bank in a net payment position (more payments made than received) becomes insolvent, it effectively has settled a payment at a fraction of the cost. The system operator will seize the collateral of the failed bank, but by construct this collateral will be insufficient to cover the full net payment position of the failed bank. Thus although default is exogenous, given default a bank would prefer to end the day in a net payment position. We refer to the difference between the collateral lost and the payment obligation settled as "default benefit" and denote it by $U_i < 0$:

$$U_i = L_i - 1,$$

where L_i is the amount of collateral pledged to the collateral pool by the failed bank.

In a situation where a bank i fails to pay the net payment balance, a loss sharing arrangement is triggered. Assuming symmetric loss sharing, each surviving bank j experiences a loss of S_j units of collateral:

$$S_j = \frac{1 - L_i}{2}$$

In CPS $S_j > 0$, but in RTGS $U_i = S_j = 0$, since RTGS is a "defaulters pay" system. Instead, $R = 0$ in CPS once the payment system opens, as the cost is sunk. Since we have assumed ex-ante identical banks and we will limit our attention to symmetric equilibria, the postscripts of S and U are abbreviated below.

3.3 Equilibrium

In the following, we find the Bayesian Nash Equilibrium (BNE) in payment timing strategies, assuming that all banks have taken part in RTGS or all of them have joined the CPS. The differences between the payment systems are fully reflected in the parameters discussed above and the structure of the game is identical irrespective of the payment system. This allows us to compare the two payments systems within the same theoretical setup.

The payment timing game has two sub-components. The first one is the choice of the first payer. Liquidity recycling is an important feature in all payment systems. In RTGS, a bank may free-ride on payments received as it can use the cash received to settle its own payment. In deferred net settlement (DNS) systems receiving a payment may relax a receiving bank's net debit cap[5] allowing it to make additional payments. Consequently, the decision to be the first payer is the key choice in all payment systems.

[5]A net debit cap is a common feature of recent DNS systems. It provides an upper bound on member banks' net debit positions to limit the credit exposure of the banks. Once a bank's net debit position hits the upper bound, the bank's payment orders are rejected until the bank receives payments that lowers the net debit position.

The choice to pay first depends on the optimal payment timing of the followers. For instance, if a bank expects that a follower hoards cash (payment inflow) by the end of the day, the first payer is at risk: It may be left with a net payment (debit) position at the closing time and will be forced to obtain costly overnight funding. The second sub-component of the game, $(\Delta^*_B, \Delta^*_C, \widetilde{\Delta}^*_C)_i$, the behavior of the followers, is therefore important as well.

We specify four equilibrium actions $(t^*_A, \Delta^*_B, \Delta^*_C, \widetilde{\Delta}^*_C)_i$ for each $i = \{A, B, C\}$. t^*_A denotes the optimal timing when to become the first payer. A bank with the smallest t^*_A becomes the first payer, if it survives by that time and has received a payment order for settlement. The latter three actions specify the followers' behavior. Δ^*_B denotes the time interval (delay) between its receipt from the first payer (t_A) and the payment timing of the second payer, conditional on its receipt from the first payer and having a payment order ready for settlement. Δ^*_C is the delay of the third payer, conditional on the payments of the first payer and the second payer and have a payment order ready for settlement. These two can be denoted as *receipt-reactive* strategies. $\widetilde{\Delta}^*_C$ denotes a *payment reactive* delay strategy – the delay of a bank after observing the first payer making the payment, but without receiving a payment. For example, if A is the first payer, bank C may pay before bank B submits a payment. In this case bank C would be executing an optimal payment reactive timing strategy. Note that in RTGS payment reactive strategy carries a cost of R . While in CPS banks can make payments to banks with net debit positions without the above mentioned cost, since it does not increase the aggregate net debit position. Effectively, a bank paying to the bank in a net debit position "takes over" the net debit position from the receiving bank.

We limit our attention to pure-strategy, symmetric Bayesian Nash Equilibria. The equilibrium action profile $(t^*_A, \Delta^*_B, \Delta^*_C, \widetilde{\Delta}^*_C)_i$ is chosen at the beginning of the day, and it does not change throughout the day. The available information at the beginning of the day is the risk parameter q, the probability distribution of the arrival timing of a payment request s_i, and the cost parameters (R, β, r, S and U). The cumulative probability distribution of s_i is strictly monotonically increasing. The hidden states of each bank i during a day are the realized value of $s_j \forall j \neq i$ and the solvency of the other banks. While we are interested in symmetric Bayesian Nash Equilibrium, we show, nevertheless, that the equilibria we obtain next are actually Bayesian perfect.

4 Payment timing equilibrium

In this section we derive a BNE action profile in payment strategies for the two payment system. We first derive the equilibrium action of $(\Delta_B^*, \Delta_C^*, \widetilde{\Delta}_C^*)_i$ in both RTGS and CPS, and then proceed to derive $(t_A^*)_i$ in each payment system.

In this section, we limit our attention to the cases of stochastic s_i, the arrival timing of a payment request. The loss functions of the banks thus include the expectation operators, which is defined as follows:

$$E[D_A(\max(s_A, t_A))] = \int_0^1 D_A(\max(s_A, t_A))\psi(s_A)ds_A \quad (2)$$

We assume that $\psi(s_A)$, the probability density function of s_A, is strictly positive for any $s_A \in [0, 1]$, and independent across the banks. We will ignore cases like $s_i = \max(s_j, t_j)$, which is a measure zero event.

4.1 Optimal receipt reactive behavior (Δ_C,Δ_B, $\widetilde{\Delta}_C$)

4.1.1 Optimal Δ_C

Optimal delay strategy Δ_C^* can be derived solving for Δ_C that minimizes the cost of the 3rd payer C, $C_C(\Delta_C)$. We can write the loss function $C_C(\Delta_C)$ conditional on (i) the 1st payer A having survived until t_A and thus having paid to B, (ii) the 2nd payer B having survived until $t_A + \Delta_B$ and having paid to C, and (iii) the third payer C having survived until $t_A + \Delta_B$:

$$\begin{aligned} C_C(\Delta_C; t_A + \Delta_B) &= D_C(\Delta_C)D_A(1 - t_A)(S + \beta) \\ &+ D_C(\Delta_C)\{(1 - D_A(1 - t_A))\}\beta \end{aligned} \quad (3)$$

The first term means that if the third payer C enters insolvency before making the payment (during Δ_C) and if the first payer A defaults by the end of the day, the third payer has to pay the cost of technical default β (as it failed to settle a payment that day) and loses a share of its collateral, S (to cover for the net payment position of the first payer). Even if C fails to make the payment, C does not lose the collateral if A, the net payer, survives by the end of the day and settles the net position (the second term). By differentiating this function, we have the following:

Lemma 1 $\Delta_C^* = \arg\min_{\Delta_C} C_C(\Delta_C, t_A + \Delta_B) = 0$, both in CPS and RTGS.

$\Delta_C^* = 0$ means that if the bank observes the other two banks make a payment, it pays immediately. The intuition is very simple. If the bank waits, there is higher chance for it to fail to make a payment by the end of the day. If it happens, it experiences a cost of technical default and risks losing its collateral in case its recipient fails too by the end of the day. S, the loss sharing rule, is equal to zero in RTGS, and the result is the same.

