Measuring the Natural Rate of Interest After COVID-19*

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Abstract

We modify the Laubach-Williams and Holston-Laubach-Williams models of the natural rate of interest to account for the extraordinary effects of the COVID-19 pandemic on the economy. We incorporate time-varying variances of shocks to control for outliers and introduce a COVID supply shock to the model. Estimation of the modified models yields three key findings. First, estimates of the natural rate of interest are similar to those from the original models during the pre-pandemic period. That is, these modifications effectively address the issues caused by COVID-19 while maintaining the basic structure and features of the models. Second, estimates of the natural rate of interest at the end of 2022 are close to the levels estimated directly before the pandemic. In particular, we do not find evidence that the era of historically low estimated natural rates of interest has ended. Third, the main longer-term consequence from the pandemic period is a reduction in the estimated natural level of output.

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1 Introduction

The COVID-19 pandemic and public health responses caused dramatic swings in economic activity around the world. The Laubach and Williams (LW, 2003) and Holston, Laubach, and Williams (HLW, 2017) models of the natural rate of interest are quite flexible and incorporate transitory and permanent shocks to supply and demand. However, the models' specifications are restrictive in two ways that are at odds with the effects of the COVID-19 experience.

First, in keeping with the standard Kalman filter method, stochastic innovations are assumed to follow a Gaussian distribution. Relative to historical experience, the pandemic is an extreme tail event. This represents a stark violation of the Gaussian assumption that would significantly distort the estimation results. Second, the models incorporate transitory shocks to supply as innovations in the equation for the inflation rate. These shocks are assumed to be serially uncorrelated, which is inconsistent with implications of a sequence of shutdowns and re-openings associated with COVID.

The goal of this paper is to design and implement an approach to modifying these models to take into account the highly unusual behavior during the COVID-19 period, while maintaining to the greatest extent the basic structure and flexibility of the original models. In that way, these models can continue to provide useful empirical tools in the future to parse the between transitory and highly persistent or permanent factors that affect the economy and provide useful benchmarks for measuring r^* in the future.

Specifically, we make two modifications to the LW and HLW models that reflect the unique characteristics of the pandemic period. First, we incorporate a persistent, but temporary, supply shock in order to capture the direct effects of restrictions related to the pandemic. This COVID shock is in addition to the transitory and permanent shocks already present in the models. Second, we allow for time-varying volatility of the shocks to output and inflation during the pandemic period consistent with the appearance of extreme outliers in the data.

We show that these modifications effectively address the two econometric issues caused by the pandemic. We find that the pattern of historically low estimates of trend GDP growth and the natural rate of interest before the pandemic persist after the COVID-19 pandemic.

2 HLW (2017) Model

We extend the HLW (2017) model of the natural rate of interest, which builds on the Laubach and Williams (2003) model. In our models, the natural rate of interest is the real interest rate consistent with output equaling its natural rate (y_t^*) and stable inflation. As in the DSGE literature (e.g. Woodford, 2003), we model the output gap and inflation dynamics as a function of the real interest rate gap, $r_t - r_t^*$, using an intertemporal IS equation and Phillips curve relationship, in line with the New Keynesian framework.

$$\widetilde{y}_t = a_{y,1}\widetilde{y}_{t-1} + a_{y,2}\widetilde{y}_{t-2} + \frac{a_r}{2}\sum_{j=1}^{2}(r_{t-j} - r_{t-j}^*) + \epsilon_{\widetilde{y},t} \tag{1}$$

$$\pi_t = b_\pi \pi_{t-1} + (1 - b_\pi)\pi_{t-2,4} + b_y \widetilde{y}_{t-1} + \epsilon_{\pi,t} \tag{2}$$

Equations 1 and 2 make up the measurement equations in our state-space model. The output gap is defined as $\widetilde{y}_t = 100 \cdot (y_t - y_t^*)$, where y_t and y_t^* are logarithms of real GDP and unobserved potential output, respectively, r_t is the real short-term interest rate, π_t denotes consumer price inflation, and $\pi_{t-2,4}$ is the average of its second to fourth lags.[1] The stochastic disturbances $\epsilon_{\widetilde{y},t}$ and $\epsilon_{\pi,t}$ capture transitory shocks to the output gap and inflation equations, respectively. We assume that they are normally distributed with standard deviations $\sigma_{\tilde{y}}$ and σ_π, respectively, and are mutually uncorrelated. These transitory shocks do not affect the natural rate of interest; rather, movements in r_t^* reflect persistent shifts in the relationship between the real short-term interest rate and the output gap (Williams, 2003).

The law of motion for the natural rate of interest is given by

$$r_t^* = c \cdot g_t + z_t$$

where g_t is the trend growth rate of the natural rate of output, y_t^*, and z_t captures other determinants of r_t^*.[2] We specify the three latent variables in our state-space model as

[1] See HLW (2017) section 2 and Appendix A for details of the model specification. We take as a starting point the open-economy New Keynesian model specification as in Galí (2008) and relax two standard restrictions to work with reduced-form IS and Phillips curve equations.

[2] Note that unlike in HLW (2017), we do not assume a one-for-one relationship between the trend growth rate of potential output and the natural rate of interest. Instead, we follow Laubach and Williams (2003) and estimate this relationship. We continue to find a coefficient close to unity. See Appendix A1 for details on changes to the HLW (2017) model.

follows. The logarithm of potential output follows a random walk with a stochastic drift g that itself follows a random walk,

$$y_t^* = y_{t-1}^* + g_{t-1} + \epsilon_{y^*,t} \tag{3}$$

$$g_t = g_{t-1} + \epsilon_{g,t} \tag{4}$$

and the component z_t capturing other determinants of r_t^* is assumed to follow a random walk as well,

$$z_t = z_{t-1} + \epsilon_{z,t} \tag{5}$$

Equations 3, 4, and 5 make up the state equations in our state-space model. We assume that the disturbances $\epsilon_{y^*,t}$, $\epsilon_{g,t}$, and $\epsilon_{z,t}$ are normally distributed with standard deviations σ_{y^*}, σ_g, and σ_z, respectively, and are serially and contemporaneously uncorrelated.

3 COVID-adjusted Model

The objective of this paper is to estimate the natural rate of interest (r^*) following the COVID-19 pandemic in a way that is consistent with the Holston, Laubach, and Williams (HLW, 2017) model, outlined in Section 2. This requires confronting two problems. The first is a statistical problem that is not unique to our model or to estimation of the natural rate of interest: COVID-19 represents an extreme tail event, with movements in GDP that are very large with respect to historical data and outliers in any standard macroeconomic model, as shown in Figure 4. In keeping with the standard Kalman filter method, stochastic innovations to the measurement equations in HLW – the IS and Phillips curve equations – are assumed to follow a Gaussian distribution. GDP and inflation realizations during the pandemic violate this Gaussian assumption, which would significantly distort the estimation results.

The second is that movements in output and inflation during the pandemic period display serial correlation. The HLW model incorporates transitory shocks to supply as innovations in equation for the inflation rate (the Phillips curve equation). Stochastic innovations to the IS and Phillips curve equations are assumed to be mutually uncorrelated. Movements in output and inflation during the pandemic period, caused in part by the sequence of shutdowns and re-openings and the broader government response associated with COVID-19, are inconsistent with a series of mutually uncorrelated shocks.

An advantage of the HLW approach is that the natural rate of interest is explicitly modeled to be affected by low-frequency non-stationary processes, and not by transitory shocks. We allow for two types of shocks in the original model: transitory shocks to the output gap and inflation equations, and permanent shocks to the latent variables. The model specification of the transitory shocks is restrictive in two ways that are at odds with the COVID-19 shock: as stated, these shocks are modeled as stochastic innovations to the measurement equations that are assumed to be Gaussian and mutually uncorrelated, while movements in GDP and inflation during the pandemic exhibit neither of these properties. These transitory shocks do *not* affect the unobserved state variables, y^*, g and z, and therefore do not affect r^*.

We distinguish these transitory disturbances from low-frequency movements in the data, which are ascribed to r^* and y^*. By parsing permanent effects from transitory ones, we are able to estimate a medium-run concept of the natural rate of interest. In updating the model to account for the COVID-19 pandemic, we do not change the specification of the latent variables permanent shocks, and our conception of r^* as a medium-run object does not change.

We modify the HLW model in two ways in order to overcome these challenges, while preserving the original model structure. The first is to introduce time-varying volatility during the pandemic period to solve the statistical problems associated with extreme outliers. Following Lenza and Primiceri (2022), we allow the variances of the stochastic innovations to the output gap and inflation equations to be higher during the COVID-19 pandemic. Importantly, we maintain the standard assumption that the stochastic innovations to the equations for potential output, its trend growth rate, and the unobserved other determinants of the natural rate of interest are normally distributed. This has the effect of down-weighting the outlier observations in our estimation procedure, while leaving the model equations and specification of the latent variables, including r^*, unchanged. While we apply this solution to the HLW model to estimate the natural rate of interest, it is a general solution that can be applied to any state-space model with unobserved components.

The choice to allow for increased volatility in the output gap and inflation innovations during the pandemic, but *not* increased volatility in the innovations to the latent variables, is consistent with the original HLW approach. We maintain the ability of the model to distinguish transitory shocks that do not affect the latent variables from low-frequency movements that do. The outsized movements in GDP and inflation during the early pandemic period, while extraordinarily large, were ultimately short-lived. Rather than explicitly modeling an increase in volatility of the unobserved latent variables, we let the data speak by modeling an explicit increase in the transitory shocks only. In the event that pandemic-induced movements in GDP and inflation are long-lasting, the Kalman filter will infer accordingly and will ascribe permanent changes to r^* and y^*.

While increasing the volatility of the transitory shocks resolves the statistical problems associated with extreme observations, there is still the problem of serial correlation. In particular, because the effects on supply of the sequence of shutdowns and re-openings associated with COVID-19 are highly serially correlated during the pandemic, they are not adequately captured by a transitory supply shock, modeled through innovations to the inflation equation, even when we account for increased innovation variances. We modify the model to incorporate a persistent, but ultimately temporary, supply shock, in addition to the transitory and permanent demand and supply shocks already present in the models, in order to capture the direct effects of economic shutdowns and restrictions related to the pandemic. This additional shock acts as an adjustment to the level of potential output in the output gap specification.

4 Outliers

Before making any adjustments to the model, we begin by estimating the standard HLW model through 2019:Q4, prior to the onset of the COVID-19 pandemic. We fix the model parameters and re-estimate the latent variables through 2022:Q4 using the parameter values from the 2019:Q4 estimated model. We also fix the initial vector of unobserved states and its covariance matrix at the 2019:Q4 values.[3] This exercise is equivalent to dropping observations beginning in 2020:Q1 through the end of the sample during the maximum likelihood estimation of model parameters, while allowing the Kalman updating procedure

[3]We store the estimated parameter vector θ from the final (stage 3) model as well as the signal-to-noise ratios λ_g and λ_z from the median unbiased estimation procedures following stages 1 and 2, respectively. See HLW (2017) for a description of the estimation procedure and footnote 6 for the initialization process of the vector of unobserved states, its conditional expectation $\xi_{1|0}$ in the first period, and the covariance matrix $P_{1|0}$.

to continue without modification through the end of the sample. In other words, we make no modifications to the state-space model, except that the model coefficient matrices and covariance matrices in the Kalman filtering procedure are fixed at their 2019:Q4 values.

