STAFF
REPORTS

NO. 1110
AUGUST 2024

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Federal Reserve Bank of New York Staff Reports, no. 1110
August 2024
https://doi.org/10.59576/sr.1110

Abstract

Binned scatter plots are a powerful statistical tool for empirical work in the social, behavioral, and biomedical sciences. Available methods rely on a quantile-based partitioning estimator of the conditional mean regression function to primarily construct flexible yet interpretable visualization methods, but they can also be used to estimate treatment effects, assess uncertainty, and test substantive domain-specific hypotheses. This paper introduces novel binscatter methods based on nonlinear, possibly nonsmooth M-estimation methods, covering generalized linear, robust, and quantile regression models. We provide a host of theoretical results and practical tools for local constant estimation along with piecewise polynomial and spline approximations, including (i) optimal tuning parameter (number of bins) selection, (ii) confidence bands, and (iii) formal statistical tests regarding functional form or shape restrictions. Our main results rely on novel strong approximations for general partitioning-based estimators covering random, data-driven partitions, which may be of independent interest. We demonstrate our methods with an empirical application studying the relation between the percentage of individuals without health insurance and per capita income at the zip-code level. We provide general-purpose software packages implementing our methods in Python, R, and Stata.

JEL classification: C14, C18, C21

Key words: partition-based semi-linear estimators, generalized linear models, quantile regression, robust bias correction, uniform inference, binning selection, treatment effect estimation

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1 Introduction

Data visualization is a crucial step in any statistical analysis. In the era of big data it has become increasingly important to have simple yet informative visual tools to guide, supplement, or in some cases even supplant, numerical statistical analyses. However, it is important to maintain statistical formality and rigor to ensure the validity of any conclusions based on the data. We seek to accomplish both of these goals—effective visualization couched in a formal framework—with binned scatter plot methods.

Often known simply as a *binscatter*, the binned scatter plot has become a popular tool for visualization in large data sets, particularly in the social and behavioral sciences. The goal is to flexibly estimate, and visualize, features of the conditional distribution of a scalar outcome y_i, which may be discrete or continuous, given a covariate or treatment variable x_i, which is scalar and continuous, while possibly also controlling for a d-dimensional vector of additional factors $\mathbf{w}_i$. For estimating the conditional mean, as in the traditional regression analysis, binning has a long history: so familiar is this approach that over 60 years ago Tukey (1961), calling it a *regressogram*, went so far as to claim that "[a]ll statisticians who handle data know how to attack the simple case of this situation where y and x are both single real numbers" (p. 682). Going on to describe the construction, Tukey writes: "[t]he x-axis is to be divided into suitable intervals, the mean of all the y-values corresponding to x-values falling in each given interval is to be found, the results are then to be plotted, either as points, each located above the center of the corresponding x-interval, or better as horizontal bars, each extending over the corresponding x-interval" (p. 682).

This simple construction (perhaps disappointingly often with plotted dots rather than horizontal lines) has recently gained popularity in statistics, economics, and data science. The prevalence of binscatter plots can be partly ascribed to its intuitive construction and compelling visualization properties: given only data on x_i and y_i, the plot is a clean and interpretable depiction of the conditional mean. Moreover, several limitations of the classical scatter plot account for the rising use of binned scatter plots in modern analyses. First, in big data sets the classical scatter plot is too dense to be informative, particularly about general "patterns" in the data which are to be modeled in subsequent analyses. Second, somewhat conversely, in cases where privacy is a concern the scatter plot is not allowed, regardless of its informational use as a visualization tool. Third, classical scatter plots do not provide a well-defined way to control for other factors, a common goal in treatment effect estimation and causal inference. Finally, particularly relevant to our setting, a scatter plot is not useful when outcomes are discrete. In contrast, a *binned* scatter plot provides a simple, yet flexible way of visualizing features of the conditional distribution of a (possibly discrete) outcome variable given a continuous covariate (or treatment) of interest, while controlling for other important factors.

Formally, a binscatter is grounded in the classical semilinear regression model. To date, however, binscatters have been available only to visualize (and estimate) conditional mean functions fitted using least squares. A common usage in this setting is comparing the nonparametric estimate to a linear fit, as a precursor to linear regression analysis. See Starr and Goldfarb (2020) for a practical review and background references. In the least squares setting, a binscatter is formally an estimator of a semilinear model for the conditional mean, nonparametric in the covariate of interest and linear in the controls, where the nonparametric component is estimated by partitioned regression. Cattaneo et al. (2024b) used that framework to derive formal statistical properties of canonical binscatter, including correcting a common mistake in empirical practice when using controls, and provide asymptotically valid confidence bands and optimal tuning parameter selection.