4.1.2 Optimal Δ_B

We need to define two different loss functions of the second payer B. This is because, once $\Delta_B > \widetilde{\Delta}_C$ i.e. B pays to C after C pays to A, the cost structure of B will be different to the case when B pays to C before C pays to A ($\Delta_B < \widetilde{\Delta}_C$). First, conditional on (i) A having made the first payment at t_A, (ii) B having survived until s_B, and (iii) C not having made the payment to A (i.e. $\max(t_A, s_C) + \widetilde{\Delta}_C > \max(t_A, s_B) + \Delta_B$), in this case, C makes its own payment at $\widetilde{\Delta}_C$ only when B fails to make the payment by Δ_B.

$$\begin{aligned} & C_B^1(\Delta_B; t_A, \widetilde{\Delta}_C) = D_B(\Delta_B)\beta \qquad\qquad (4) \\ + \quad & (1 - D_B(\Delta_B))E[D_C(\max(t_A + \Delta_B, s_C) + \Delta_C)]D_A(1 - t_A)S \\ + \quad & D_B(\Delta_B)E[D_C(\max(t_A, s_C) + \widetilde{\Delta}_C)]D_A(1 - t_A)S \\ + \quad & D_B(\Delta_B)E[1 - D_C(\max(t_A, s_C) + \widetilde{\Delta}_C]E[D_C(1 - \max(t_A, s_C) - \widetilde{\Delta}_C)]S \end{aligned}$$

B has to give up its collateral in two cases. First, when C fails to pay and A enters insolvency after its payment (the second and the third terms of the RHS). Second, when B fails to pay, and when C enters insolvency after making its payment to A (the fourth term).

If condition (iii) is changed to $\max(t_A, s_C) + \widetilde{\Delta}_C < \max(t_A, s_B) + \Delta_B)$, the loss function is:

$$
\begin{aligned}
& C_B^2(\Delta_B; t_A, \widetilde{\Delta}_C) = D_B(\Delta_B)\beta \qquad\qquad \textbf{(5)} \\
+ \quad & E[D_C(\max(t_A, s_C) + \widetilde{\Delta}_C)]D_A(1 - t_A)S \\
+ \quad & D_B(\Delta_B)E[1 - D_C(\max(t_A, s_C) + \widetilde{\Delta}_C)]E[D_C(1 - \max(t_A, s_C) - \widetilde{\Delta}_C)]S
\end{aligned}
$$

Differentiating the loss function w.r.t. Δ_B, we have the following lemma.

Lemma 2 $\Delta_B^* = 0$ for any $\widetilde{\Delta}_C^*$, both in CPS and RTGS.

See the appendix for the proof. $\Delta_B^* = 0$ means that the second payer B makes the payment immediately after receiving the payment.

4.1.3 Optimal $\widetilde{\Delta}_C$

Consider the situation that bank A made its payment at t_A, bank C has survived by t_A, and it has not yet received a payment from bank B. Conditional on this and assuming $\max(t_A, s_C) + \widetilde{\Delta}_C < \max(t_A, s_B) + \Delta_B$, i.e. C makes its payment before B's payment to C, bank C's loss function $\widetilde{C}_C(\widetilde{\Delta}_C; t_A + \Delta_B^*)$ is defined as follows:

$$
\begin{aligned}
& \widetilde{C}_C^1(\widetilde{\Delta}_C; t_A, \Delta_B^*) = (1 - D_C(\widetilde{\Delta}_C))R \qquad\qquad \textbf{(6)} \\
& + D_C(\widetilde{\Delta}_C)\beta \\
& + (1 - D_C(\widetilde{\Delta}_C))E[D_B(\max(t_A, s_B) + \Delta_B)]E[D_C(1 - \max(t_A, s_C) - \widetilde{\Delta}_C)]U \\
& + (1 - D_C(\widetilde{\Delta}_C))E[D_B(\max(t_A, s_B) + \Delta_B)]E[1 - D_C(1 - \max(t_A, s_C) - \widetilde{\Delta}_C)]r \\
& + D_C(\widetilde{\Delta}_C)D_A(1 - t_A)S
\end{aligned}
$$

The third and fourth terms of the RHS are the costs when C ends up being a net payer at the end of the day. The other terms are obvious from the previous sections.

If $\max(t_A, s_C) + \widetilde{\Delta}_C > \max(t_A, s_B) + \Delta_B$, i.e. C's payment is relatively late compared with B, the loss function is:

$$
\begin{aligned}
&\widetilde{C}_C^2(\widetilde{\Delta}_C; t_A, \Delta_B^*) = &\textbf{(7)}\\
&+E[1 - D_B(\max(t_A, s_B) + \Delta_B)]E[1 - D_C(\max(t_A, s_B) + \Delta_B - \max(t_A, s_C))]F_C(\Delta_C; t_A, \Delta_B, s_C)\\
&+E[1 - D_B(\max(t_A, s_B) + \Delta_B)]E[D_C(\max(t_A, s_B) + \Delta_B - \max(t_A, s_C))]\beta\\
&+E[D_B(\max(t_A, s_B) + \Delta_B)]D_C(\widetilde{\Delta}_C)\beta\\
&+E[D_B(\max(t_A, s_B) + \Delta_B)]\{1 - D_C(\widetilde{\Delta}_C)\}E[1 - D_C(1 - \max(t_A, s_C) - \widetilde{\Delta}_C)]r\\
&+E[D_B(\max(t_A, s_B) + \Delta_B)]\{1 - D_C(\widetilde{\Delta}_C)\}E[D_C(1 - \max(t_A, s_C) - \widetilde{\Delta}_C)]U\\
&+E[1 - D_B(\max(t_A, s_B) + \Delta_B)]E[D_C(\max(t_A, s_B) + \Delta_B - \max(t_A, s_C))]D_A(1 - t_A)S\\
&+E[D_B(\max(t_A, s_B) + \Delta_B)]D_C(\widetilde{\Delta}_C)D_A(1 - t_A)S\\
&+E[D_B(\max(t_A, s_B) + \Delta_B)]\{1 - D_C(\widetilde{\Delta}_C)\}R
\end{aligned}
$$

The first term of the function means that, if B successfully pays to C before C pays to A, C's expected loss function becomes equivalent to the equation 3. The second and the third term mean that C fails to make its payment (i) when C has been insolvent at the time B pays to C ($\max(t_A, s_B) + \Delta_B$), or (ii) when B fails to pay to C, and C enters insolvency before $\max(t_A, s_C) + \widetilde{\Delta_C}$. The fourth and fifth terms are nearly equivalent to equation 6. If C fails to make the payment to A and A enters insolvency with a net payment position, C has to share the loss S.

By taking the derivative of these two loss functions, we have the following lemma.

Lemma 3 $\widetilde{\Delta}_C^* = 0$ in CPS. In RTGS, $\widetilde{\Delta}_C^* \in \{0, \max(t_A, s_B) - \max(t_A, s_C)\}$.

See the appendix for the proof. As explained in the proof, $\widetilde{\Delta}_C^*$ is not well specified for RTGS due to a technical difficulty. But $\widetilde{\Delta}_C^*$ is, however, not very important on the equilibrium path of RTGS as we will see below.