The final step of the Kalman filtering procedure to estimate the vector of unobserved states at time t ($\hat{\xi}_{t|t}$, given the information set at time t) is given by the updating equation. This equation adds the initial estimate ($\hat{\xi}_{t|t-1}$, given information at time $t-1$) to the vector of one-step-ahead prediction errors for the IS and Phillips curve equations, pre-multiplied by the Kalman gain matrix, K_t.

$$\hat{\xi}_{t|t} = \hat{\xi}_{t|t-1} + K_t \underbrace{(y_t - \mathbf{H}'\hat{\xi}_{t|t-1} - \mathbf{A}'x_t)}_{\text{one-step-ahead prediction error}} \tag{6}$$

In this initial exercise, the coefficient matrices $\mathbf{H}'$ and $\mathbf{A}'$ on the state vector $\hat{\xi}_{t|t-1}$ and data $\mathbf{x}_t$, respectively, are fixed at their 2019:Q4 values and the resulting one-step-ahead prediction errors (also known as forecast errors) are large. Note that the one-step-ahead prediction errors are the residuals to the IS and Phillips curve equations, using the forecast of $\mathbf{y}_t$ (the vector of the output gap and inflation) based on the data at time t and information at time $t-1$, corresponding to the state vector $\hat{\xi}_{t|t-1}$:

$$\mathbf{y}_t - \mathbb{E}[\mathbf{y}_t|\mathbf{x}_t, \zeta_{t-1}] = \mathbf{y}_t - (\mathbf{H}'\mathbf{x}_t + \mathbf{A}'\hat{\xi}_{t|t-1}) \tag{7}$$

During the pandemic period, the large forecast errors translate directly to the estimated vector of unobserved variables (y^*, g, z), so that the data during this period has a sizable effect on the estimated natural rate of interest r^*. As expected, even when outliers are excluded during parameter estimation, using the Kalman filter with extreme outliers present significantly distorts the estimation results, resulting in large swings in the estimates of the latent variables. This is in conflict with the specification of these latent variables as reflecting lower-frequency movements.

Following Harvey and Koopman (1992), we make use of the auxiliary residuals to the measurement and state equations in order to detect outliers. Because ours is an unobserved-components model, we are able to look at auxiliary residuals as an alternative to the one-step-ahead prediction errors that are commonly used for diagnostic testing of time-series models.[4]

Auxiliary residuals are smoothed estimators of the disturbances to the measurement and state equations (ϵ_t and η_t, respectively). They have the advantage of a direct interpretation and test for outliers: under the assumption that the stochastic innovations are from a Gaussian distribution, standardized auxiliary residuals greater than 2 (in absolute value) indicate either the presence of outliers or a structural change, and subsequent testing can distinguish between the two cases. We use the algorithm in Koopman and Durbin (2000) to obtain the standardized auxiliary residuals to the IS and Phillips curve equations

$$\widetilde{\epsilon}_t = \mathbb{E}[\epsilon_t|\mathbf{y}_T, \mathbf{x}_T, \zeta_T] \tag{8}$$

as well as to the equations for the unobserved state variables

$$\widetilde{\eta}_t = \mathbb{E}[\eta_t|\mathbf{y}_T, \mathbf{x}_T, \zeta_T] \tag{9}$$

Figures 6 and 7 show the standardized auxiliary residuals to the measurement equations, $\widetilde{\epsilon}_t/\hat{\sigma}_\epsilon$, while figure 8 displays the standardized auxiliary residuals to the y^* equation, from $\widetilde{\eta}_t/\hat{\sigma}_\eta$. For the estimated model with parameters fixed at pre-pandemic values, there are extreme outliers in the output gap equation and in the equation for potential output across all economies in our sample.

5 Implementation of COVID-Adjusted Model

5.1 Increased Shock Volatility during COVID-19

In the HLW framework, transitory shocks to demand and supply are modeled as stochastic innovations to the output gap and inflation equations; that is, the IS and Phillips curve equations. Following the standard Kalman filter approach, these stochastic innovations are assumed to be Gaussian and mutually uncorrelated. The innovation variances, $\sigma_{\tilde{y}}$ and σ_π, are assumed to be constant over the sample and are estimated by maximum likelihood, together with the remaining model parameters.

The COVID-19 shock violates the Gaussian assumption. As shown in Figure 4, it is an extreme outlier that would significantly distort the estimation results. In order to proceed

[4] See also Harvey et al. (1999).

with estimating the natural rate of interest after the COVID-19 pandemic, we need a way to account for the substantially higher volatility associated with disturbances to the output gap and inflation equations during the pandemic period.

We build on an insight from Lenza and Primiceri (2022): if the timing of increased volatility is known – as is the case for the COVID-19 pandemic – we can introduce time-varying volatility in the model directly by applying a scale factor to the innovation variances during the period of increased volatility. We apply this insight to our unobserved components model in order to estimate the natural rate of interest, but our approach can generalize to any state-space model with latent variables that can be estimated using the Kalman filter.

In particular, we introduce three new model parameters, κ_{2020}, κ_{2021}, and κ_{2022}. These are the variance scale parameters for 2020, 2021, and 2022, respectively, which multiply the variances of the innovations to the output gap and inflation equations. We define the vector κ_t of variance scale parameters at time t, that takes the values

$$\kappa_t = \begin{cases} \kappa_{2020} & \text{2020:Q2} \leq t \leq \text{2020:Q4} \\ \kappa_{2021} & \text{2021:Q1} \leq t \leq \text{2021:Q4} \\ \kappa_{2022} & \text{2022:Q1} \leq t \leq \text{2022:Q4} \\ 1 & \text{otherwise} \end{cases} \tag{10}$$

We estimate the three variance scale parameters by maximum likelihood together with the other model parameters, with the constraints $\kappa_{2020} \geq 1$, $\kappa_{2021} \geq 1$, and $\kappa_{2022} \geq 1$[5]. κ_t takes the value of 1 before the pandemic period and in 2023 and beyond.

The covariance matrix of the stochastic innovations to the output gap and inflation equations is now time-varying and is given by

$$\mathbf{R}_t = \kappa_t^2 \cdot \mathbf{R} = \begin{bmatrix} (\kappa_t \sigma_{\tilde{y}})^2 & 0 \\ 0 & (\kappa_t \sigma_\pi)^2 \end{bmatrix} \tag{11}$$

with time-varying innovation variances to the IS curve and Phillips curve equations of $(\kappa_t \sigma_{\tilde{y}})^2$ and $(\kappa_t \sigma_\pi)^2$, respectively.

It is straightforward to see that outside the pandemic period, the innovation variances

[5]The restriction that $\kappa_t \geq 1$ is necessary to ensure that the likelihood estimation cannot *down-weight* the variance of certain observations, which would in effect allow it to place more weight on favorable observations. Instead, the estimated κs can only increase the innovation variances during the pandemic period.

are specified exactly as in HLW (2017), so that the innovation variances to the output gap equation, $\sigma_{\tilde{y}}^2$, and inflation equation, σ_{π}^2, are constant over the sample prior to 2020 and after 2022. During the year 2021, for example, the innovation variances take the values $(\kappa_{2021} \cdot \sigma_{\tilde{y}})^2$ and $(\kappa_{2021} \cdot \sigma_{\pi})^2$, respectively. Therefore κ_t is a ratio of the standard deviations of the disturbances to the measurement equations (the output gap and inflation equations) at time t relative to the standard deviations in the non-pandemic sample. If $\kappa_t > 1$, as we find for all economies in our sample, this implies that the innovation variances are greater at time t than in the non-pandemic sample.

Introducing the innovation variance scaling factors has the effect of down-weighting extreme outlier observations in parameter estimation as well as in estimation of the unobserved state variables via the Kalman filter. When $\kappa_t > 1$, the diagonal matrix $\mathbf{R}_t$ of the disturbances to the output gap and inflation equations is larger relative to the case where $\kappa_t = 1$, and the resulting Kalman gain is smaller. The Kalman gain dictates the weight placed on the one-step-ahead prediction error – the difference between realized values of the output gap and inflation in a period and the model's predicted values given information in the prior period – in updating the filtered estimates of the latent variables. As the innovation variances in a given period become large, the Kalman gain is small, so that the Kalman filter places little weight on these new observations and the state vector estimates (that is, y_t^*, g_t, z_t and therefore r_t^*) remain close to the estimates from the prior period. In the limit as the innovation variances tend toward infinity, the Kalman gain approaches zero, so that no weight is placed on the time-t observations in estimating the state vector. In effect, the model does not make use of time-t information, so that the forecast of the state vector at time t given the time-t information set is unchanged from the forecast given the information set at time $t-1$. The same holds for parameter estimation: when κ_t is large, the model forecast error in this period is down-weighted when computing the log likelihood function, such that the data in this period have relatively little impact on the set of parameters that maximize the log likelihood function.

We see this approach as preferable to outright discarding the COVID-19 outliers by treating them as missing data for several reasons. It is not our goal to provide reliable estimates of r^* during the COVID-19 pandemic, and we treat these estimates with extreme caution. Rather, our objective is to deliver a framework for estimating of the natural rate of interest that is consistent with our approach in HLW, so that we are able to parse permanent changes to r^* from transitory shocks once the pandemic has abated. While the timing of the onset of the pandemic is clear, selecting an end date for the set of observations to discard would not be straightforward, and estimation of r^* may be sensitive to this choice. Additionally, a binary decision to drop or keep pandemic-related observations necessitates treating the entire period universally. Our approach is flexible in that we *allow* for increased variance during the three years following the onset of the pandemic, but we do not *impose* higher variances. We also do not impose any relationship between κ_{2020}, κ_{2021}, and κ_{2022}. By estimating these parameters together with the remaining model parameters, including the innovation variances during the non-pandemic period, our approach instead allows the data to inform the choice of variance scaling factors, so that more extreme outliers are more heavily down-weighted. Indeed, when we allow the model to treat the later quarters of the pandemic differently from the earlier quarters, we find that it chooses to do so. Finally, it is well-known that estimates of the natural rate of interest are highly uncertain. Excluding data associated with the COVID-19 pandemic would understate the true uncertainty about future r^* estimates (Lenza and Primiceri, 2022). Our approach preserves estimation of the model standard errors over the full sample.

5.2 COVID-adjusted Potential Output

The direct effects of COVID-19 on the economy are incorporated in the model as an adjustment to the natural rate of output in the output gap specification. We introduce one new variable, denoted d_t, as a proxy for the direct effects of the government restrictions and shutdowns implemented in response to the pandemic. We set this COVID indicator variable equal to the quarterly average of the COVID-19 Stringency Index from the Oxford COVID-19 Government Response Tracker (OxCGRT) for each country or region, as shown in Figure 5.[6] The stringency index, which ranges between 0 and 100 with larger numbers indicating stricter restrictions, aggregates measures of government containment and closure policies such as school and workplace closures, travel restrictions, bans or limits on public gatherings, and shutdowns of public transportation.

We choose this indicator because it is comprehensive and publicly available for all of the economies in our sample. We recognize that such an index of government responses cannot

[6] See Hale et al. (2021). We use the national weighted average of the stringency indices for vaccinated and unvaccinated populations, and our results are robust to using the index for vaccinated individuals only.

capture the full set of behavioral responses or compliance; nonetheless, it should provide a reasonable first-order approximation to the time-series properties of the direct effects of the pandemic and associated public health actions on economies. For the Euro Area, we use a GDP-weighted stringency index with 2019 GDP weights. As the OxCGRT project suspended data collection at the end of 2022, we assume a constant decay in the indicator variable beginning in 2023:Q1, reaching zero in 2024:Q4. The COVID indicator is set equal to zero through 2019:Q4 in all economies. It is assumed to be an exogenous variable in the model.