The restriction to least squares semilinear regression to estimate the conditional mean has limited the applicability of binscatter methods. For one, important features of the data, such as spread or variability, cannot be visualized. Further, existing methods (and theory) can be misleading in settings where the outcome is discrete or in another way restricted. For example, in the empirical illustration we use throughout, we study uninsuredness rates using a fractional outcome model, most naturally fitted using quasi-likelihood methods based on the logistic link. Last but not least, binscatter methods for quantile regression analysis are currently lacking in the literature, despite of their usefulness for empirical work.

This paper introduces and studies a broad class of binscatter M-estimation methods, in models allowing for (i) a nonlinear and/or nonsmooth loss function and (ii) a nonlinear link function. Our results provide for the use of binned scatter plots for various visualization goals and different data types, particular leading cases being semiparametric conditional quantile regression and generalized partially linear models. We make several methodological and theoretical contributions: (i) we propose a feasible method for optimal tuning parameter selection to choose the appropriate number of bins; (ii) we provide (pointwise and) uniform inference to construct confidence bands and hypothesis tests for parametric specifications and shape restrictions, and (iii) we develop group-wise comparisons for continuous treatment effects or for treatment effect heterogeneity. Developing these methods relies on novel technical work: allowing for a large class of binning methods, including random binning, we prove new uniform (in x) Bahadur representations and strong approximations, and thus uniform distribution theory, for the broad class of nonlinear semiparametric models considered. Obtaining these results for nonlinear, nonsmooth models, with data-dependent partitions and additional covariates, represents the main technical contributions of our paper, some of which may be of independent interest.

Our proposed nonlinear binscatter methods help restore, and in cases such as discrete outcomes or additional controls, surpass, the utility of the conventional scatter plot. We offer principled ways to visually assess patterns in the data, quantify uncertainty, and develop hypothesis tests about the findings. Our results on quantile regression allow researchers to assess the spread of the conditional distribution, detect outliers or influential observations in the data, and study a larger class of treatment effects, formalizing and expanding common practices based on the classical scatter plot in small data sets, all while controlling for additional important factors. Our confidence bands properly quantify and communicate the uncertainty around the estimated function of interest, and can also be used to guide further analyses. We also develop formal uniform hypothesis testing procedures regarding those functions, to assess shape constraints and parametric specifications. Causal inference is an important application area of our uniform inference results: studying treatment effect heterogeneity for binary treatments or the dose response function for a continuous treatment without imposing a functional form (e.g., to evaluate important hypotheses such as monotonicity in the dosage).

The paper proceeds as follows. We next discuss the connections between our work and the existing literature. Section 2 introduces binned scatter plots, defines the statistical model, and clarifies the parameters of interest. Section 3 gives details on our theoretical contributions, which are then used in Sections 4 and 5 to deliver tuning parameter selection and uniform inference. We illustrate our methods and results with a running empirical application using zip code-level data from the American Community Survey (ACS). The dependent variable, y_i, is the percentage of individuals without health insurance, and the independent variable of interest, x_i, is per capita income. Section 6 concludes. The online Supplemental Appendix (SA hereafter) contains additional technical and implementation details, all mathematical proofs, and further discussion of how our technical contributions improve on the related literature. General-purpose software in `Python`, `R`, and `Stata`, as well as replication files, are available at https://nppackages.github.io/binsreg/. See Cattaneo et al. (2024a) for an introduction.

1.1 Related Literature

This paper contributes to several strands of the literature. First, from a practical point of view, our work builds upon and extends existing binned scatter plot methods available for applied research. See Starr and Goldfarb (2020) for a review of that literature and Cattaneo et al. (2024b) for formal results concerning least squares semilinear binscatter. Our main methodological contribution is to introduce nonlinear binscatter methods, constructed using a general, possibly nonsmooth semilinear M-estimation approach. As a result, we propose a broad array of new binscatter methods for generalized linear models (e.g., Logit or Probit), robust semiparametric regression (e.g., Huber or trimmed least squares), and quantile regression.