4.2 Optimal first payer timing in RTGS

Based on the receipt reactive actions Δ_B, Δ_C and $\widetilde{\Delta}_C$, we find the optimal timing to submit the first payment in RTGS, t_A, t_B and t_C. We only consider the choice of A, t_A, for notational simplicity, which does not lose any generality (i.e. denoting the first payer as A).

t_A is defined as the payment timing that bank A prefers, if possible. The payment is actually made when s_A arrives, i.e. $\max(s_A, t_A)$. In other word, if $t_A = 0$, bank A wants to make the payment as soon as possible. if $t_A = 1$, bank A will make the payment at t=1 irrespective of s_A.

In RTGS the first payer incurs intra-day liquidity cost as $R > 0$. If banks submit payments simultaneously, we assume that only one bank bears the cost of R and others can free-ride on the first payer's cash. This makes the banks' loss function discontinuous in t_A. For given $t_B = t_C$, bank A may find it optimal to submit its payment slightly later than the others, to lower the probability to be the first payer. In fact, we have the following lemma.

Lemma 4 Given that $t_B = t_C$, A's loss function $C_A(t_A)$ is discontinuous at $t_A = t_B = t_C$ in RTGS, and $C_A(t_A = t_B) > C_A(t_A = t_B + \varepsilon)$ for sufficiently small ε.

See the appendix for the proof. Lemma 4 eliminates almost all possible symmetric equilibria, $t_i = [0, 1)$ for any i, because a bank always finds it optimal to slightly delay its payment. We have, therefore, unique symmetric equilibrium candidate, $t_i = 1$ for any i.

Thus all we need to confirm is $t_A = 1$ is the best response to the given actions $t_B = t_C = 1$. The loss function assuming $t_A < t_B = t_C = 1$ is (note that $e^{-q_A \max(s_A, t_A)} = 1 - D(\max(s_A, t_A))$):

$$\begin{aligned} C_A(t_A) \quad = \quad & E\left[e^{-q_A \max(s_A, t_A)}\right] R \\ & + E\left[e^{-q_A \max(s_A, t_A)} \cdot \left(1 - e^{-q_C \max(s_A, t_A, s_C)}\right) \cdot e^{-q_A(1 - \max(s_A, t_A))}\right] r \\ & + E\left[1 - e^{-q_A \max(s_A, t_A)}\right] \beta \end{aligned}$$

It is obvious that $C_A(t_A)$ is strictly increasing against t_A as long as $\beta > R$, showing $C_A(t_A = 1; t_B = t_C = 1) < C_A(t_A = 0; t_B = t_C = 1)$ is sufficient to prove that $t_i = 1$ for any i is the symmetric NBE.

The loss function given of the first payer is obviously continuous in terms of β and r. This is because the probability of the technical default is $1 - e^{-q \max(t_A, s_C)}$, and because the probability of bearing the cost r is independent of whether A is the first payer or not (the probability of r is the probability that A is left being a net payer by the end of the day: so the order of payments does not matter here).

Let's compare the costs to bank A of two possible outcomes: $t_i = 1, \forall i$ and $t_A = 0$, $t_{B,C} = 1$. The cost of managing liquidity to the bank A in those two situations is:

$$
\begin{aligned}
&C_A\left(t_A = 0; t_{B,C} = 1\right) = \\
&E\left[e^{-q_A \cdot s_A}\right] \cdot R + \\
&+E\left[e^{-q_A \cdot s_A} \cdot \left(1 - e^{-q_C \cdot \max(s_A, s_C)}\right) \cdot e^{-q_A \cdot (1 - s_A)}\right] \cdot r \\
&+\left(1 - e^{-q_A \cdot s_A}\right) \cdot \beta
\end{aligned}
\tag{8}
$$

$$
\begin{aligned}
&C_A\left(t_A = 1; t_{B,C} = 1\right) = \\
&e^{-q_A} \cdot e^{-q_B} \cdot e^{-q_C} \cdot \left\{\frac{1}{3} \cdot R + \frac{2}{3} \cdot 0\right\} \\
&+e^{-q_A} \cdot e^{-q_B} \cdot \left(1 - e^{-q_C}\right) \cdot \left\{\frac{1}{2} \cdot (R + r) + \frac{1}{2} \cdot (R + r)\right\} \\
&+e^{-q_A} \cdot \left(1 - e^{-q_B}\right) \cdot e^{-q_C} \cdot \left\{\frac{1}{2} \cdot R + \frac{1}{2} \cdot 0\right\} \\
&+e^{-q_A} \cdot \left(1 - e^{-q_B}\right) \cdot \left(1 - e^{-q_C}\right) \cdot (R + r) \\
&+(1 - e^{-q_A}) \cdot \beta
\end{aligned}
\tag{9}
$$

Comparing the two expressions we can derive:

$$
\begin{aligned}
&C_A\left(t_A=0; t_{B,C}=1\right)-C_A\left(t_A=1; t_{B,C}=1\right)= \\
&E\left[\left(e^{-q_C}-e^{-q_C \cdot \max(s_A,s_C)}\right)\right]\cdot r+ \\
&+\left(e^{-q_A}-e^{-q_A\cdot s_A}\right)\cdot\beta \\
&+\left\{e^{-q_A\cdot s_A}-e^{-q_A}\cdot\left(1-\frac{1}{2}\cdot e^{-q_C}-\frac{1}{6}\cdot e^{-q_B}\cdot e^{-q_C}\right)\right\}\cdot R
\end{aligned} \tag{10}
$$

If $q_i = 0$ for any i, Equation **(10)** is strictly positive and thus $t_i = 1$ is optimal. Since Equation **(10)** is also continuous w.r.t. q_i, there exists a set of $q_i \geq 0$ that makes $t_i = 1$ optimal.

Lemma 5 For any s_i $(i \in \{A, B, C\})$, r, β and R, $\exists \hat{q}$ s.t. for $\forall \max[q_{A,}q_{B,}q_C] \leq \hat{q}$, $t_i^* = 1$ in RTGS.

Since $t_i^* = 1$ holds for any $s_i \in [0, 1]$ as long as $\max[q_{A,}q_{B,}q_C] \leq \hat{q}$, the proposition above holds for both stochastic and deterministic payment arrival cases.

4.3 Uniformly distributed payment order arrivals

If $s_i \overset{\text{iid}}{\sim} U[0, 1]$, then it can be shown that:

$$
\begin{aligned}
&E[C_A\left(t_A=0; t_{B,C}=1\right)-C_A\left(t_A=1; t_{B,C}=1\right)]= \\
&\frac{e^{-3d}\left(3de^dR+dR+6(d+1)e^{2d}(r-R+\beta)-6e^{3d}(r-R+\beta)\right)}{6d}
\end{aligned} \tag{11}
$$

which is strictly positive, if

$$
R > \frac{6e^{2d}\left(-d+e^d-1\right)(r+\beta)}{3e^d d+d-6(d+1)e^{2d}+6e^{3d}} \tag{12}
$$

LEMMA HERE?????

Intuitively, if the intra-day cost of default is significant, banks will delay their payment until the end of the day. The threshold value of the intra-day cost of liquidity is given in Equation **(12)**.