We incorporate the COVID variable as an adjustment to the natural rate of output, within the output gap specification. In particular, the COVID-adjusted natural rate of output is given by

$$y^*_{t,COVID-adj.} = y^*_t + \frac{\phi}{100} d_t$$

where y^*_t is the standard natural rate of output, d_t is the COVID-19 indicator, and ϕ is an estimated parameter that translates the COVID variable d_t into effects on output. The output gap is correspondingly modified, with COVID-adjusted potential output replacing the standard natural rate of output:

$$\widetilde{y}_{t,COVID-adj.} = 100(y_t - y^*_{t,COVID-adj.}) = 100(y_t - y^*_t) - \phi d_t$$

where y_t is the log of real GDP. We estimate the parameter ϕ together with the remaining model parameters by maximum likelihood, including the variance-scaling parameters $\kappa_{2020}, \kappa_{2021}, \kappa_{2022}$.

The COVID-adjusted output gap replaces the standard output gap in the measurement equations, which are the IS and Phillips curve equations. The state equations, which are the model equations for the unobserved latent variables, are unchanged. This includes the equation for the underlying natural rate of output,

$$y^*_t = y^*_{t-1} + g_{t-1} + \epsilon_{y^*,t}$$

so that the laws of motion for the natural rate of output, y^*_t, and its trend growth rate, g_t, are identical to the standard HLW specification. Accordingly, the law of motion for the natural rate of interest is unchanged.

6 Estimation Results

In this section, we report the estimation results using data through the end of our sample and compare those to estimates from the pre-pandemic period.

6.1 Parameter estimates

Table 1 reports the estimates of model parameters for the three economies. For comparison, the corresponding estimates from the model estimated through 2019:Q4 are reported in the appendix table A1. The parameter estimates are broadly similar to estimates from the pre-pandemic period. Note that in all cases the estimated values of λ_g and λ_z are strictly positive.

The estimated values of the parameter on the COVID shock variable, ϕ, is statistically significant and, as discussed below, has an economically large effect. Consistent with the observations of enormous outliers early in the pandemic, the estimated values of the parameter κ are sizable for 2020Q2-Q4. The parameter estimates for κ in 2021 and 2022 range between one and two, and are generally statistically significant.

6.2 Model Residuals

Figures 6 and 7 show the standardized auxiliary residuals, described in Section 4, to the IS and Phillips curve equations, with the horizontal lines indicating two standard deviations. The yellow lines show residuals from the version of the model with parameters held fixed at 2019:Q4 values. In all economies, IS equation residuals in 2020 indicate extreme outliers to the model. The magnitude of the estimated κ_{2020} parameters is consistent with these extreme outliers. The blue lines show auxiliary residuals from the modified HLW model. With the inclusion of the variance scale parameters and the adjustment to potential output, these residuals are of similar magnitude to the pre-pandemic period, and no longer indicate the presence of outliers. Figure 8 displays the standardized auxiliary residuals for the potential output equation. Again, the massive outliers in the standard version of the model are no longer present in the COVID-adjusted model, despite the fact that the specification of potential output and its stochastic innovation are unchanged.

6.3 Estimates of r^*, g, and y^*

Estimation of the modified HLW model reveals three key findings. First, the modified estimation procedure yields results that are overall quite similar to those from the original model during the pre-pandemic period. Second, the current estimates of the natural rate of interest are similar to those estimated directly before the pandemic. Third, the estimates of the natural rate of output at the end of 2022 are much lower than predicted before the pandemic.

The current HLW estimates of the natural rate of interest in the United States are shown in Figure 1. For comparison, the figure also shows estimates using a version of the model that holds the model parameter values fixed at estimates using data through the end of 2019, and does not include the COVID modifications (i.e. $\phi = 0, \kappa_t = 1$ for all periods). The two sets of estimates are very similar through 2019, but differ sharply during the acute period of the pandemic, when the estimates from the unmodified model exhibit large swings due to the presence of outliers. Interestingly, the two estimates are very close to each other at the end of the sample.

Based on the estimates of r^* for Canada, the Euro Area, and the United States, there are no clear signs of a significant reversal of the decline in the natural rate of interest estimates that is evident in prior decades. Figures 2 and 3 show the corresponding estimates of the natural rate of interest for Canada and the Euro Area.

In fact, in all three economies, the estimates of the natural rate of interest in 2022 are within a few tenths of a percentage point of the corresponding estimates in 2019.

The estimates of trend growth in potential output are slightly lower in 2022 than in 2019. Table 2 reports the estimates of the trend growth rate, g, in specified years. It also reports forecasts of trend growth rates from various sources. Interestingly, for each economy, the HLW estimates and the corresponding forecasts are tightly clustered.

The largest differences between model estimates pre- and post-pandemic relate to the level of each economy's natural rate of output. Figure 9 compares the model projections of the natural rate of output based on estimates using data through the fourth quarter of 2019 to current estimates. For example, at the end of 2022, the COVID-adjusted level of the natural rate of output in the United States is 4.2 percent below the pre-pandemic projection, with nearly half of that shortfall explained by the COVID shock measure and the remainder a permanent change in the natural level of output.

In summary, according to the model estimates, the main longer-term consequence from the pandemic period is a reduction in the natural rate of output, but the imprint on the natural rate of interest appears to be relatively modest. We do not find evidence from the HLW estimates that the era of historically low estimated natural rates of interest has come to an end.

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Table 1: Parameter Estimates

Parameter	United States	Canada	Euro Area
Sample	1961-2022	1961-2022	1972-2022
λ_g	0.073	0.061	0.039
λ_z	0.021	0.011	0.023
$\sum a_y$	0.936	0.946	0.945
a_r	-0.079	-0.077	-0.037
	(4.215)	(2.947)	(1.757)
b_y	0.073	0.049	0.088
	(3.003)	(1.821)	(2.310)
c	1.128	1.109	0.953
	(3.574)	(2.794)	(1.155)
ϕ	-0.085	-0.062	-0.135
	(2.199)	(3.442)	(6.557)
$\kappa_{2020Q2-Q4}$	9.033	9.377	18.609
	(2.351)	(2.134)	(2.384)
κ_{2021}	1.791	1.087	1.960
	(2.941)	(1.835)	(2.150)
κ_{2022}	1.676	1.613	1.557
	(2.060)	(1.293)	(1.231)
$\sigma_{\tilde{y}}$	0.452	0.579	0.318
σ_π	0.787	1.343	0.965
σ_{y^*}	0.500	0.448	0.382
σ_g	0.145	0.109	0.060
σ_z	0.118	0.085	0.201
$\sigma_{r^*} = \sqrt{c^2\sigma_g^2 + \sigma_z^2}$	0.202	0.148	0.209
S.E. (sample ave.)			
r^*	1.140	1.603	2.739
g	0.428	0.413	0.264
y^*	1.612	2.474	1.662
S.E. (final obs.)			
r^*	1.565	1.858	3.805
g	0.689	0.639	0.405
y^*	2.914	3.679	2.878

Notes: t statistics are in parentheses; σ_g is expressed at an annual rate.

Table 2: Trend Growth Estimates

	1990	2007	2019	2022	Change 1990-2007	Change 2007-2019	Change 2019-2022
United States							
HLW g estimates	3.3	2.8	2.1	1.8	−0.5	−0.7	−0.2
Consensus Forecasts	2.4	2.8	2.0	1.9	0.4	−0.8	−0.1
IMF World Economic Outlook	2.6	3.0	1.6	1.8	0.4	−1.5	0.2
Blue Chip Financial Forecasts	2.5	2.8	2.0	1.9	0.4	−0.8	−0.1
Canada							
HLW g estimates	3.4	2.5	1.8	1.5	−0.9	−0.7	−0.3
Consensus Forecasts	2.8	2.4	1.8	1.8	−0.4	−0.7	0.0
IMF World Economic Outlook	3.2	2.7	1.7	1.7	−0.5	−1.0	0.0
Euro Area							
HLW g estimates	2.7	2.0	1.2	1.0	−0.7	−0.9	−0.1
Consensus Forecasts	*n/a*	1.9	1.3	1.2	*n/a*	−0.6	−0.1
IMF World Economic Outlook	*n/a*	2.0	1.3	1.4	*n/a*	−0.7	0.1

Notes: Consensus Forecasts data are the mean of panellists' trend estimates of expected GDP growth in the 6 to 10 years following. IMF estimates are the 5-year ahead forecast for real GDP growth. Blue Chip estimates are the mean long-years ahead forecast for real GDP. For each of these three forecasts, numbers are the average of the Spring and Fall publications for each year. HLW estimates are yearly averages. Numbers may not sum due to rounding.
Sources: Consensus Economics Inc, London; IMF World Economic Outlook; Blue Chip Financial Forecasts.