Second, our theoretical results contribute to the literature on series/sieve estimation in general, and partitioning-based methods in particular (i.e., piecewise polynomials and splines approximations). See Györfi et al. (2002) for a textbook introduction, and Shen et al. (1998), Huang (2003), Belloni et al. (2015), Cattaneo and Farrell (2013), Cattaneo et al. (2020), as well as references therein, for prior convergence rates and distribution theory. These prior works studied uniform estimation and inference for linear piecewise polynomials and spline series regression without data-driven partitioning and without additional covariates, often imposing strong regularity conditions. Our primary technical contributions as compared to the recent literature are (i) allowing for general, possibly nonlinear and nonsmooth M-estimation, (ii) allowing for random partitions and hence random basis functions in the series estimator, (iii) controlling for other factors in a semilinear model, and (iv) obtaining novel strong approximations and uniform inference under weaker conditions than those previously available. A substrand of the series estimation literature studies quantile regression, the closest antecedent to our work being Belloni et al. (2019). Unlike that prior work, we consider general nonlinear, possibly nonsmooth, M-estimation problems and allow for random partitions and additional controls, and our technical results are obtained under weaker regularity conditions, which in particular permit the use of piecewise constant fitting necessary for a binned scatter plot. Finally, none of the results in Cattaneo et al. (2024b) are applicable to the large class of nonlinear binscatter estimators considered in this paper, because they only consider least squares semilinear binscatter models. Further details of how each of our individual theoretical results improves on the extant literature is given throughout the SA.

Finally, our paper also contributes to the literature on data visualization, which has become an increasingly active field of study in recent years due to the rise of big data and machine learning methods. Our results speak directly to this literature, and in particular to the need for clear and explicit depictions of uncertainty, both in terms of variance and estimation error (Healy, 2018). These are crucial in data visualization in science and research contexts as this "builds trust and credibility" (Schwabish, 2021, p.189).

2 Setup

A binned scatter plot is designed to provide a flexible, nonparametric estimate of a regression function. The construction and interpretation of a binned scatter plot is simple and intuitive, which drives their appeal for applied work. But as we will see, there are some subtleties when binned scatter plots are applied to nonlinear, nonsmooth models—especially when controlling for additional covariates.

To describe the construction, it is helpful to first make precise the model and objects of interest. Our goal is to learn a regression function (which need not be the conditional mean) that features in the conditional distribution of a scalar outcome y_i, which may be discrete or continuous, given a covariate or treatment variable x_i, which is scalar and continuous, while possibly also controlling for a d-dimensional vector of additional factors $\mathbf{w}_i$. In applications, the goal is to flexibly study the relationship of y_i to x_i, but not necessarily to discover (or allow for) heterogeneity or nonlinearity in $\mathbf{w}_i$. Further, $\mathbf{w}_i$ is often a large-dimensional set of controls, such as fixed effects or factor variables. Consequently, we assume the regression function depends on the scalar index $\theta_0(x_i, \mathbf{w}_i) := \mu_0(x_i) + \mathbf{w}_i'\boldsymbol{\gamma}_0$, for an unknown function μ_0 and vector $\boldsymbol{\gamma}_0$, and is thus partially linear in nature. This specification is directly interpretable, and in cases where d is moderate or large, empirically convenient.

The model is defined by the following structure, which determines how the scalar index $\theta_0(x_i, \mathbf{w}_i)$ relates to the outcome y_i. Let θ be a generic value of the index. For a loss function $\rho(y; \eta(\theta))$ and inverse link function $\eta(\theta)$, let

$$(\mu_0(\cdot), \boldsymbol{\gamma}_0) = \underset{\mu \in \mathcal{M}, \boldsymbol{\gamma} \in \mathbb{R}^d}{\arg\min} \ \mathbb{E}\big[\rho(y_i; \eta(\mu(x_i) + \mathbf{w}_i'\boldsymbol{\gamma}))\big], \tag{2.1}$$

where we assume the solution is unique, and $\rho(y;\eta(\theta))$, $\eta(\theta)$, and the function class $\mathcal{M}$ obey typical boundedness and smoothness restrictions discussed in Section 3. For different choices of $\rho(\cdot)$ and $\eta(\cdot)$ this formulation covers a large class of problems including generalized linear models, robust regression, quantile regression, and nonlinear least squares. We illustrate with some leading specific examples.

Example 1 (Least Squares Regression)**.** Setting $\eta(\theta) = \theta$ and $\rho(y;\eta) = (y-\eta)^2$ recovers semiparametric least squares regression for partially linear models. ⌟

Example 2 (Logistic Regression)**.** Assume that the binary outcome y_i, conditional on x_i and $\mathbf{w}_i$, is distributed Bernoulli with probability $\eta(\mu(x_i) + \mathbf{w}_i'\boldsymbol{\gamma})$, where $\eta(\theta) = (1+\exp(-\theta))^{-1}$, then $\rho(y;\eta) = -y\log(\eta) - (1-y)\log(1-\eta)$. ⌟

Example 3 (Huber Regression)**.** Semiparametric robust partially linear regression sets $\eta(\theta) = \theta$ and $\rho(y;\eta) = (y-\eta)^2\mathbb{1}(|y-\eta| \leq \tau) + \tau(2|y-\eta| - \tau)\mathbb{1}(|y-\eta| > \tau)$ for a user-specified $\tau > 0$. ⌟