4.4 Optimal first payer timing in CPS

In CPS, we find the optimal t_A based on $R = 0$, $S > 0$ and $U < 0$. Since the strategy shifts to Δ_B, Δ_C and $\widetilde{\Delta}_C$ immediately after a bank submits the payment order, t_A, the choice of t_A is made conditional on (i) no other payment having been made, and (ii) the payment request having arrived. We limit our attention on the symmetric equilibria.

t_A is defined as the payment timing that bank A prefers, if possible. The payment is actually made when s_A arrives, i.e. $\max(s_A, t_A)$. In other word, if $t_A = 0$, bank A wants to make the payment as soon as possible. if $t_A = 1$, bank A will make the payment at t=1 irrespective of s_A.

In CPS, we cannot regard the deterministic arrival timing of a payment request as a special case of the stochastic arrival case for the choice t_A. We will solve the stochastic arrival case first, and then show that the deterministic case has the same equilibrium.

4.4.1 Stochastic arrival cases

The loss function when payment orders arrive from their customers at a random timing is defined as follows:

$$\begin{aligned}
C(t_A) &= E[\{1 - D_A(max(s_A, t_A))\}D_A(1 - max(s_A, t_A))D_C(max(s_C, s_A, t_A))]U \quad \textbf{(13)} \\
&+ E[\{1 - D_B(max(s_B, s_A, t_A))\}D_B(1 - max(s_B, s_A, t_A))D_A(max(s_A, t_A))]S \\
&+ E[\{1 - D_C(max(s_C, s_A, t_A))\}D_C(1 - max(s_C, s_A, t_A))D_B(max(s_B, s_A, t_A))]S \\
&+ E[\{1 - D_A(-q_A)\}D_C(max(s_C, s_A, t_A))]r \\
&+ E[D_C(max(s_A, t_A))]\beta
\end{aligned}$$

This assumes that s_B and s_C have the same distribution, which is independent across banks, and that s_A is known for bank A. We also assume q_i is equal across banks and $U = -2S$. We need to consider two cases: $t_A \geq E[s_i]$ and $t_A \leq E[s_i]$. By taking the derivative of the loss function for the two cases, we have the following proposition.

Lemma 6 There exists a unique equilibrium action $t_A^* = 0$ in CPS, if s_i is stochastic and has an identical and independent distribution.

See the appendix for the proof.

4.4.2 Deterministic arrival cases

If the banks know s_A,s_B and s_C ex-ante, we can no longer keep the assumption $E[s_B] = E[s_C]$, and have to consider two cases $s_B > s_C$ and $s_B < s_C$ separately(if $s_B = s_C$, the result is the same as above). Assume further that $t_B = t_C = 0$, i.e. B and C want to be the first payer as soon as possible, to see whether $t_A = 0$ is the best response to the actions.

First, If $t_A > \max(s_B, s_C)$, t_A matters only when B and C fail to make their payments. The relevant part of the loss function is therefore,

$$\begin{aligned} = \quad & E[D_C(s_C)D_C(s_B)\{1 - D_A(\max(s_A, t_A))\}D_A(1 - \max(s_A, t_A))]U \qquad \mathbf{(14)} \\ + \quad & E[D_C(s_C)D_C(s_B)\{1 - D_A(1)\}r \\ + \quad & E[D_C(s_C)D_C(s_B)D_A(\max(s_A, t_A))]\beta \end{aligned}$$

It is strictly increasing in t_A, as long as $E[s_i] > 0$.

Second, if $s_C < t_A < s_B$, t_A matters only when C fails to make its payment. The relevant part of the loss function, which is relevant to t_A, is therefore,

$$
\begin{aligned}
&= E[D_C(s_C)\{1 - D_A(\max(s_A, t_A))\}D_A(1 - \max(s_A, t_A))]U \qquad \textbf{(15)}\\
&+ E[D_C(s_C)\{1 - D_A(1)\}]r\\
&+ E[D_C(s_C)D_A(\max(s_A, t_A))]\beta
\end{aligned}
$$

It is obviously strictly increasing in t_A.

Third, if $s_B < t_A < s_C$, t_A matters only when B fails to make its payment. The relevant part of the loss function is therefore,

$$
\begin{aligned}
&= E[D_B(s_B)\{1 - D_A(\max(s_A, t_A))\}D_C(s_C)D_A(1 - \max(s_A, t_A))]U \qquad \textbf{(16)}\\
&+ E[D_B(s_B)\{1 - D_A(1)\}D_C(s_C)]r\\
&+ E[D_B(s_B)\{1 - D_C(s_C)\}D_C(1 - s_C)]S\\
&+ E[D_B(s_B)D_A(\max(s_A, t_A))]\beta
\end{aligned}
$$

It is also strictly increasing in t_A.

Lastly, if $t_A < \max(s_B, s_C)$,

$$
\begin{aligned}
&= E[\{1 - D_A(\max(s_A, t_A))\}D_A(1 - \max(s_A, t_A))D_C(s_C)]U \qquad \textbf{(17)}\\
&+ E[\{1 - D_B(s_B)\}D_B(1 - s_B)D_A(1 - \max(s_A, t_A))]S\\
&+ E[\{1 - D_C(s_C)\}D_C(1 - s_C)D_B(s_B)]S\\
&+ E[\{1 - D_A(1)\}D_C(s_C)]r\\
&+ E[D_A(\max(s_A, t_A))]\beta
\end{aligned}
$$

Again, it is obviously increasing in t_A. This completes the proof of the following lemma.

Lemma 7 If the arrival timing of payment requests s_i are deterministic variables and common knowledge ex-ante, there is an equilibrium $t_A^* = 0$ in CPS.

4.5 Payment timing equilibria in RTGS and CPS

From lemmas 1- 3, 6 and 7 now we complete proving the following propositions.

Proposition 8 The BNE action profile of CPS is $t_A^* = 0, \Delta_B^* = 0, \Delta_C^* = 0, \widetilde{\Delta}_C^* = 0$ for any bank i, irrespective of whether payment requests arrive randomly or not.

That is to say, in CPS, the banks make payments as soon as possible, for any risk parameter q and cost parameters.

Likewise, from the lemmas 1- 3 and 5, we have the following proposition.

Proposition 9 The BNE action profile of RTGS is $t_A^* = 1, \Delta_B^* = 0, \Delta_C^* = 0, \widetilde{\Delta}_C^* = 0$ for any bank i, irrespective of whether payment requests arrive randomly or not, if the risk parameter q is smaller than $\hat{q}$.

In RTGS, no bank wants to be the first payer, in order to avoid the cost of intra-day liquidity R. But once a bank makes its payment, the other banks make their payments as soon as possible, using the cash received from the first payer.

Although we use Bayesian Nash equilibrium to describe the banks' behavior, the BNE we obtained here is actually Bayesian perfect. The hidden states of the game are, in the case of the stochastic version of the model, (i) whether a bank has already received a payment request or not and (ii) whether a bank has been stricken by a failure (and unable to make any payments) or not. If it is deterministic, only the latter is the hidden state. The observable signal of the game is a bank's payment, and it is possible to infer the hidden information from the actions made. But the first half of the payment game, the choice of the first payer t_i, is terminated immediately after the first payment is made. The banks thus cannot condition their choices of t_i based on the other banks' payment behavior observed. Namely there is only one information set in the first half of the game, and thus sequential rationality and Bayesian updates, the main conditions of Bayesian Nash equilibrium, are trivially satisfied.