Figure 1: Estimates: United States

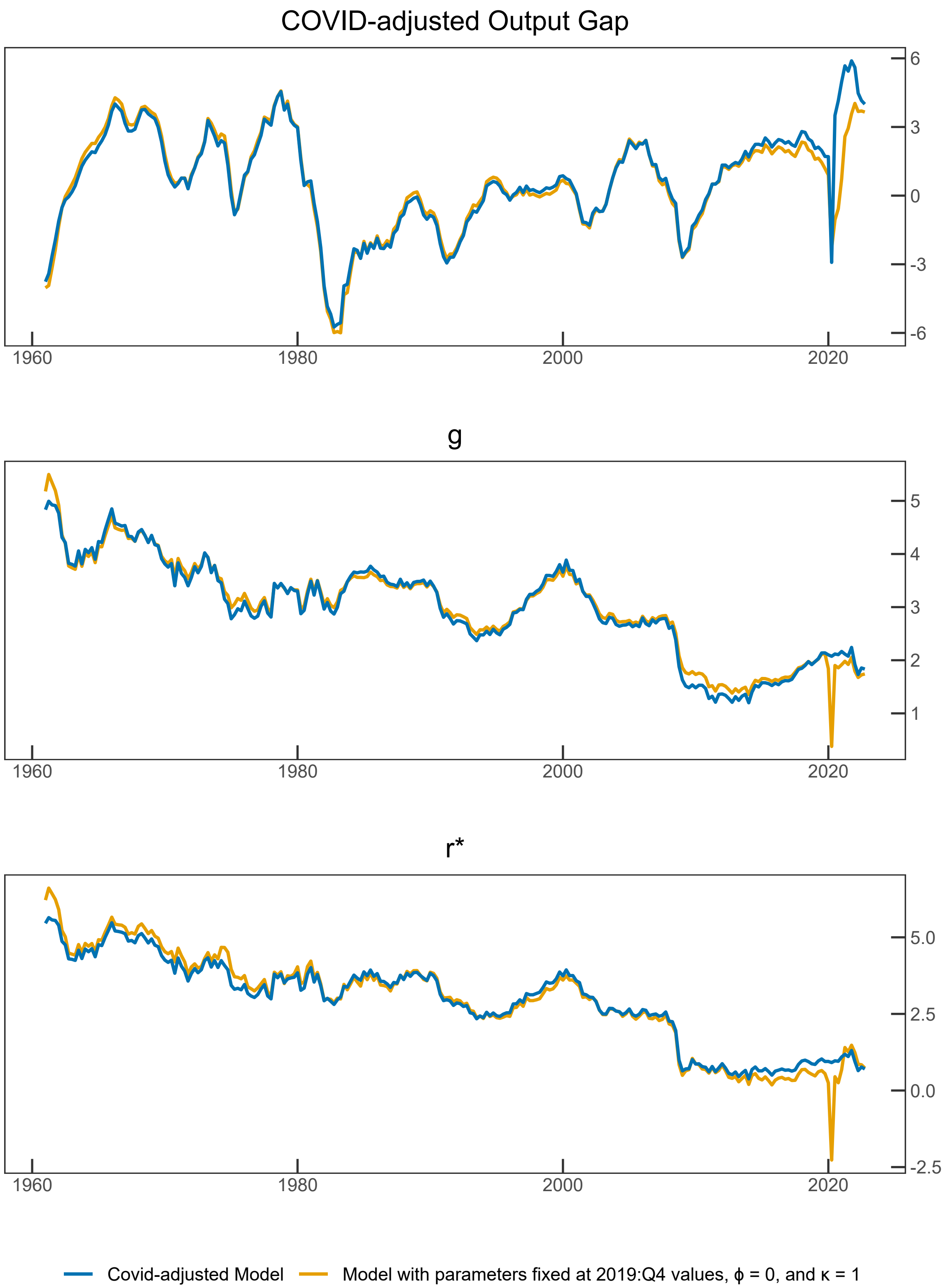


Figure 2: Estimates: Canada

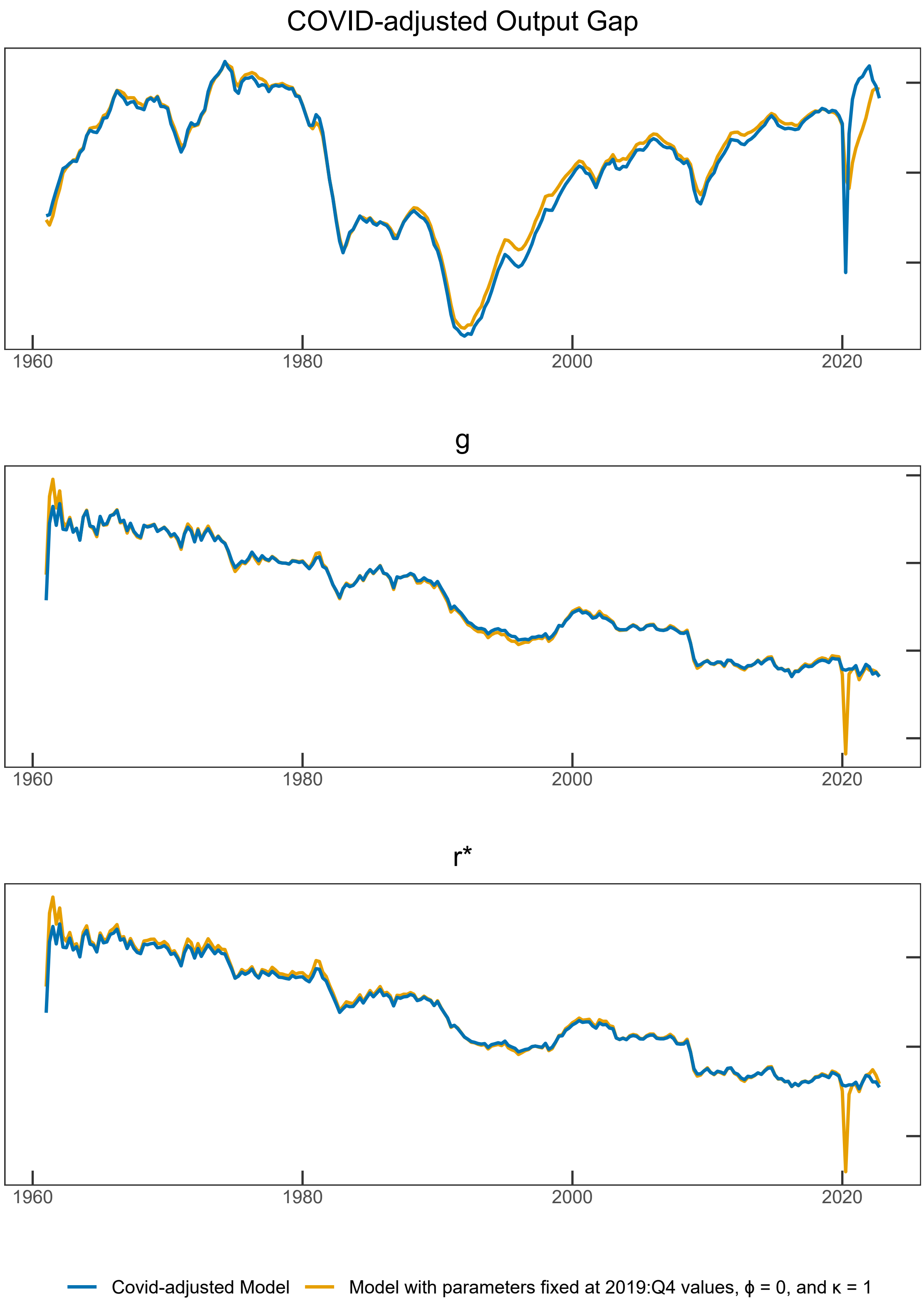


Figure 3: Estimates: Euro Area

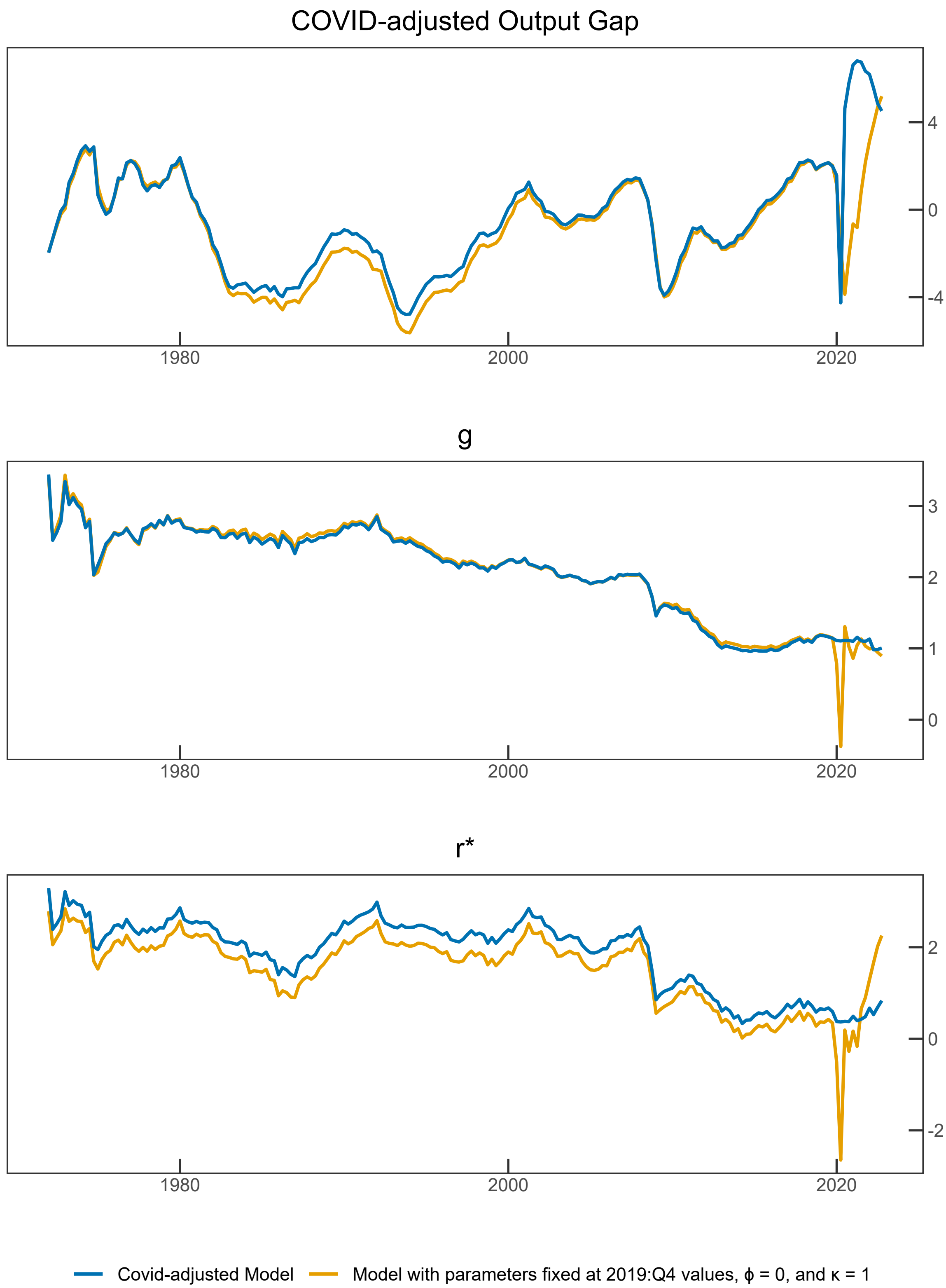


Figure 4: Comparison of Data Across Economies

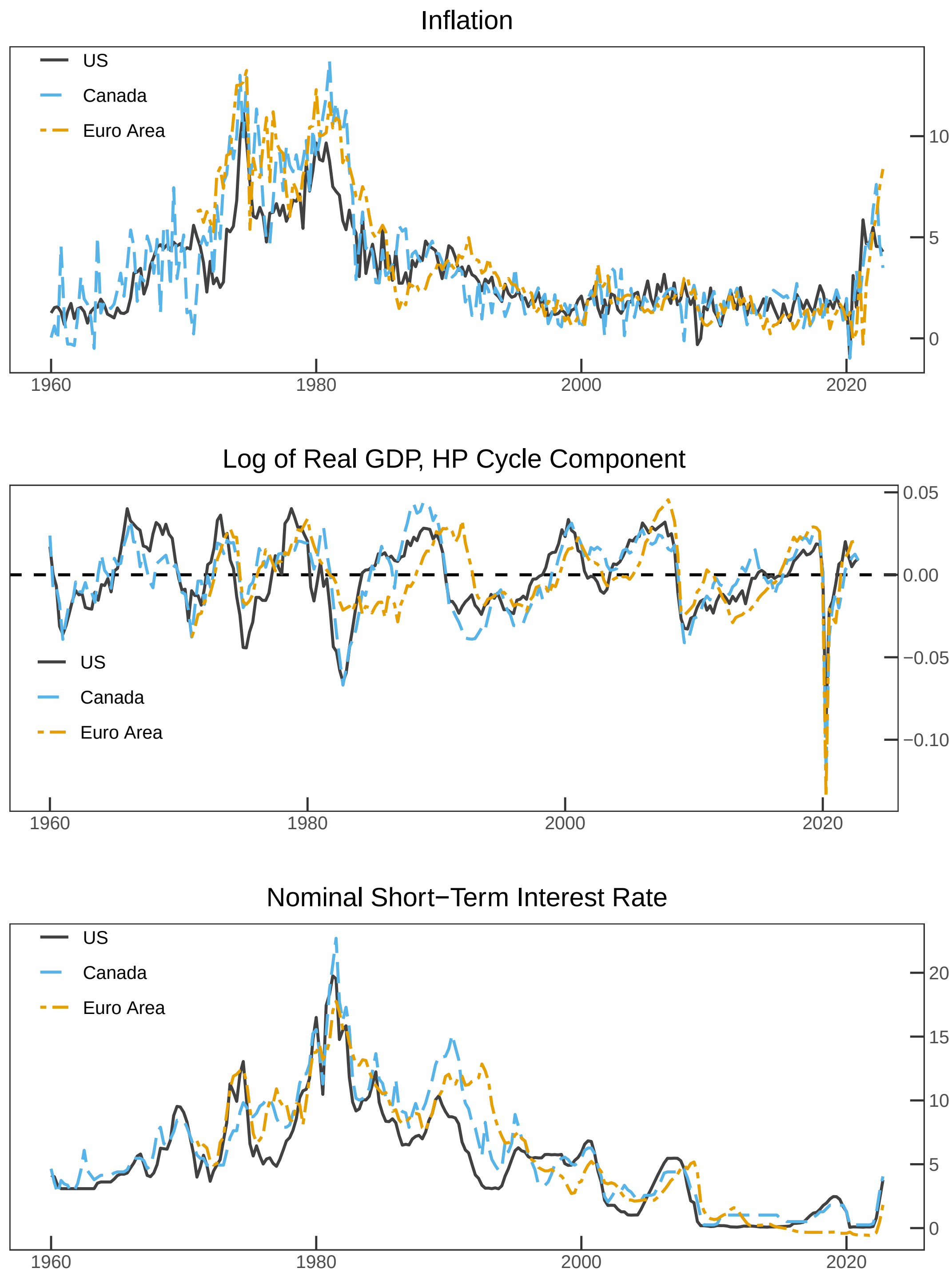


Figure 5: COVID Stringency Index, Oxford COVID-19 Government Response Tracker

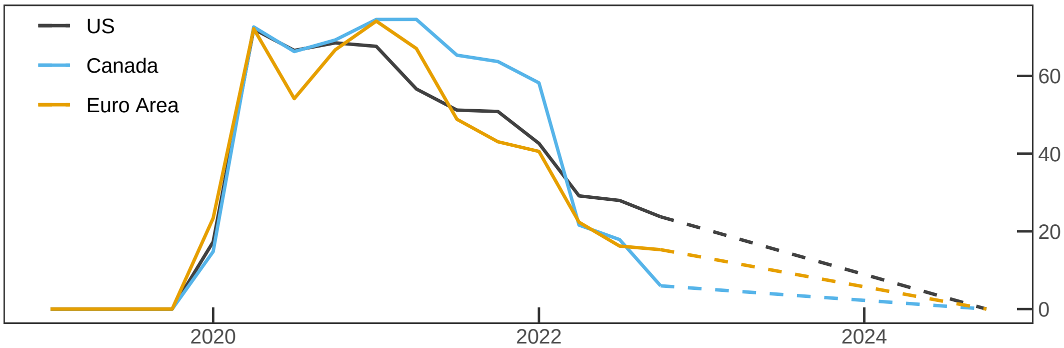


Figure 6: Standardized Auxiliary Residuals, IS Curve

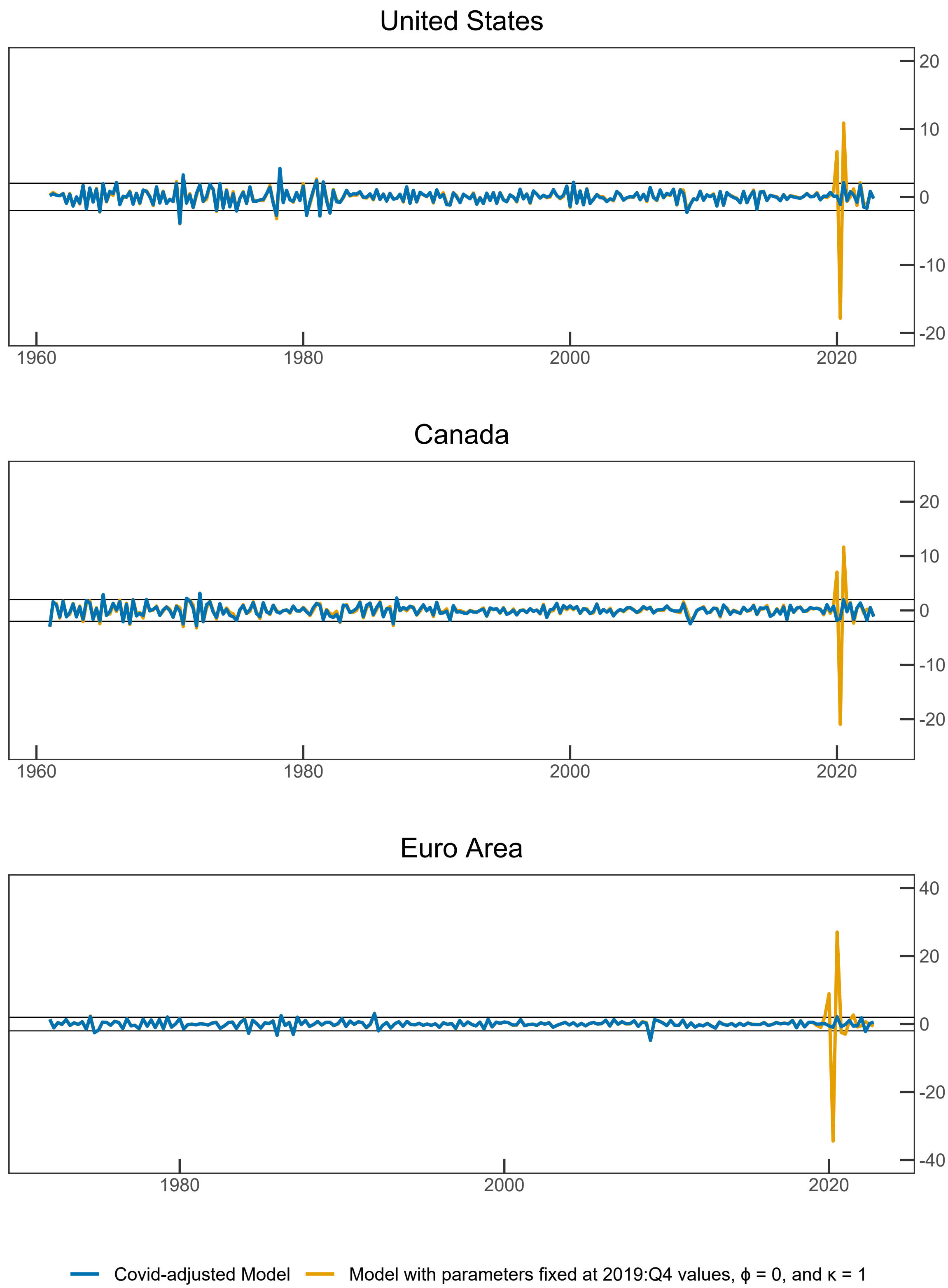


Figure 7: Standardized Auxiliary Residuals, Phillips Curve

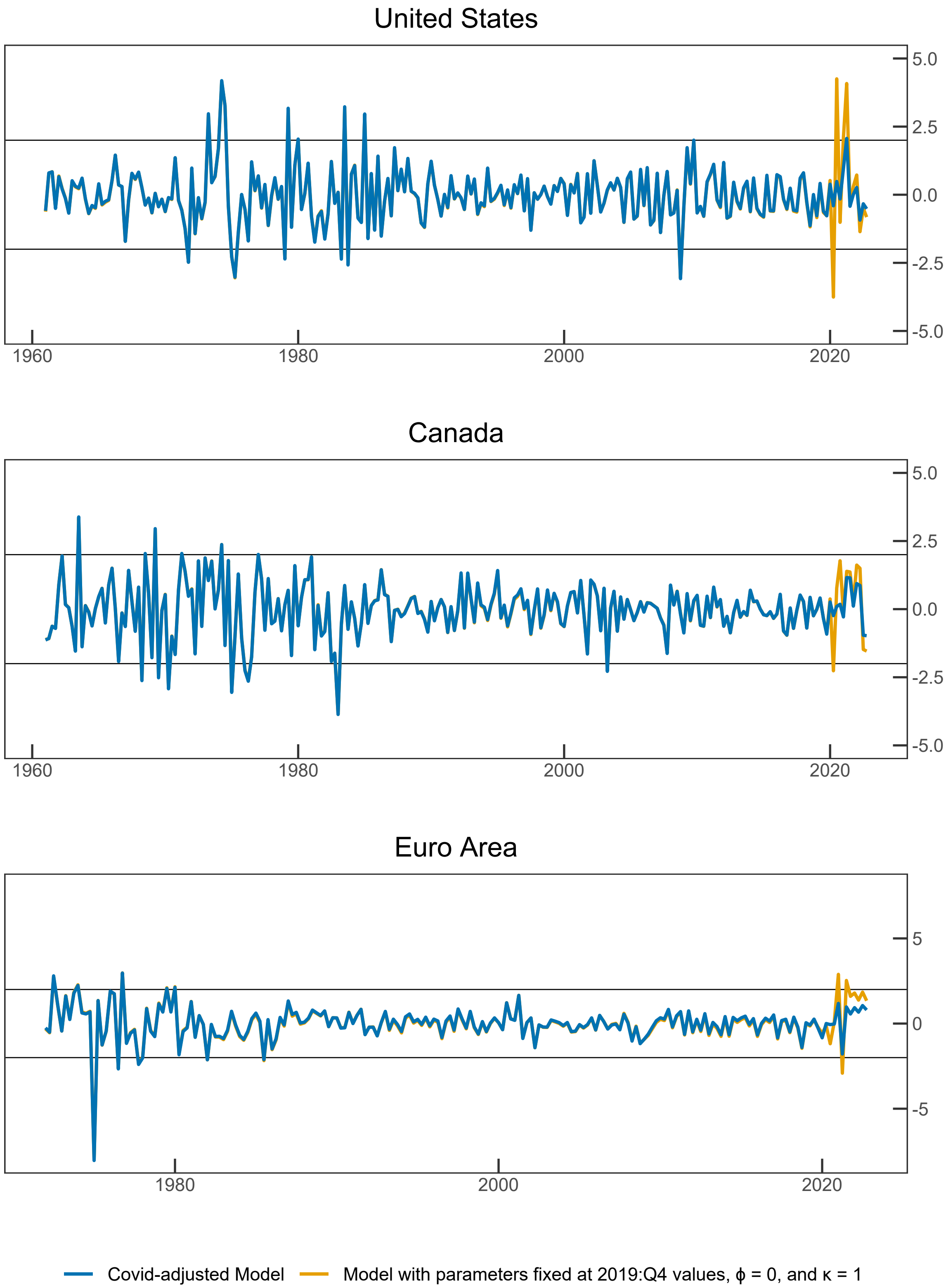


Figure 8: Standardized Auxiliary Residuals, Potential Output

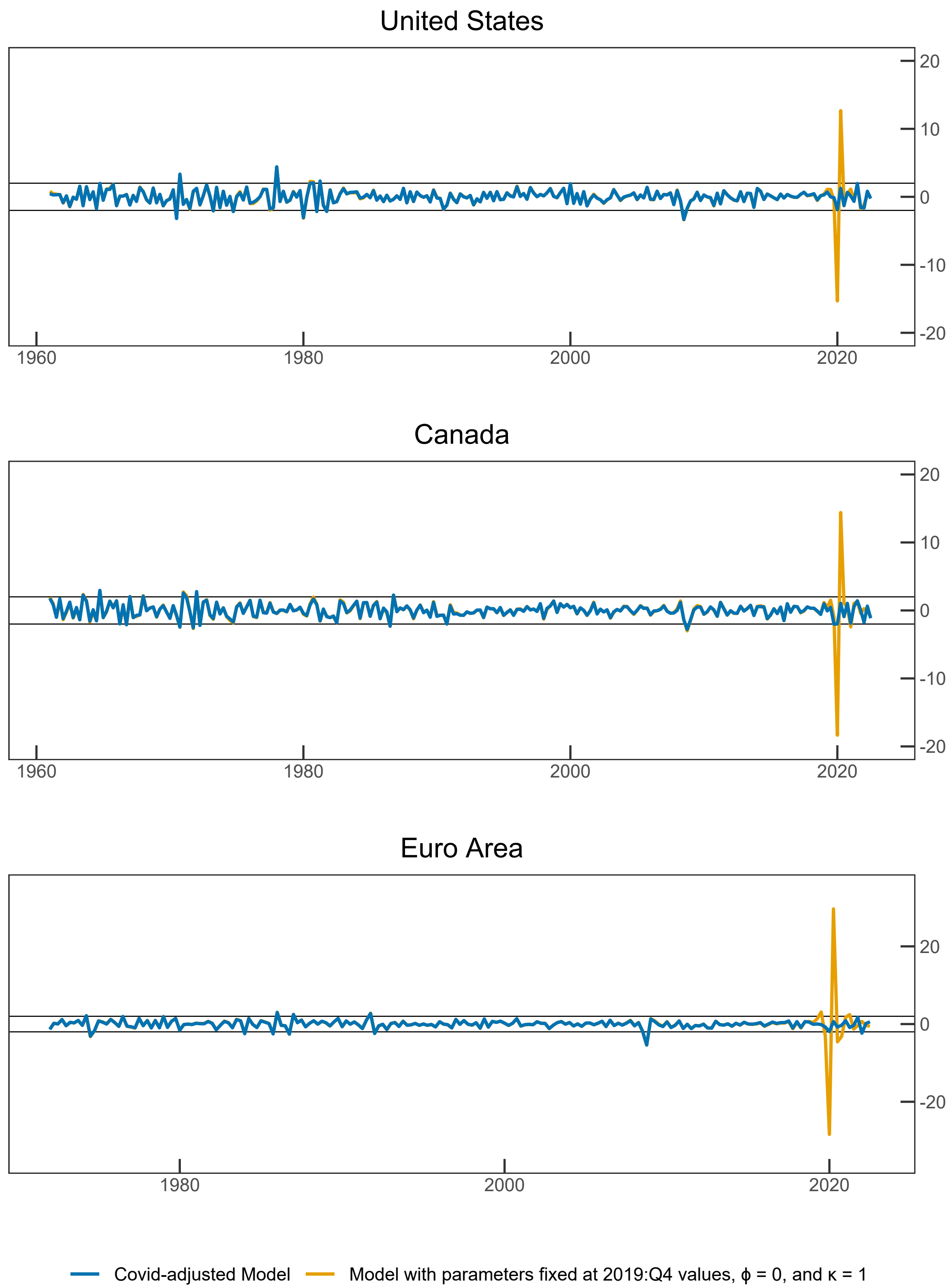
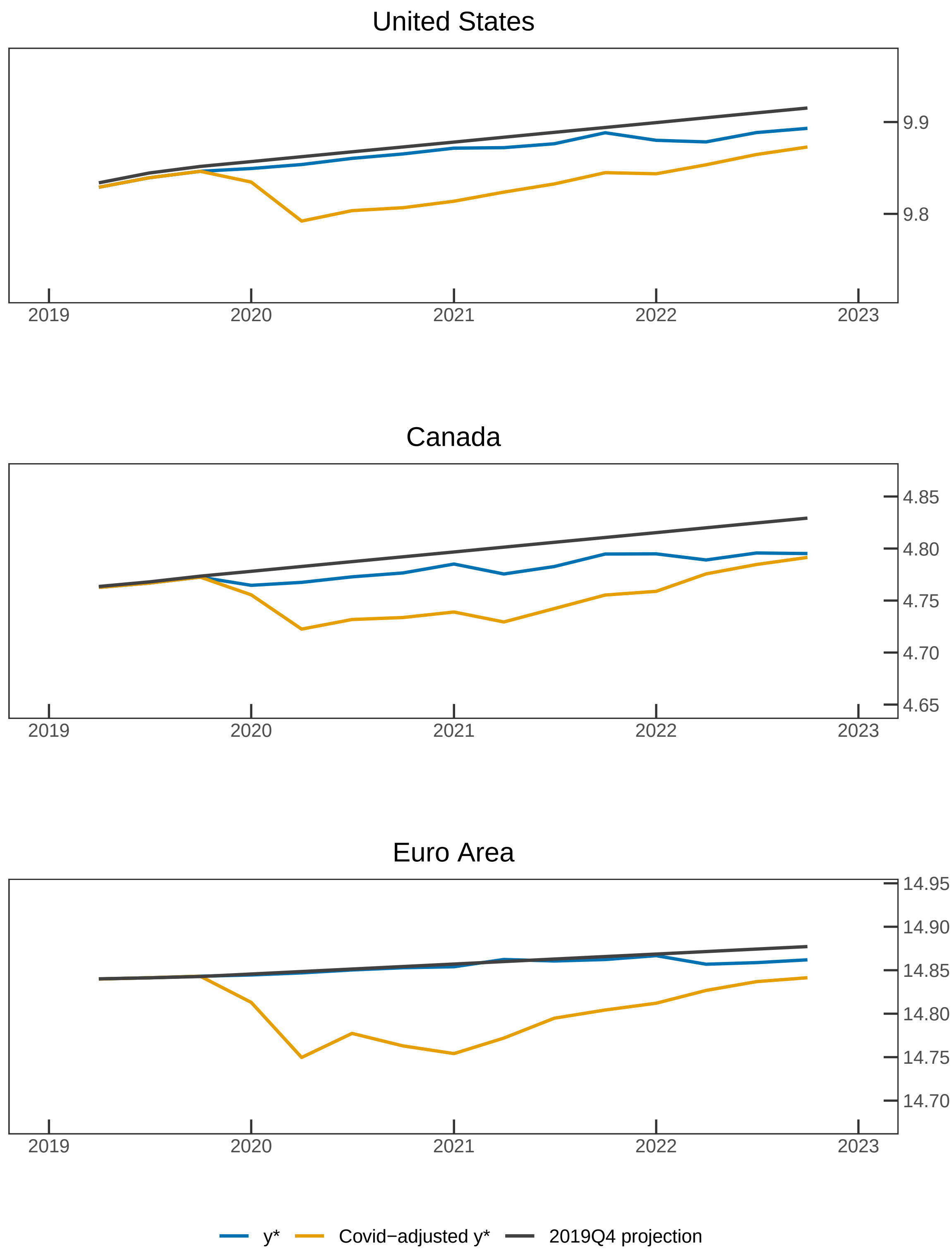


Figure 9: Potential Output



Appendix A1: State-Space Models

This section presents the COVID-adjusted and original HLW models in state-space form.[7] Our estimation procedure has three stages.[8] The first and second stage models represent versions of the final stage model, and each of the models can be cast in state-space form:

$$\mathbf{y}_t = \mathbf{A}' \cdot \mathbf{x}_t + \mathbf{H}' \cdot \xi_t + \epsilon_t \tag{12}$$

$$\xi_t = \mathbf{F} \cdot \xi_{t-1} + \eta_t \tag{13}$$

Here, $\mathbf{y}_t$ is a vector of contemporaneous endogenous variables, while $\mathbf{x}_t$ is a vector of exogenous and lagged exogenous variables. ξ_t the vector of unobserved state variables. In the HLW (2017) and Laubach and Williams (2003) models, the vectors of stochastic disturbances ϵ_t and η_t are assumed to be Gaussian and mutually uncorrelated, with mean zero and covariance matrices $\mathbf{R}$ and $\mathbf{Q}$, respectively. The covariance matrix $\mathbf{R}$ is assumed to be diagonal. In the COVID-adjusted model, we modify the covariance matrix $\mathbf{R}_t$ to be time-varying.