Example 4 (Quantile Regression)**.** Set $\rho(y;\eta) = [\tau - \mathbb{1}(y < \eta)](y-\eta)$ with $\eta(\theta) = \theta$ for a user-specified quantile $\tau \in (0,1)$. ⌟

The key statistical challenge is (uniform in x) recovery of the function $\mu_0(x)$ for estimation and inference. Once accomplished, we can cover a wide variety of objects derived from (2.1). For concreteness we will focus on the following three objects, as they are of primary practical importance:

(i) the level of the regression function, $\vartheta_0(x,\mathsf{w}) = \eta(\mu_0(x) + \mathsf{w}'\boldsymbol{\gamma}_0) = \eta(\theta_0(x,\mathsf{w}))$,

(ii) the nonparametric component itself (or its derivative), $\mu_0^{(v)}(x) = \frac{d^v}{dx^v}\mu_0(x)$, $v \geq 0$, and

(iii) the marginal effect $\zeta_0(x,\mathsf{w}) = \frac{\partial}{\partial x}\eta(\mu_0(x) + \mathsf{w}'\boldsymbol{\gamma}_0) = \eta^{(1)}(\theta_0(x,\mathsf{w}))\mu_0^{(1)}(x)$,

where $h^{(1)}(u) = \frac{d}{du}h(u)$ denotes the derivative of a function with respect to its scalar argument and w is a user-chosen evaluation point for the additional controls. Typically, w

is chosen as the mean or median, or for discrete variables or fixed effects, set to a baseline category. The role of w in both plotting and inference introduces some important nuances that are discussed below.

Each of these parameters corresponds to different empirical questions. The level, $\vartheta_0(x, \mathsf{w})$, is directly useful for visualization of the relationship of y_i to x_i and is commonly used in causal inference. If the variable x_i is a continuous treatment, our results yield a nonparametric estimate of the dose response function, while controlling for relevant factors $\mathbf{w}_i$. A plot of $\vartheta_0(x, \mathsf{w})$ shows this function for the subgroup defined by $\mathbf{w}_i = \mathsf{w}$. We can also obtain separate dose response functions for different subgroups of the data to be used in multi-sample comparisons. On the other hand, if x_i is a pre-treatment variable, the same multi-sample results provide an analysis of treatment effect heterogeneity for discrete (often binary) treatments, and our uniform inference allows for discovery of treatment effect heterogeneity. Finally, for visualization, obtaining $\vartheta_0(x, \mathsf{w})$ in the quantile regression case can be used to assess the spread of the conditional distribution (especially for quantiles close to zero and one) or robust measures of central tendancy, as would be done with a classical scatter plot.

The nonparametric component, $\mu_0(x)$, is most often studied to assess its functional form, generally against a parsimonious parametric specification to be considered for later analyses or for a shape restriction that is of substantive interest, such as monotonicity or convexity. Historically, a common use of binscatter was to visually (and informally) assess if $\mu_0(x)$ is well-approximated by a linear model, and if so, proceeding under that specification for the empirical results. Our results provide rigor to such practice, and expand the idea to a much richer class of models and hypotheses.

The marginal (or partial) effect $\zeta_0(x, \mathsf{w})$ is a standard object in economic analysis in nonlinear models. In binary choice models it is common to study how the probability of $y = 1$ changes as a function of x. The marginal effect at the average, obtained by setting w to the sample mean, is a standard way to summarize nonlinear models by giving the effect for the "average" individual. For example, we show that the marginal effect of income changes sign in our application, indicating a changing response in the uninsuredness rate as a result of Medicaid. Comparing marginal effects across groups for heterogeneity analysis, in causal or noncausal settings, is a common goal in social science applications with nonlinear models.

2.1 Estimation

Given an i.i.d. sample $(y_i, x_i, \mathbf{w}_i')'$, $i = 1, \ldots, n$, the binscatter estimator is constructed by solving the empirical analogue of (2.1) using a partitioning-based approximation to the unknown function $\mu_0(x)$. This nonparametric approximation requires two choices: the partitioning of the support of x_i and the estimation within each bin.

To fix ideas, it is useful to begin with the simplest case where local constant fitting is used and $\mathbf{w}_i$ is absent. First, the support of x_i is divided into $J < n$ disjoint bins. J is the main tuning parameter for this nonparametric estimation problem, and its choice is crucial both visually and statistically. To describe the estimator, we will take $J < n$ as given at present, and return to its choice in Section 4 below.