The signal obtained in the first half of the game, the timing and the identity of the first payer, can be used in the second half of the game (the choice of Δ_B, Δ_C and $\widetilde{\Delta_C}$). The identity of the first payer creates three information sets (being the second payer, being the third payer, or being the second payer before receiving a payment), and the corresponding optimal behavior is chosen at each one of the sets (Δ_B, Δ_C and $\widetilde{\Delta_C}$ respectively). We have seen that another signal, the payment timing of the first payer, does not affect to the choice of the optimal actions. Again, sequential rationality and Bayesian updates are satisfied, although it is not explicitly shown. The Bayesian perfectness holds for both stochastic and deterministic cases, and for both CPS and RTGS. C.f. Fudenberg and Tirole (1986).

5 Participation

In this section we investigate conditions under which banks are willing to participate in collateral pool. We assume that a system operator cannot discriminate between the banks and sets equal terms for all banks. Note, that as we discussed before, CPS system fails to operate if at least one bank does not find it beneficial to join.

When considering participating in CPS, banks take into consideration the direct costs of participation (cost of collateral pledged) and the indirect costs that are the result of the payment timing game played. The way the indirect costs enter the cost function of an individual bank depends crucially on the payment order arrival time s_i. Intuitively, the bank that receives payment orders later in the day may find it less attractive to participate in a collateral pool as it would not be able to use it until the payment order is received. In the meantime its counterparties may settle their payments against the liquidity available in the pool exposing the late payer to the default probability of one of the early payers.

We assume that the system operator required banks willing to participate in a collateral pool to pledge an equal amount of collateral. Clearly, this does not have to be the case. Especially if some banks have a higher probability of default or differ from other banks in some other way (ie receive payment orders at different time of the day). Therefore the condition that we derive next is a sufficient condition for participating in CPS but clearly not a necessary one.

It can be shown that the cost of participating in CPS and RTGS is, respectively:

$$C_{RTGS} = D_a(1)\beta + (1 - D_a(1))D_c(1)r + (1 - D_a(1))\left(1 - \frac{1}{2}D_c(1) - \frac{1}{6}D_b(1)D_c(1)\right)R$$

$$C_{CPS} = \left\{\begin{array}{r} D_a(s_a)\cdot\beta \\ +[(1 - D_a(s_a))D_a(1 - s_a)]D_c(s_c)\cdot -U \\ +[(1 - D_c(s_c))D_c(1 - s_c)]D_b(s_b)\cdot S_c \\ +(1 - D_b(s_b))D_b(1 - s_b)D_a(s_a)\cdot S_b \\ +(1 - D_a(1))D_c(s_c)\cdot r \end{array}\right\} + L\cdot R$$

where U is the 'default benefit', and S_b, S_c denote loss sharing costs due to the failure of banks b, c respectively. If we maintain symmetry in collateral postings, $U = 2/3$ (unit payment settled and 1/3 units of collateral lost), while $S_b = S_c = 1/3$.

Proposition 10 $q < \overline{q}$ with $\overline{q} = \ln\left[\frac{1}{2}(2 + \sqrt{6})\right]$ is a sufficient condition for $C_{CPS} < C_{RTGS}$, if $s_i = 0$.

Proof. First let's investigate the case when all banks receive their payment orders at the beginning of the day, $s_i = 0$. The cost of participating in RTGS is:

$$C_{RTGS}(s = 0) = e^{-q}\left(1 - e^{-q}\right)r + e^{-q}\left(1 - \frac{e^{-2q}}{6} - \frac{e^{-q}}{2}\right)R + \left(1 - e^{-q}\right)\beta$$

while the cost of participating in CPS is:

$$C_{CPS}(s = 0) = \frac{1}{3}R$$

It is clear that if the coefficient in front of R in C_{RTGS} is less than 1/3 a bank would be better off joining the collateral pool than settling payments in RTGS. Therefore we can establish an upper bound on parameter q in the probability of default function specified in Equation **(1)**:

$$
\begin{aligned}
& e^{-q}\left(1-\frac{e^{-2q}}{6}-\frac{e^{-q}}{2}\right)<1/3 \\
& \Rightarrow \\
& q<\ln\left[\frac{1}{2}(2+\sqrt{6})\right]
\end{aligned}
$$

If the default probability is sufficiently small, the direct and indirect costs of collateral (loss sharing and default benefit) are smaller in CPS than in RTGS. If $q > \overline{q}$ the collateral costs in CPS are larger than in RTGS, but a trade-off between higher collateral costs and lower expected costs of overnight liquidity and technical default arise. If the latter dominate, CPS may be still preferred. Another interpretation is that if the cost of overnight liquidity increases, CPS is more likely to be preferred. The same applies to the cost of technical default.

Proposition 11 $q < \overline{q}$ with $\overline{q} = \ln\left[\frac{1}{2}(2+\sqrt{6})\right]$ is a sufficient and necessary condition for $C_{CPS} < C_{RTGS}$, if $s_i = 1$.

Proof. If all banks receive their payment orders at the end of the day C_{CPS} and C_{RTGS} are identical in terms of r and β. The only difference is in terms of the collateral cost R:

$$
\begin{aligned}
C_{RTGS}(s_i=1) &= e^{-q}\left(1-e^{-q}\right)r+\left(1-e^{-q}\right)\beta+e^{-q}\left(1-\frac{e^{-2q}}{6}-\frac{e^{-q}}{2}\right)R \\
C_{CPS}(s_i=1) &= e^{-q}\left(1-e^{-q}\right)r+\left(1-e^{-q}\right)\beta+\frac{R}{3}
\end{aligned}
$$

Thus $e^{-q}\left(1-\frac{e^{-2q}}{6}-\frac{e^{-q}}{2}\right)<1/3$ or $q<\ln\left[\frac{1}{2}(2+\sqrt{6})\right]$ is a necessary condition for $C_{CPS} < C_{RTGS}$.

5.1 Uncertain payment order arrival timing

If banks don't know the payment order arrival time, then their participation decision is based on comparing the expected costs of participating in either system. Let's assume that the payment order arrival time is independent across banks and uniformly distributed over a unit interval: $s_i \overset{\text{iid}}{\sim} U[0,1]$.

The expected cost of participating in RTGS is identical to the deterministic case of $s_i = s$ as, regardless of the payment order arrival time, banks choose to delay their payments until the end of the day. Therefore:

$$E[C_{RTGS}] = e^{-q}\left(1-e^{-q}\right)r + e^{-q}\left(1 - \frac{e^{-2q}}{6} - \frac{e^{-q}}{2}\right)R + \left(1-e^{-q}\right)\beta$$

The expected cost of participating in CPS for bank A (expected costs are symmetric across banks) is:

$$\begin{aligned} E[C^a_{CPS}] &= \iiint_{0\le s_a,s_b,s_c\le 1} C_{CPS}(s_a,s_b,s_c)ds_a\,ds_b\,ds_c \\ &= \frac{e^{-2q}\left(3r + 3e^q((q-1)r+\beta) + e^{2q}(qR + 3(q-1)\beta)\right)}{3q} \end{aligned}$$

Proposition 12 CPS is preferred to RTGS if $q < \overline{q}$ and payment order arrival time $s_i \overset{\text{iid}}{\sim} U[0,1]$. If $q > \overline{q}$, $R < \overline{R}$ is sufficient to ensure CPS is still preferred.
$\overline{q} = \ln\left[\frac{1}{2}(2+\sqrt{6})\right]$ and $\overline{R} = \frac{6e^q(-q+e^q-1)(r+e^q\beta)}{q(1+3e^q-6e^{2q}+2e^{3q})}$.