Each model has a corresponding vector of parameters to be estimated by maximum likelihood. Because maximum likelihood estimates of $sigma_g$ and $sigma_z$, which are the standard deviations of the innovations to the g_t and z_t equations, are likely to be biased towards zero due to the pile-up problem (see Section 2.2 of HLW), we use Stock and Watson's (1998) median unbiased estimator to obtain estimates of two ratios, $\lambda_g \equiv \frac{\sigma_g}{\sigma_{y^*}}$ and $\lambda_z \equiv \frac{a_r \sigma_z}{\sigma_{\tilde{y}}}$. We estimate λ_g following the first stage model and λ_z following the second stage model, and impose these ratios in subsequent stages of the estimation, including when estimating the remaining model parameters by maximum likelihood. The COVID-adjusted model includes five additional parameters: the coefficient ϕ that translates the COVID indicator variable d_t into effects on output; the three variance scale parameters κ_{2020}, κ_{2021}, and κ_{2022}; and the coefficient c on trend growth g_t in the r_t^* equation, which appears only in the stage 3 model and is estimated in LW (2003) but fixed at unity in HLW (2017).

In addition to estimating the relationship between the trend growth rate of potential output and r^*, we make two minor technical changes to the model that are not related to the COVID-19 pandemic.[9] First, we include a second lag of trend growth, g_{t-2}, in the stage

[7]Notation follows Hamilton (1994) and is consistent with the corresponding R programs.

[8]See HLW (2017) for a full description of our estimation procedure.

[9]We also explicitly include g_t and z_t in the vector of unobserved state variables in addition to two lags