Coupled with a choice of J is a method to divide the support. A major theoretical innovation of our work is that the bin breakpoints themselves can be data-dependent, distinct from a data-driven choice of J. The partition is denoted by $\widehat{\Delta} = \{\widehat{\mathcal{B}}_1, \widehat{\mathcal{B}}_2, \ldots, \widehat{\mathcal{B}}_J\}$, with the bins and breakpoints denoted by

$$\widehat{\mathcal{B}}_j = \begin{cases} \big[\widehat{\tau}_{j-1}, \widehat{\tau}_j\big) & \text{if } j = 1, 2, \ldots, J-1, \\ \big[\widehat{\tau}_{J-1}, \widehat{\tau}_J\big] & \text{if } j = J. \end{cases}$$

The breakpoints $\{\widehat{\tau}_j\}_{j=1}^{J}$ result from a user-chosen, possibly data-driven, partitioning method. Our theoretical results cover any random partition that is independent of the outcomes y_i's (given x_i's and $\mathbf{w}_i$'s), and "quasi-uniform", which intuitively requires the bins to be sufficiently similar. The formal condition is stated in the next section. The simplest approach is evenly-spaced breakpoint locations ($\widehat{\tau}_j = x_{\mathtt{min}} + j(x_{\mathtt{max}} - x_{\mathtt{min}})/J$). The most popular choice, however, is to use the empirical quantiles of x_i, setting $\widehat{\tau}_j = \widehat{F}^{-1}(j/J)$ with $\widehat{F}(u) = \frac{1}{n}\sum_{i=1}^{n} \mathbb{1}(x_i \leq u)$ the standard empirical cumulative distribution function, and $\widehat{F}^{-1}$ its generalized inverse. In the SA we show that both of these satisfy our generic assumptions. Other possible methods include certain adaptive regression trees and related partitioning methods, such as those with the so-called "X-property" or via sample splitting; see Devroye et al. (2013), Zhang and Singer (2010), and references therein. For concreteness, we will use quantile spacing for empirical analysis throughout the paper.

Given a partition $\widehat{\Delta}$, the binscatter estimate is formed by fitting the sample analogue of (2.1) within each bin, using only an intercept. In the simple case of least squares regression, this is identical to computing the sample average of y_i for observations in each bin, exactly as Tukey (1961) described, yielding a piecewise constant approximation to the unknown conditional expectation. The same method is followed for all other models. For example, in the case of binary data or fractional outcomes, a logistic regression of y_i on a constant is fit for each bin. For the median, or any other quantile, one simply computes the empirical quantile of y_i using observations only within the bin.

Formally, we define the basis functions $\widehat{\mathbf{b}}_0(x) = [\mathbb{1}_{\widehat{\mathcal{B}}_1}(x), \mathbb{1}_{\widehat{\mathcal{B}}_2}(x), \cdots, \mathbb{1}_{\widehat{\mathcal{B}}_J}(x)]'$, consisting of indicators for each bin. We then obtain