Proof. Let's define

$$G = E[C_{CPS}] - E[C_{RTGS}].$$

It can be shown that

$$\frac{\partial G}{\partial R} = \frac{1}{6}e^{-3q}\left(e^q\left(2e^q\left(-3+e^q\right)+3\right)+1\right)$$

with $\partial G/\partial R > 0$ if $q > \overline{q}$ and $\partial G/\partial R < 0$ if $q < \overline{q}$.

Lets find the threshold value $\overline{R}$ for which $G(R) = 0$:

$$\overline{R} = \frac{6e^q\left(-q+e^q-1\right)\left(r+e^q\beta\right)}{q\left(1+3e^q-6e^{2q}+2e^{3q}\right)}$$

The numerator of the expression above is always positive, while the sign of the denominator depends on q. It can be shown that the denominator is negative only for $0 < q < \overline{q}$, so that $\overline{R} < 0$ if $q < \overline{q}$, while $\overline{R} > 0$ if $q > \overline{q}$. Since $R > 0$, it is sufficient to establish that $E[C_{CPS}] < E[C_{RTGS}]$ if $q < \overline{q}$. Moreover, $E[C_{CPS}] < E[C_{RTGS}]$ if $q > \overline{q}$ and $R < \overline{R}$.

If banks are not certain about the payment order arrival time their willingness to participate in CPS depends on the cost of intra-day liquidity and probability of bank failure. If the probability of a bank failure is not too large, or, if it is, if the intra-day cost of liquidity is small, then banks are better off joining the collateral pool than settling payments in RTGS.

6 Discussion

The collateral pool provides an incentive for early payment submission. While everybody paying early is a socially desirable equilibrium, banks in RTGS would rather pay late since they expect the other banks to pay late too. By collecting collateral before a payment system opens, the payment system can incentivise the member banks to pay early. Two different mechanisms work here. First, the cost of liquidity becomes sunk and thus the banks do not need to wait for the other banks' payment inflow to recycle the cash. Second, by making an early payment, the bank can minimize the risk of loss-sharing. The incentive mechanism comes at a cost - there is a risk that due to some other reasons (that are exogenous and unrelated to strategic payment timing) a counterparty will default intra-day putting the surviving collateral pool participants' collateral at risk.

Intra-day default matters not only for banks participating in CPS. Although in RTGS banks have no reason to worry about loss sharing, they take into consideration a possibility that they will not be able to settle payments themselves, or will have to obtain overnight funding after failing to receive a payment.

Clearly the key decision for a bank is to weigh the costs and benefits of CPS in comparison to RTGS and decide if it is beneficial to participate in a collateral pool. We show that if intra-day liquidity costs are significant, banks in RTGS will settle their payments late in the day. The higher is the possibility of an intra-day default, the lower is the threshold, $\bar{q}$, of intra-day liquidity costs for banks to pay late.

In CPS banks will settle their payments as early as possible. If, on the one hand, banks have received all of their payment requests at the beginning of they day, in equilibrium all payments are settled at the beginning of the day, and thus there is no risk of being exposed to loss sharing. If, on the other hand, banks do not know at which point of the day a payment obligation will arise, there is a chance that other participants will be able to utilize the collateral pool before a bank even receives a payment order. We show that even faced with such uncertainty banks prefer to participate in a collateral pool, if the probability of bank's failure is not too high.

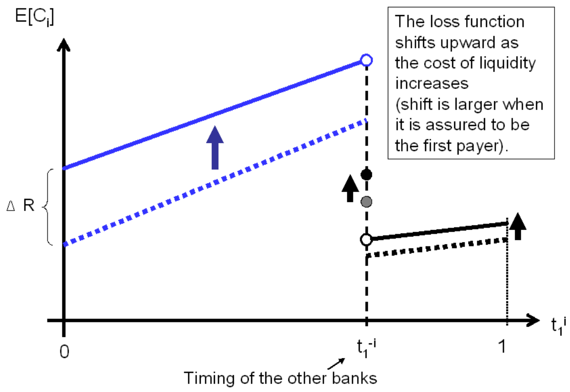

Chart 2: Equilibrium payment timing in RTGS: change in R

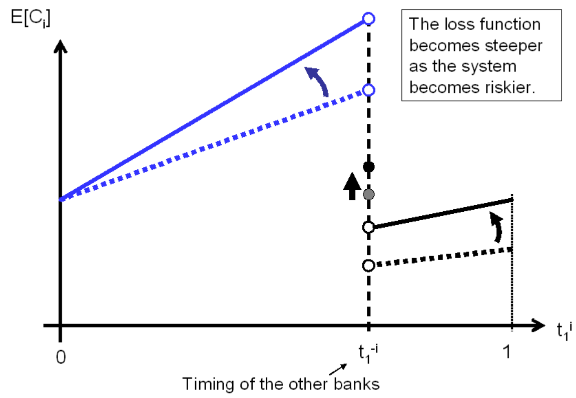

Chart 3: Equilibrium payment timing in RTGS: change in q

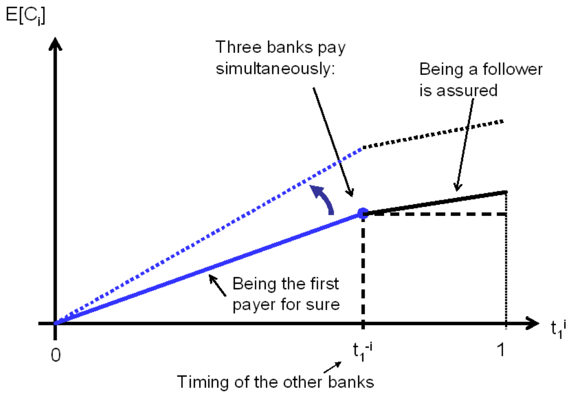

Chart 4: Equilibrium payment timing in CPS

In RTGS, as shown in Chart 2, the loss function of the first payer (solid line), given the strategies of the other banks, t_1^{-i}, is discontinuous in t_1. In the first segment of the cost function the bank will be the first payer (unless it enters insolvency before payment), and thus the cost of intra-day liquidity R is almost[6] certain. In the last segment of the cost function, the bank becomes the follower (unless the other banks fail to make the payment) and is unlikely to incur the intra-day liquidity cost, R. At the discontinuity all banks are equally likely to be the first payers. Clearly, as can be seen from the Chart, every bank has an incentive to delay its payment slightly later than the others. This is the reason why the symmetric equilibrium is $t_1^i = 1$. If the expected cost for bank i is smaller at $t_1^i = 1$ than at $t_1^i = 0$, given that $t_1^{-i} = 1$, $t_1^i = 1$ is also a unique equilibrium.

As the intra-day liquidity cost (R) increases, the condition can be satisfied easily. Chart 2 (dashed line) shows how the loss function changes when the intra-day liquidity cost R increases by ΔR. The expected loss at $t_1^i = 0$ shifts upward by the same amount ΔR. The loss function in the region $t_1 < t_1^{-i}$ shifts upward. [7] The cost in the region $t_1^i \geq t_1^{-i}$ also increases, but the change is smaller than ΔR, because banks may be able to free-ride on a payment inflow from another bank. To summarize, a late payment equilibrium is more likely as R increases.