2 IS equation, consistent with the IS equation specification in the stage 3 model. Second, we correct the stage 2 state-space model so that the y_t^* equation is $y_t^* = y_{t-1}^* + g_{t-1} + \epsilon_{y^*,t}$ as expressed in the paper; previously, the stage 2 y_t^* equation included the second lag of trend growth, g_{t-2}, rather than the first lag in error (Buncic, 2021, 2022). These modifications to the model are highlighted in blue text in the following sections, while changes in response to the COVID-19 pandemic are highlighted in red. These changes have minor effects on our estimates of the unobserved state variables, including r^*.

7.1 The COVID-Adjusted State-Space Models

7.1.1 The COVID-adjusted Stage 1 Model

The first-stage model, which corresponds to the *rstar.stage1.R* program, can be represented by the following matrices:

$$\mathbf{y}_t = [y_t, \pi_t]' \tag{14}$$

$$\mathbf{x}_t = [y_{t-1}, y_{t-2}, \pi_{t-1}, \pi_{t-2,4}, d_t, d_{t-1}, d_{t-2}]' \tag{15}$$

$$\xi_t = [y_t^*, y_{t-1}^*, y_{t-2}^*]' \tag{16}$$

$$\mathbf{H}' = \begin{bmatrix} 1 & -a_{y,1} & -a_{y,2} \\ 0 & -b_y & 0 \end{bmatrix}, \quad \mathbf{A}' = \begin{bmatrix} a_{y,1} & a_{y,2} & 0 & 0 & \phi & -\phi a_{y,1} & -\phi a_{y,2} \\ b_y & 0 & b_\pi & 1-b_\pi & 0 & -\phi b_y & 0 \end{bmatrix}$$

$$\mathbf{F} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad \mathbf{Q} = \begin{bmatrix} \sigma_{y^*}^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{R}_t = \begin{bmatrix} (\kappa_t \sigma_{\tilde{y}})^2 & 0 \\ 0 & (\kappa_t \sigma_\pi)^2 \end{bmatrix}$$

The vector of parameters to be estimated by maximum likelihood is as follows:

$$\theta_1 = \left[a_{y,1}, a_{y,2}, b_\pi, b_y, g, \sigma_{\tilde{y}}, \sigma_\pi, \sigma_{y*}, \phi, \kappa_{2020Q2-Q4}, \kappa_{2021}, \kappa_{2022}\right]$$

of each variable as in HLW. This is purely an accounting change and has no effect on the estimates.