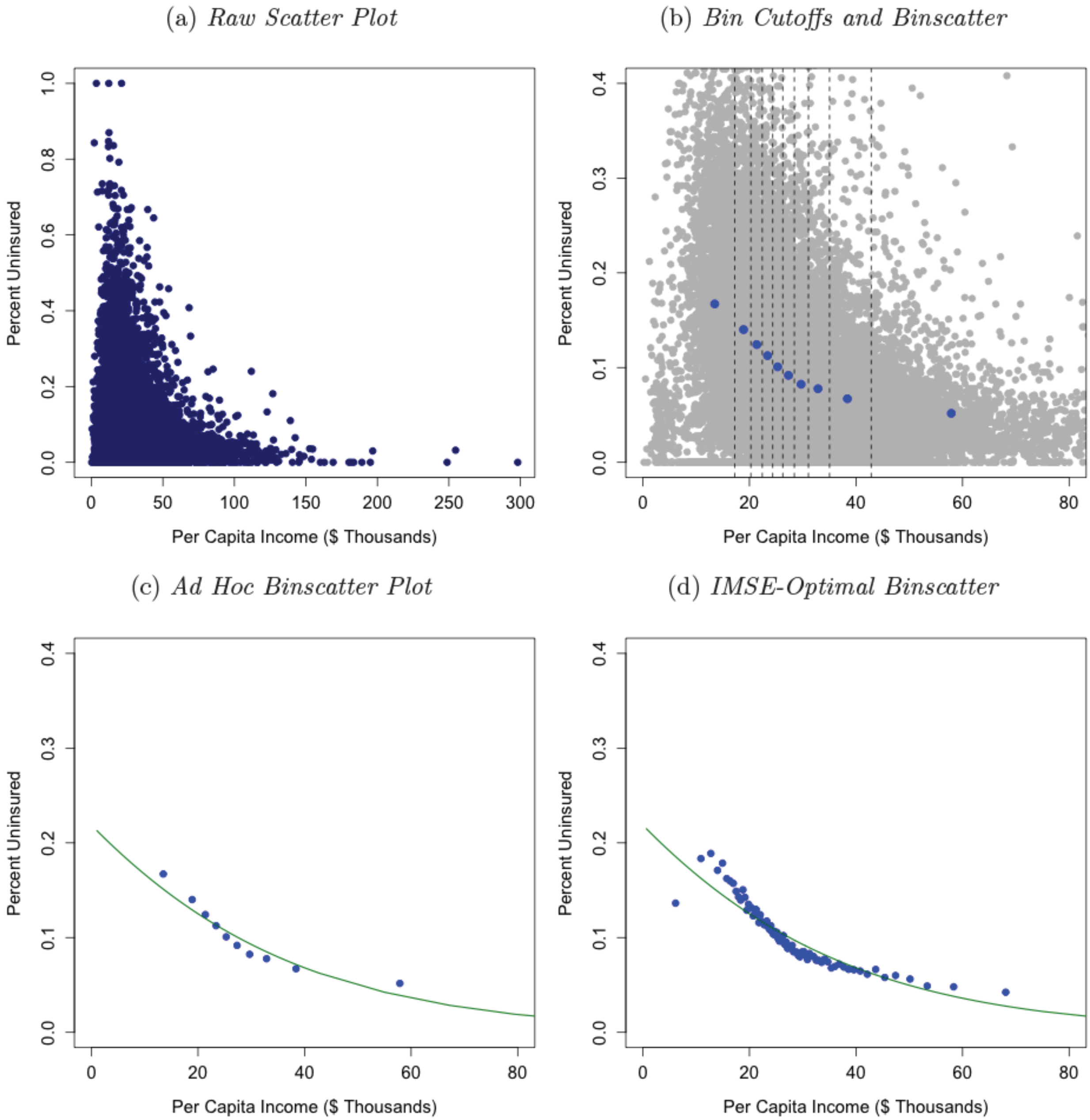
$$\widehat{\mu}(x) = \widehat{\mathbf{b}}_0(x)'\widehat{\boldsymbol{\beta}}, \qquad \widehat{\boldsymbol{\beta}} = \underset{\boldsymbol{\beta}\in\mathbb{R}^J}{\arg\min} \sum_{i=1}^{n} \rho\Big(y_i;\ \eta\big(\widehat{\mathbf{b}}_0(x_i)'\boldsymbol{\beta}\big)\Big). \tag{2.2}$$

A graphical illustration of this procedure is shown in Figure 1. The data are obtained from the ACS using the 5-year survey estimates beginning in 2013 and ending in 2017 (available from the Census Bureau website). All analyses are performed at the zip code tabulation area level for the United States (excluding Puerto Rico). The dependent variable, y_i, is the percentage of individuals without health insurance, and the independent variable of interest, x_i, is per capita income. The fractional nature of the outcome motivates the use of logistic quasi-maximum likelihood for estimation and inference (Papke and Wooldridge, 1996). Figure 1(a) shows the classical scatter plot of the raw data. This data set has about 32,000 observations, far from the millions commonly encountered, and already this plot fails to be useful for assessing the functional form: the visualization is dominated by a dense cloud of data with a few outlying observations. Figure 1(b) shows a binned scatter plot being constructed, with the raw data in the background. The dots are the fitted values of applying (2.2) following Example 2, i.e., we show $\widehat{\vartheta}(x) = \eta(\widehat{\mu}(x))$. Figure 1(c) isolates the binscatter and overlays a linear fit (i.e., a global logistic quasi-likelihood with $\mu_0(x)$ assumed linear in x). The linear approximation to $\mu_0(x)$ appears satisfactory at first, but this is because the nonparametric estimate is undersmoothed. Figure 1(d) presents the estimate using the optimal number of bins (Section 4), and shows that the informal analysis, relying on an ad hoc choice of J, would miss an important feature of the data: the presence of the Medicaid program which provides subsidized health insurance for limited-income individuals. As a preview, Table 1 below shows that formal tests reject polynomial parametric specifications and reject the hypothesis that the uninsurance rate is monotonically decreasing with per capita income.

Graphs like Figures 1(c) and (d) have a long tradition in statistics and data science, and have recently become ubiquitous in applied microeconomics. Visually assessing functional forms is the typical use. Importantly, in this case the visualization shows an estimate of $\vartheta_0(x, \mathsf{w})$, not $\mu_0(x)$ directly. Further, although the binned scatter plot invites the viewer to "connect the dots" smoothly, the actual estimator is piecewise constant, which generally gives a less appealing visualization but underpins any formal analysis.