[6]It could fail before making a payment

[7]To be precise, the loss function becomes slightly flatter, because the marginal benefit of “not making a payment due to intra-day failure” decreases as R increases.

If the risk indicator q increases, the loss function becomes steeper as Chart 3 (dashed line) shows. This is because settlement delay is more likely to cause the bank's own payment failure (cost of technical default β) and because the bank is more likely to be left with the net payment position at the end of the day, which costs r. The expected loss in the region of $t_1^i \geq t_1^{-i}$ may also increase, at least when the level of q is low.[8] Since the loss at $t_1 = 0$ is unchanged for any q,the condition to have a symmetric equilibrium is likely to be violated if q is very large.

If the condition is violated, we may have an asymmetric equilibrium that one bank pays as soon as possible while others, although with $t_1^{-i} = 1$, follow immediately. Thus an early payment equilibrium can arise when the system is exposed to high risk (large q) or lower intra-day cost of liquidity R. This outcome is opposite to Mills Jr. and Nesmith (2008).

We do not discuss the asymmetric equilibrium in detail, since it is less robust. Bank i has an incentive to pay at $t_1^i = 0$ because the bank believes that other banks would make their first payment at $t_1^{-i} = 1$. But if the bank i deviates from $t_1^i = 0$ and pays later in the day, other banks will find it optimal to pay early instead. The asymmetric equilibrium is thus supported by an empty threat – "we will delay, thus if you do not pay early, all payments are delayed until the last minute".

In CPS, as shown in Chart 4, the loss function of the first payer is continuous (as $R = 0$), thus it is clear that the symmetric Bayesian Nash Equilibrium is $t_1^i = 0$ for all i. The equilibrium is unchanged even if q increases, as shown by the dashed line.

Another interesting feature of a collateral pool is that a failure of an institution does not necessarily create a liquidity sink (a situation when all remaining participants do not have sufficient liquidity to send payments). In CPS banks can keep on settling payments as institutions in net debit positions would be still able to receive payments from the other banks. Effectively, by paying a bank in a net debit position, an intra-day credit is granted to the payer and at the same time extinguished for the payee. In aggregate, the total amount of intra-day credit granted does not change.

[8] The sign of the first order condition can be positive and negative, depending on the parameters.

In this paper we compare collateral pool arrangement against plain vanilla RTGS. In practice, many RTGS systems have facilities to net offsetting payments (otherwise referred to as liquidity saving mechanisms). But liquidity saving mechanisms work best if payment orders are submitted for settlement early as more offsetting opportunities can be identified. Therefore, since CPS incentivises early payments it may help to make liquidity saving mechanisms more efficient. Even in the absence of LSMs, the collateral pool arrangement has an advantage as liquidity savings can be achieved without any delay, while payments submitted to the offsetting facility may have to wait for an offsetting possibility to arise.

7 Summary

In this paper we investigate collateral pool settlement (CPS) payment systems. We develop a theoretical model that we use to compare CPS and Real Time Gross Settlement (RTGS) payment systems in terms of optimal bank behavior and costs. Using a unified analytical framework we show that, unlike in an RTGS system, the CPS system provides incentives for banks to submit payments early.

Under quite general circumstances payment system participants are willing to participate in a collateral pool. We show that even if banks are not certain about the time at which their payment order arrive, they prefer to participate in a collateral pool if the probability of a bank failure intra-day is not too large and the cost of intra-day liquidity is sufficiently high. We derive the explicit threshold values.

In our model CPS also provides an additional way for banks to resolve a liquidity sink situation. While in RTGS banks may run out of liquidity to make payments if some participant becomes a liquidity sink, some payments in CPS can still be made. Specifically, a bank with a negative net payment balance can continue to receive payments from the other participants as it does not increase the aggregate net payment balance.

This paper also contributes to the theoretical payments literature by providing an alternative to the usual trade-off between paying early and nonpecuniary ‘delay-cost’. In our model delay costs arise explicitly due to the probability of a bank failure intra-day. We show that a late payment equilibrium arises in RTGS if intra-day liquidity is costly enough. The late payment equilibrium becomes less likely if a probability of a bank failure increases. Therefore we confirm the result in the literature that an RTGS payment system may result in a late payment equilibrium, but we show that it may not be necessarily the case if the intra-day cost of liquidity is very low.

In this paper we assume that payments are perfectly offsetting ex-ante. It would be interesting to extend the analysis to cases where payments are not perfectly offsetting and to allow for banks to operate in an environment of parallel streams of CPS and RTGS. The model could also be extended by making payment flows to failing institutions endogenous.

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Appendix

In the following proofs, we do not use the notation $D_i(t)$, but use $1 - e^{-q_i t}$ for notational clarity.

Proof of Lemma 2

The first order condition of the loss function given in Equation **(4)** is:

$$\begin{aligned}
& \frac{\partial C_B^1(\Delta_B; t_A, \widetilde{\Delta}_C)}{\partial \Delta_B} = q_B e^{-q_B \Delta_B} \beta \\
+ \quad & (-q_B) e^{-q_B \Delta_B} E[1 - e^{-q_C(\max(t_A+\Delta_B, s_C)+\Delta_C)}](1 - e^{-q_A(1-t_A)})S \\
+ \quad & e^{-q_B \Delta_B} \frac{\partial E[1 - e^{-q_C(\max(t_A+\Delta_B, s_C)+\Delta_C)}]}{\partial \Delta_B}(1 - e^{-q_A(1-t_A)})S \\
+ \quad & q_B e^{-q_B \Delta_B} E[1 - e^{-q_C(\max(t_A, s_C)+\widetilde{\Delta}_C)}](1 - e^{-q_A(1-t_A)})S \\
+ \quad & q_B e^{-q_B \Delta_B} E[1 - e^{-q_C(\max(t_A, s_C)+\widetilde{\Delta}_C)}](1 - e^{-q_C(1-\max(t_A, s_C)-\widetilde{\Delta}_C)})S
\end{aligned}$$

Since $\max(t_A + \Delta_B, s_C) + \Delta_C < \max(t_A, s_C) - \widetilde{\Delta}_C$ from the assumption, the sum of the second term and the fourth term is strictly positive. The other terms are obviously positive, so the FOC is strictly positive.

The first order condition of the loss function given in Equation **(5)** is:

$$\begin{aligned}
& \frac{\partial C_B^2(\Delta_B; t_A, \widetilde{\Delta}_C)}{\partial \Delta_B} = q_B e^{-q_B \Delta_B} \beta \\
+ \quad & q_B e^{-q_B \Delta_B} E[e^{-q_C(\max(t_A, s_C)+\widetilde{\Delta}_C)}] E[1 - e^{-q_C(1-\max(t_A, s_C)-\widetilde{\Delta}_C)}]S
\end{aligned}$$

Since in any cases Δ_B is optimal to be as small as possible, by the definition of Δ_B, $\Delta_B^* = 0$.