7.1.2 The COVID-adjusted Stage 2 Model

The second-stage model, which corresponds to the *rstar.stage2.R* program, can be represented by the following matrices:

$$\mathbf{y}_t = [y_t, \pi_t]' \tag{17}$$

$$\mathbf{x}_t = [y_{t-1}, y_{t-2}, r_{t-1}, r_{t-2}, \pi_{t-1}, \pi_{t-2,4}, 1, d_t, d_{t-1}, d_{t-2}]' \tag{18}$$

$$\xi_t = \left[y_t^*, y_{t-1}^*, y_{t-2}^*, g_t, g_{t-1}, g_{t-2}\right]' \tag{19}$$

$$\mathbf{H}' = \begin{bmatrix} 1 & -a_{y,1} & -a_{y,2} & 0 & \frac{a_g}{2} & \frac{a_g}{2} \\ 0 & -b_y & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{A}' = \begin{bmatrix} a_{y,1} & a_{y,2} & \frac{a_r}{2} & \frac{a_r}{2} & 0 & 0 & a_0 & \phi & -\phi a_{y,1} & -\phi a_{y,2} \\ b_y & 0 & 0 & 0 & b_\pi & 1-b_\pi & 0 & 0 & -\phi b_y & 0 \end{bmatrix}$$

$$\mathbf{F} = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \quad \mathbf{Q} = \begin{bmatrix} \sigma_{y^*}^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & (\lambda_g \sigma_{y^*})^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{R}_t = \begin{bmatrix} (\kappa_t \sigma_{\tilde{y}})^2 & 0 \\ 0 & (\kappa_t \sigma_\pi)^2 \end{bmatrix}$$

The vector of parameters to be estimated by maximum likelihood is as follows:

$$\theta_2 = \left[a_{y,1}, a_{y,2}, a_r, a_0, a_g, b_\pi, b_y, \sigma_{\tilde{y}}, \sigma_\pi, \sigma_{y*}, \phi, \kappa_{2020Q2-Q4}, \kappa_{2021}, \kappa_{2022}\right]$$

7.1.3 The COVID-adjusted Stage 3 Model

The third-stage model, which corresponds to the *rstar.stage3.R* program, can be represented by the following matrices:

$$\mathbf{y}_t = [y_t, \pi_t]' \tag{20}$$

$$\mathbf{x}_t = [y_{t-1}, y_{t-2}, r_{t-1}, r_{t-2}, \pi_{t-1}, \pi_{t-2,4}, d_t, d_{t-1}, d_{t-2}]' \tag{21}$$

$$\xi_t = \left[y_t^*, y_{t-1}^*, y_{t-2}^*, g_t, g_{t-1}, g_{t-2}, z_t, z_{t-1}, z_{t-2}\right]' \tag{22}$$

$$\mathbf{H}' = \begin{bmatrix} 1 & -a_{y,1} & -a_{y,2} & 0 & -4c\cdot\frac{a_r}{2} & -4c\cdot\frac{a_r}{2} & 0 & \frac{-a_r}{2} & \frac{-a_r}{2} \\ 0 & -b_y & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{A}' = \begin{bmatrix} a_{y,1} & a_{y,2} & \frac{a_r}{2} & \frac{a_r}{2} & 0 & 0 & \phi & -\phi a_{y,1} & -\phi a_{y,2} \\ b_y & 0 & 0 & 0 & b_\pi & 1-b_\pi & 0 & -\phi b_y & 0 \end{bmatrix}$$

$$\mathbf{F} = \begin{bmatrix} 1&0&0&1&0&0&0&0&0\\ 1&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&1&0 \end{bmatrix}, \quad \mathbf{Q} = \begin{bmatrix} \sigma_{y^*}^2&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&(\lambda_g\sigma_{y^*})^2&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&\left(\frac{\lambda_z\sigma_{\tilde{y}}}{a_r}\right)^2&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0 \end{bmatrix},$$

$$\mathbf{R}_t = \begin{bmatrix} (\kappa_t\sigma_{\tilde{y}})^2 & 0 \\ 0 & (\kappa_t\sigma_\pi)^2 \end{bmatrix}$$

The vector of parameters to be estimated by maximum likelihood is as follows:

$$\theta_3 = \left[a_{y,1}, a_{y,2}, a_r, b_\pi, b_y, \sigma_{\tilde{y}}, \sigma_\pi, \sigma_{y*}, \phi, c, \kappa_{2020Q2-Q4}, \kappa_{2021}, \kappa_{2022}\right]$$

The law of motion for the natural rate of interest is $r_t^* = c\cdot g_t + z_t$.

7.2 The HLW (2017) State-Space Models

7.2.1 The Stage 1 Model

The first-stage model, which corresponds to the *rstar.stage1.R* program, can be represented by the following matrices:

$$\mathbf{y}_t = [y_t, \pi_t]' \tag{23}$$

$$\mathbf{x}_t = [y_{t-1}, y_{t-2}, \pi_{t-1}, \pi_{t-2,4}]' \tag{24}$$

$$\xi_t = \left[y_t^*, y_{t-1}^*, y_{t-2}^*\right]' \tag{25}$$

$$\mathbf{H}' = \begin{bmatrix} 1 & -a_{y,1} & -a_{y,2} \\ 0 & -b_y & 0 \end{bmatrix}, \quad \mathbf{A}' = \begin{bmatrix} a_{y,1} & a_{y,2} & 0 & 0 \\ b_y & 0 & b_\pi & 1-b_\pi \end{bmatrix}$$

$$\mathbf{F} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad \mathbf{Q} = \begin{bmatrix} \sigma_{y^*}^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{R} = \begin{bmatrix} \sigma_{\widetilde{y}}^2 & 0 \\ 0 & \sigma_\pi^2 \end{bmatrix}$$

The vector of parameters to be estimated by maximum likelihood is as follows:

$$\theta_1 = \left[a_{y,1}, a_{y,2}, b_\pi, b_y, g, \sigma_{\widetilde{y}}, \sigma_\pi, \sigma_{y*}\right]$$

7.2.2 The Stage 2 Model

The second-stage model, which corresponds to the *rstar.stage2.R* program, can be represented by the following matrices:

$$\mathbf{y}_t = [y_t, \pi_t]' \tag{26}$$

$$\mathbf{x}_t = [y_{t-1}, y_{t-2}, r_{t-1}, r_{t-2}, \pi_{t-1}, \pi_{t-2,4}, 1]' \tag{27}$$

$$\xi_t = \left[y_t^*, y_{t-1}^*, y_{t-2}^*, g_{t-1}\right]' \tag{28}$$

$$\mathbf{H}' = \begin{bmatrix} 1 & -a_{y,1} & -a_{y,2} & a_g \\ 0 & -b_y & 0 & 0 \end{bmatrix}, \quad \mathbf{A}' = \begin{bmatrix} a_{y,1} & a_{y,2} & \frac{a_r}{2} & \frac{a_r}{2} & 0 & 0 & a_0 \\ b_y & 0 & 0 & 0 & b_\pi & 1-b_\pi & 0 \end{bmatrix}$$

$$\mathbf{F} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{Q} = \begin{bmatrix} \sigma_{y^*}^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & (\lambda_g \sigma_{y^*})^2 \end{bmatrix}, \quad \mathbf{R} = \begin{bmatrix} \sigma_{\tilde{y}}^2 & 0 \\ 0 & \sigma_\pi^2 \end{bmatrix}$$

The vector of parameters to be estimated by maximum likelihood is as follows:

$$\theta_2 = \left[a_{y,1}, a_{y,2}, a_r, a_0, a_g, b_\pi, b_y, \sigma_{\tilde{y}}, \sigma_\pi, \sigma_{y*}\right]$$

7.2.3 The Stage 3 Model

The third-stage model, which corresponds to the *rstar.stage3.R* program, can be represented by the following matrices:

$$\mathbf{y}_t = [y_t, \pi_t]' \tag{29}$$

$$\mathbf{x}_t = [y_{t-1}, y_{t-2}, r_{t-1}, r_{t-2}, \pi_{t-1}, \pi_{t-2,4}]' \tag{30}$$

$$\xi_t = \left[y_t^*, y_{t-1}^*, y_{t-2}^*, g_{t-1}, g_{t-2}, z_{t-1}, z_{t-2}\right]' \tag{31}$$

$$\mathbf{H}' = \begin{bmatrix} 1 & -a_{y,1} & -a_{y,2} & \frac{-a_r}{2} & \frac{-a_r}{2} & \frac{-a_r}{2} & \frac{-a_r}{2} \\ 0 & -b_y & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{A}' = \begin{bmatrix} a_{y,1} & a_{y,2} & \frac{a_r}{2} & \frac{a_r}{2} & 0 & 0 \\ b_y & 0 & 0 & 0 & b_\pi & 1-b_\pi \end{bmatrix}$$

$$\mathbf{F} = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \quad \mathbf{Q} = \begin{bmatrix} \left(1+\lambda_g^2\right)\sigma_{y^*}^2 & 0 & 0 & (\lambda_g \sigma_{y^*})^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ (\lambda_g \sigma_{y^*})^2 & 0 & 0 & (\lambda_g \sigma_{y^*})^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \left(\frac{\lambda_z \sigma_{\tilde{y}}}{a_r}\right)^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\mathbf{R} = \begin{bmatrix} \sigma_{\tilde{y}}^2 & 0 \\ 0 & \sigma_\pi^2 \end{bmatrix}$$

The vector of parameters to be estimated by maximum likelihood is as follows:

$$\theta_3 = \left[a_{y,1}, a_{y,2}, a_r, b_\pi, b_y, \sigma_{\widetilde{y}}, \sigma_\pi, \sigma_{y*}\right]$$

The law of motion for the natural rate of interest is $r_t^* = g_t + z_t$.

Appendix A2: Data

For each economy, we require data for real GDP, inflation, and the short-term nominal interest rate, as well as a procedure to compute inflation expectations to calculate the ex ante real short-term interest rate r_t.[10] The variable y_t refers to the logarithm of real GDP. The inflation measure is the annualized quarterly growth rate of the specified consumer price series. With the exception of the United States, for which core personal consumption expenditure (PCE) price data are available over the entire sample, the inflation series is constructed by splicing the core price index with an all-items price index. We use a four-quarter moving average of past inflation as a proxy for inflation expectations in constructing the ex ante real interest rate. Short-term interest rates are expressed on a 365-day annualized basis.

For the United States, we use real GDP and core PCE data published by the Bureau of Economic Analysis. Inflation is constructed using the price index for PCE excluding food and energy, referred to as core PCE inflation. The short-term interest rate is the annualized nominal federal funds rate, available from the Board of Governors. Because the federal funds rate frequently fell below the discount rate prior to 1965, we use the Federal Reserve Bank of New York's discount rate, part of the IMF's International Financial Statistics Yearbooks (IFS), prior to 1965. All U.S. data can be downloaded from the St. Louis Fed's Federal Reserve Economic Data (FRED) website.[11]

Canadian real GDP data is taken from the IMF's IFS. The short-term nominal interest rate is the Bank of Canada's target for the overnight rate, taken as the end-of-period value for each month and aggregated to quarterly frequency. Since the Bank of Canada began treating the target rate as its key interest rate in May 2001, we use the bank rate as the short-term interest rate prior to that date. We use the Bank of Canada's core Consumer Price Index to construct our inflation series. Prior to 1984, we use CPI containing all items. With the exception of real GDP, all data is from Statistics Canada.[12]

Euro Area data is from the Area Wide Model (AWM), available from the Euro Area

[10] A detailed description of our data and programs, as well as replication materials for the standard HLW model, is available on the Federal Reserve Bank of New York's website.