We expand on (2.2) in two ways: adding the covariates $\mathbf{w}_i$ and enriching the set of allowable basis functions. The covariates can be directly incorporated into the loss, exactly as they are in (2.1). Moreover, the additively separable and linear nature of the controls makes this generalization straightforward empirically. Importantly, the presence of controls invalidates bin-by-bin estimation, as the coefficients $\boldsymbol{\gamma}_0$ are global parameters. The SA discusses different approaches to estimating μ_0 and $\boldsymbol{\gamma}_0$.

Figure 1: **Illustration of Nonlinear Binned Scatter Plots.** This figure illustrates the construction of a nonlinear binned scatter plot using the ACS data. All analyses are performed at the zip code tabulation area level for the United States (excluding Puerto Rico). The dependent variable is the percentage of individuals without health insurance and the independent variable of interest is per capita income.



Next, we replace the piecewise constant approximation based on $\widehat{\mathrm{b}}_0(x)$ with an order-p polynomial approximation in each bin that is $(s-1)$-times continuously differentiable at the breakpoints of the bins, with the convention that $s = 0$ corresponds to discontinuous fits and

$s = 1$ is continuous but nondifferentiable. This change to the basis gives additional flexibility that is crucial for bias reduction and derivative estimation, the latter being instrumental for studying shape restrictions and specification testing. By construction, $p \geq s \geq 0$, and derivative estimation requires that the derivative of interest, $v \geq 0$, is no larger than p. The general basis is defined as $\widehat{\mathbf{b}}_{p,s}(x) := \widehat{\mathbf{T}}_s[\widehat{\mathbf{b}}_0(x)' \otimes (1, x, \ldots, x^p)']$, where $\otimes$ denotes the Kronecker product and $\widehat{\mathbf{T}}_s$ is a $[(p+1)J - (J-1)s] \times (p+1)J$ transformation matrix ensuring that the $(s-1)$-th derivative of the estimate is continuous. The exact form of $\widehat{\mathbf{T}}_s$ is available in Section SA-5.2 of Cattaneo et al. (2024b), and we note that $\widehat{\mathbf{T}}_s$ also depends on the random partitions. When $s = 0$, $\widehat{\mathbf{T}}_s$ is the identity matrix, and the fit is a piecewise polynomial of degree p. The piecewise constant fit (as bars or as dots) corresponds to $s = p = 0$. Another popular choice are cubic B-splines, obtained by setting $s = p = 3$. On account of its popularity and to simplify notation, we will assume throughout the paper that $s = p$ and use the notation $\widehat{\mathbf{b}}_p(x) := \widehat{\mathbf{b}}_{p,p}(x)$. The SA treats the general case of $s \leq p$.

The generalized, covariate-adjusted binscatter can now be defined. We first solve

$$\begin{bmatrix} \widehat{\boldsymbol{\beta}} \\ \widehat{\boldsymbol{\gamma}} \end{bmatrix} = \underset{\boldsymbol{\beta},\boldsymbol{\gamma}}{\arg\min} \sum_{i=1}^{n} \rho\Big(y_i;\ \eta\big(\widehat{\mathbf{b}}_p(x_i)'\boldsymbol{\beta} + \mathbf{w}_i'\boldsymbol{\gamma}\big)\Big). \tag{2.3}$$

Using (2.3) the estimators of the three functions of interest are:

$$\widehat{\vartheta}_p(x, \widehat{\mathsf{w}}) = \eta(\widehat{\theta}_p(x, \widehat{\mathsf{w}})), \tag{2.4}$$

$$\widehat{\mu}_p^{(v)}(x) = \widehat{\mathbf{b}}_p^{(v)}(x)'\widehat{\boldsymbol{\beta}}, \tag{2.5}$$

$$\widehat{\zeta}_p(x, \widehat{\mathsf{w}}) = \eta^{(1)}(\widehat{\theta}_p(x, \widehat{\mathsf{w}}))\widehat{\mu}_p^{(1)}(x), \tag{2.6}$$

respectively, where $\widehat{\theta}_p(x, \widehat{\mathsf{w}}) = \widehat{\mu}_p(x) + \widehat{\mathsf{w}}'\widehat{\boldsymbol{\gamma}}$, is the plug-in estimator of the true index $\theta_0(x, \mathsf{w})$, and $\widehat{\mathsf{w}}$ (non-random or generated based on $\{\mathbf{w}_i\}_{i=1}^n$) is a consistent estimator for the desired evaluation point w. We will often make the polynomial order p explicit, as this is needed for clarity when constructing confidence bands and hypothesis tests in Section 5; dependence on $\widehat{\Delta}$ and choice of J is suppressed, but also important.