Proof of Lemma 3

The first order derivative of the loss function given in Equation **(6)** is:

$$\begin{aligned}
\frac{\partial \widetilde{C}^1_C(\widetilde{\Delta}_C)}{\partial \widetilde{\Delta}_C} &= (-q_C)e^{-q_C\widetilde{\Delta}_C}R \\
&+ q_C e^{-q_C\widetilde{\Delta}_C}\beta \\
&- q_C e^{-q_C\widetilde{\Delta}_C}E[1-e^{-q_B(\max(t_A,s_B)+\Delta_B)}]U \\
&+ q_C e^{-q_C\widetilde{\Delta}_C}1-e^{-q_A(1-t_A)}S
\end{aligned}$$

In CPS, where $R=0$, $U<0$ and $S>0$, $\frac{\partial \widetilde{C}_C(\widetilde{\Delta}_C)}{\partial \widetilde{\Delta}_C}>0$. In RTGS, where $\beta>R>0$, $U=0$ and $S=0$, $\frac{\partial \widetilde{C}_C(\widetilde{\Delta}_C)}{\partial \widetilde{\Delta}_C}>0$ as well.

The first order derivative of the loss function given in Equation **(7)** is:

$$\begin{aligned}
\frac{\partial \widetilde{C}^2_C(\widetilde{\Delta}_C)}{\partial \widetilde{\Delta}_C} &= -q_C E[1-e^{-q_B(\max(t_A,s_B)+\Delta_B)}]e^{-q_C\widetilde{\Delta}_C}(R+U) \qquad \textbf{(A-1)} \\
+ \quad & q_C E[1-e^{-q_B(\max(t_A,s_B)+\Delta_B)}]e^{-q_C\widetilde{\Delta}_C}\beta \\
+ \quad & q_C E[1-e^{-q_B(\max(t_A,s_B)+\Delta_B)}]e^{-q_C\widetilde{\Delta}_C}1-e^{-q_A(1-t_A)}S
\end{aligned}$$

In CPS, $\frac{\partial \widetilde{C}^2_C(\widetilde{\Delta}_C)}{\partial \widetilde{\Delta}_C}>0$ since $R+U<0$. Thus, $\widetilde{\Delta}_C$ takes its possible minimum, $\max(t_A,s_B)+\Delta_B-\max(t_A,s_C)$.

$\widetilde{C}^1_C \le \widetilde{C}^2_C$ for any possible s_B and s_C, because the third and fourth terms of $\widetilde{C}^1_C$ and the fourth and fifth terms of $\widetilde{C}^2_C$ are equal, and the second and fifth terms of C_{C^1} are equal to zero when $\widetilde{\Delta}_C=0$, and $R=0$. The optimal $\widetilde{\Delta}_C$ is, therefore, $\widetilde{\Delta}^*_C=0$ in CPS.

In RTGS, however, $\widetilde{C}^1_C$ and $\widetilde{C}^2_C$ are hard to compare and thus difficult to find the global minimum. $\widetilde{\Delta}^*_C$ can take, therefore, two possible optimum 0 and

$\max(t_A, s_B) + \Delta_B^* - \max(t_A, s_C) = \max(t_A, s_B) - \max(t_A, s_C)$.

Since $\widetilde{\Delta}_C^*$ is not very important on the equilibrium path of RTGS as we will see below,

Proof of Lemma 4

The probability of paying R, when $t_A = t_B = t_C$ is:

$$\begin{aligned}
\Pr[s_A &< t_A]\Pr[s_B < t_B]\Pr[s_C < t_C]e^{-qt_A}\left(1 - \frac{1}{2}e^{-qt_C} - \frac{1}{6}e^{-qt_B}e^{-qt_C}\right) \\
+\Pr[s_A &< t_A]\Pr[s_B < t_B]\Pr[s_C > t_C]e^{-qt_A} \\
+\Pr[s_A &< t_A]\Pr[s_B > t_B]\Pr[s_C < t_C]e^{-qt_A}\left\{e^{-qt_B}\cdot\frac{1}{2} + (1 - e^{-qt_B})\cdot 1\right\} \\
+\Pr[s_A &< t_A]\Pr[s_B > t_B]\Pr[s_C > t_C]e^{-qt_A} \\
+\Pr[s_A &> t_A]\cdot X
\end{aligned}$$

X is the probability that A becomes the first payer, conditional on A receives a payment request after t_A. We do not formulate X, because the infinitesimal delay ε from t_A does not change the probability X as long as $s_A > t_A$.

The probability of paying R, when $t_A = t_B + \varepsilon = t_C + \varepsilon$ and ε approaches zero, is:

$$\begin{aligned}
\Pr[s_A &< t_A]\Pr[s_B < t_B]\Pr[s_C < t_C]e^{-qt_A}(1 - e^{-qt_B})(1 - e^{-qt_C}) \\
+\Pr[s_A &< t_A]\Pr[s_B < t_B]\Pr[s_C > t_C]e^{-qt_A}(1 - e^{-qt_B}) \\
+\Pr[s_A &< t_A]\Pr[s_B > t_B]\Pr[s_C < t_C]e^{-qt_A}(1 - e^{-qt_C}) \\
+\Pr[s_A &< t_A]\Pr[s_B > t_B]\Pr[s_C > t_C]e^{-qt_A} \\
+\Pr[s_A &> t_A]\cdot X
\end{aligned}$$

Since $1 - \frac{1}{2}e^{-qt_C} - \frac{1}{6}e^{-qt_B}e^{-qt_C} > (1 - e^{-qt_B})(1 - e^{-qt_C})$, we have
$C_A(t_A = t_B) > C_A(t_A = t_B + \varepsilon)$ if $\varepsilon \to 0$. This is sufficient to prove the discontinuity.

Proof of Lemma 6

The first order derivative of the loss function when $t_A \geq E[s_i]$ is:

$$
\begin{aligned}
& \frac{\partial C(t_A)}{\partial t_A} = \{2qe^{-2q\max(s_A,t_A)} - qe^{-q\max(s_A,t_A)} - qe^{-(q+q\max(s_A,t_A))}\}U \\
+ \quad & \{2qe^{-2q\max(s_A,t_A)} - qe^{-q\max(s_A,t_A)} - qe^{-(q+q\max(s_A,t_A))}\}S \\
+ \quad & \{2qe^{-2q\max(s_A,t_A)} - qe^{-q\max(s_A,t_A)} - qe^{-(q+q\max(s_A,t_A))}\}S \\
+ \quad & qe^{-(q+q\max(s_A,t_A))}r \\
+ \quad & qe^{-q\max(s_A,t_A)}\beta
\end{aligned}
$$

The derivative is always strictly positive if $t_A \geq s_A$. The first order derivative of the loss function when $t_A \leq E[s_i]$ is:

$$
\begin{aligned}
\frac{\partial C(t_A)}{\partial t_A} &= E[-qe^{-qt_A}(1-e^{-qs_C})]U \\
&+ E[qe^{-qt_A}(e^{-qs_B}-e^{-q})]S \\
&+ qe^{-(q+qt_A)}r \\
&+ qe^{-qt_A}\beta
\end{aligned}
$$

For any $s_B \in [0,1]$ and $s_C \in [0,1]$, the derivative is also strictly positive if $t_A \geq s_A$. Since the bank tries to minimise t_A under the condition of $t_A = s_A$, $t_A = s_A$ for any $s_A \in [0,1]$. By definition, it means that $t_A^* = 0$.