[11] Mnemonics are as follows. Real GDP: GDPC1; Core PCE: PCEPILFE; Federal Funds Rate: FEDFUNDS; FRBNY Discount Rate: INTDSRUSM193N.

[12] Mnemonics from Statistics Canada are as follows. Core CPI: v41690926 (Table 326-0022); CPI: v41690914 (Table 326-0022); v41690973 (Table 326-0020); Bank Rate: v122530 (Table 176-0043); Target Rate: v39079 (Table 176-0048). Real GDP is IFS series "Gross Domestic Product, Real, Seasonally adjusted, Index".

Business Cycle Network (Fagan et al., 2001). The inflation measure is based on the core price index, HICP excluding energy (series HEX) beginning in 1988; prior to 1988 we use the overall price index HICP. The nominal short-term interest rate is the three-month rate (series STN) and the real GDP mnemonic is YER. At the time of publication, the final update to the Area Wide Model was in 2017; we update the three series from the ECB's Statistical Data Warehouse.[13]

[13] Mnemonics for the SDW are as follows. Real GDP: MNA.Q.Y.I8.W2.S1.S1.B.B1GQ._Z._Z._Z.EUR.LR.N; Core HICP: ICP.M.U2.N.XE0000.4.INX; Nominal Short-term Rate: FM.Q.U2.EUR.RT.MM.EURIBOR3MD_.HSTA. Because data availability is longer for the non-seasonally adjusted price series, we use those and seasonally adjust them.

Appendix A3: Pre-COVID Estimates from Modified and Standard Models

Table A1 reports parameter estimates from the model estimated through 2019:Q4. Figures 10-12 compare estimates from the original HLW (2017) model and the modified model through 2019:Q4. Both use the current data vintage at the time of publication.

Table A1: Parameter Estimates, Sample Ending 2019Q4

Parameter	United States	Canada	Euro Area
Sample	1961-2019	1961-2019	1972-2019
λ_g	0.053	0.052	0.036
λ_z	0.031	0.016	0.032
$\sum a_y$	0.941	0.951	0.950
a_r	-0.067	-0.065	-0.037
	(3.973)	(2.969)	(1.798)
b_y	0.076	0.047	0.062
	(3.077)	(1.697)	(1.713)
c	1.198	1.130	0.816
	(3.484)	(2.716)	(0.888)
$\sigma_{\tilde{y}}$	0.344	0.395	0.289
σ_π	0.794	1.359	0.972
σ_{y^*}	0.568	0.581	0.397
σ_g	0.121	0.121	0.058
σ_z	0.157	0.096	0.246
$\sigma_{r^*} = \sqrt{c^2\sigma_g^2 + \sigma_z^2}$	0.213	0.167	0.251
S.E. (sample ave.)			
r^*	1.236	1.625	3.480
g	0.395	0.422	0.260
y^*	1.559	2.470	1.875
S.E. (final obs.)			
r^*	1.656	1.844	4.822
g	0.546	0.574	0.357
y^*	1.971	2.761	2.429

Notes: t statistics are in parentheses; σ_g is expressed at an annual rate. Sample through 2019Q4, using current data vintage at time of publication.

Figure 10: Modified Model vs. HLW (2017) Model: US

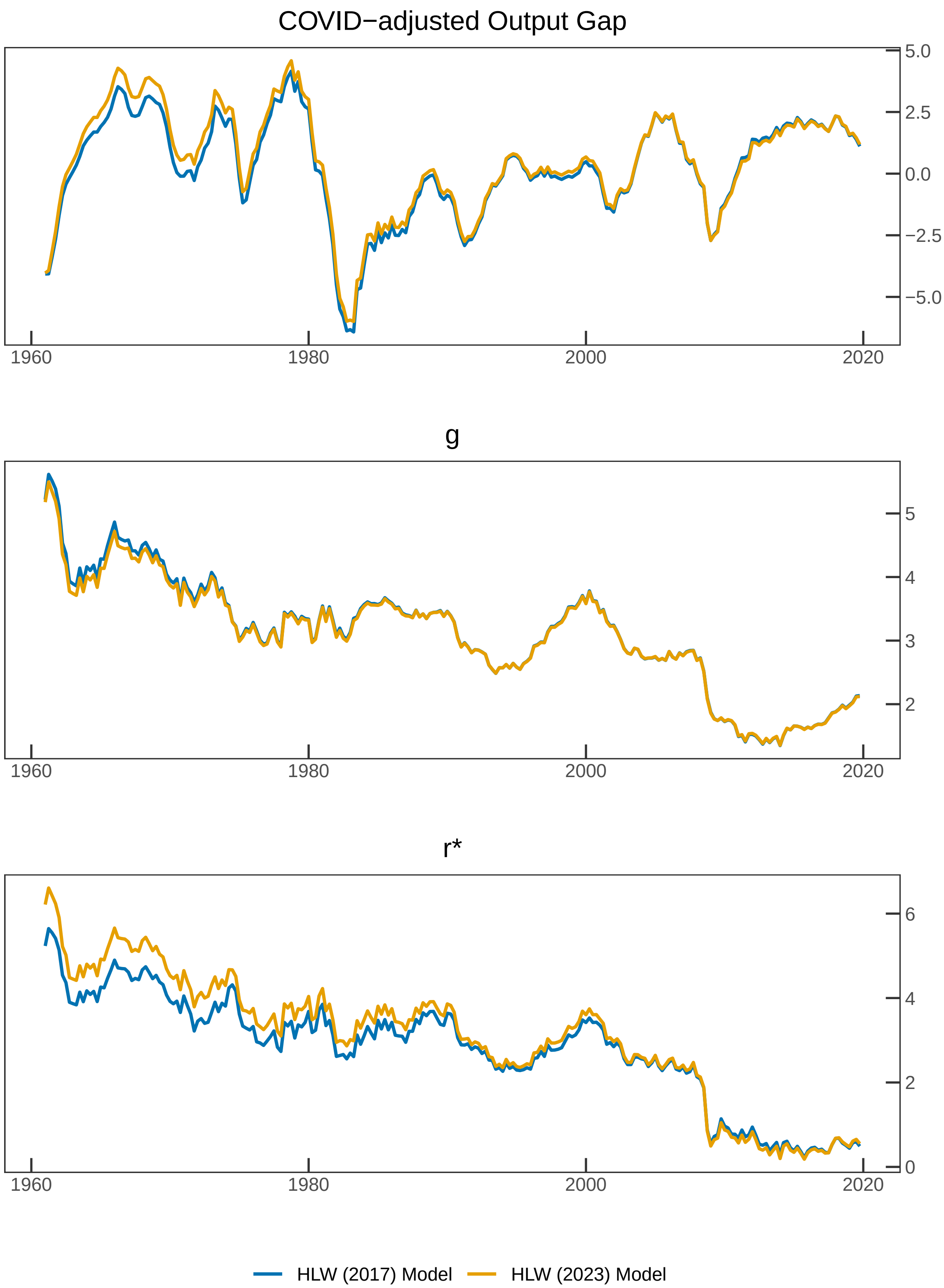


Figure 11: Modified Model vs. HLW (2017) Model: Canada

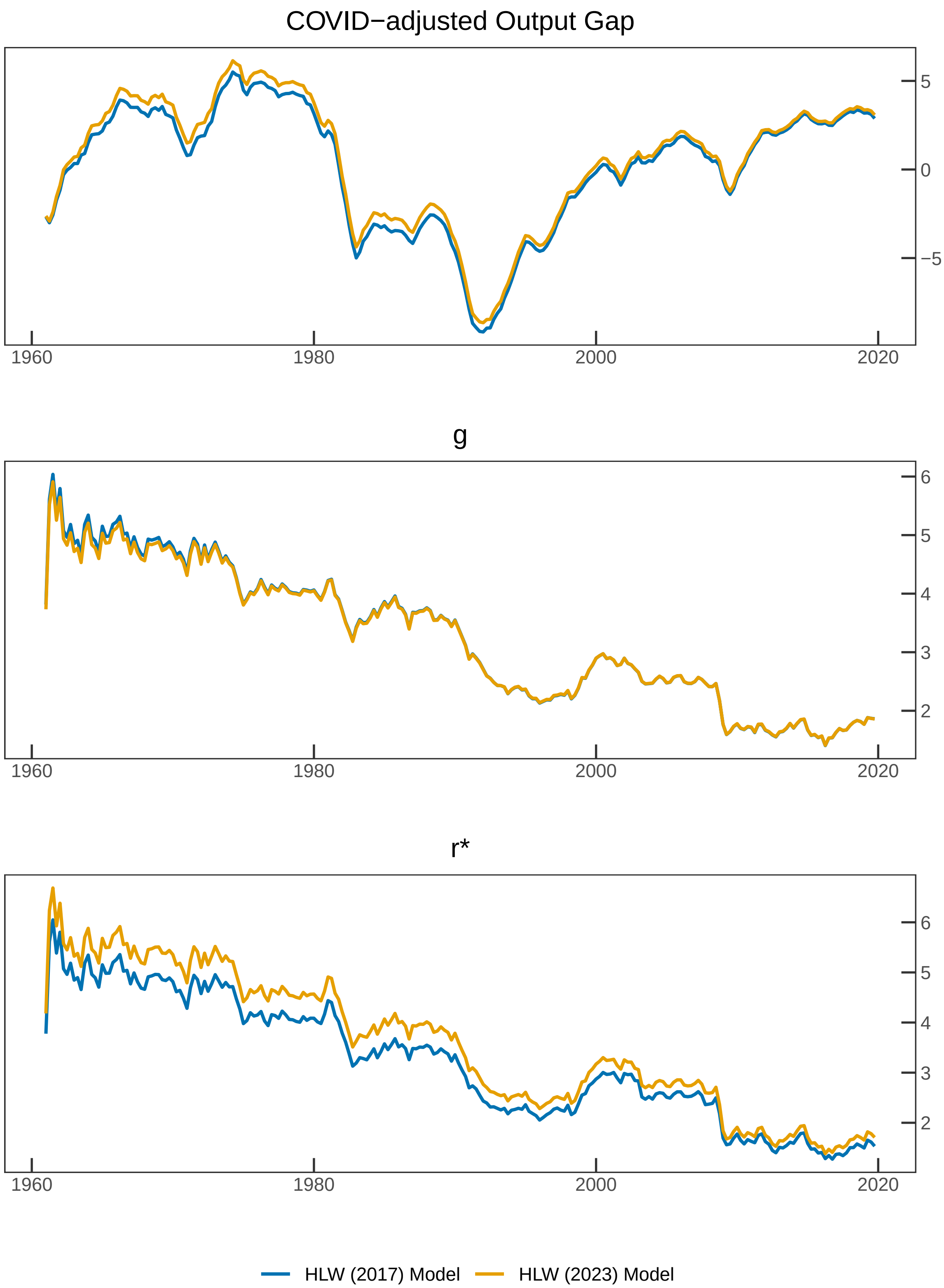


Figure 12: Modified Model vs. HLW (2017) Model: Euro Area