It is worth mentioning that nonlinear binscatter methods can be constructed for both fixed $J < \infty$ and large $J \to \infty$ as $n \to \infty$, naturally leading to different interpretations. The functions of interest defined at the beginning of this section cannot be recovered when J is fixed, but coarsened versions thereof will be, and these objects can have an interesting interpretation: if the parition "settles" as $n \to \infty$ to some fixed Δ_0 with associated fixed basis $\mathbf{b}_p(x)$ (see Assumption 4 below for a precise definition), then the probability limit of the fixed-J binscatter is the solution to (2.1) where the function class $\mathcal{M}$ is restricted to be $\mathcal{M} = \{\mu(x) = \mathbf{b}_p(x)'\boldsymbol{\beta} : \boldsymbol{\beta} \in \mathbb{R}^K, K = \dim(\mathbf{b}_p(x))\}$. This is most natural with $p = 0$ and quantile-spacing, because the binscatter shows average outcomes across quantiles of a continuous covariate. For $p = 0$ and $J = 100$, for example, the results allow for comparison of y_i across percentiles of x_i (possibly controlling for $\mathbf{w}_i$), which is standard for x_i variables such as test scores or measures of wealth. All our estimation and inference results remain valid when J is fixed, provided the target parameter is adjusted accordingly; see Cattaneo et al. (2024a,b) for further discussion of fixed-J binscatter methods. For the remainder of the paper, we will consider only the case $J \to \infty$ as $n \to \infty$ to streamline the presentation.

3 Theory

This section presents two main novel technical results: a uniform Bahadur representation and a feasible strong approximation. The methodological results in subsequent sections—tuning parameter selection, confidence bands, and hypothesis testing—are built from these results. To conserve space and notation, in this section we only show the results for $\widehat{\vartheta}_p(x, \widehat{\mathsf{w}})$. The analogous results for $\widehat{\mu}_p^{(v)}(x)$ and $\widehat{\zeta}_p(x, \widehat{\mathsf{w}})$ are deferred to the SA (specific references below) and are conceptually similar. The SA also gives other important technical results and additional discussion of how our theory improves on the existing literature.

First, we state the assumptions required. The class of data generating processes is restricted by the following.

Assumption 1 (Data Generating Process)**.**

(i) $\{(y_i, x_i, \mathbf{w}_i') : 1 \leq i \leq n\}$ *are i.i.d. random vectors satisfying* (2.1) *and supported on* $\mathcal{Y} \times \mathcal{X} \times \mathcal{W}$*, where* $\mathcal{X}$ *is a compact interval and* $\mathcal{W}$ *is a compact set.*

(ii) *The marginal distribution of* x_i *has a Lipschitz continuous (Lebesgue) density bounded away from zero on* $\mathcal{X}$*.*

(iii) *The conditional distribution of* y_i *given* $(x_i, \mathbf{w}_i')$ *has a (conditional) density with respect to some sigma-finite measure uniformly bounded over its support and* $\mathcal{X} \times \mathcal{W}$*.*

This assumption is fairly standard. It restricts attention to cross-sectional data and bounded covariates with minimal regularity imposed on the underlying joint distribution. Requiring x_i to be continuously distributed is natural given the visualization and estimation goals, but our results can also be applied to discrete x_i by taking each mass point as its own bin to conduct simultaneous estimation and inference over those support points.

The following assumption restricts the class of statistical models.

Assumption 2 (Statistical Model)**.**

(i) $\rho(y; \eta)$ *is absolutely continuous with respect to* $\eta \in \mathbb{R}$ *and admits a derivative* $\psi(y, \eta) := \psi^\dagger(y - \eta)\psi^\ddagger(\eta)$ *almost everywhere.* $\psi^\ddagger(\cdot)$ *is continuously differentiable and strictly positive or negative. If the conditional distribution of* y_i *given* $(x_i, \mathbf{w}_i')$ *does not have a Lebesgue density, then* $\psi^\dagger(\cdot)$ *is Lipschitz continuous, otherwise it is piecewise Lipschitz with finitely many discontinuity points.*

(ii) $\rho(y; \eta(\theta))$ *is convex with respect to* θ *and* $\eta(\cdot)$ *is strictly monotonic and three-times continuously differentiable.*

(iii) $\mathbb{E}[\psi(y_i, \eta(\theta_0(x_i, \mathbf{w}_i)))|x_i, \mathbf{w}_i] = 0$*.* $\sigma^2(x, \mathbf{w}) := \mathbb{E}[\psi(y_i, \eta(\theta_0(x_i, \mathbf{w}_i)))^2|x_i = x, \mathbf{w}_i = \mathbf{w}]$ *is bounded away from zero uniformly over* $\mathcal{X} \times \mathcal{W}$*.* $\mathbb{E}[\eta^{(1)}(\theta_0(x_i, \mathbf{w}_i)^2\sigma^2(x_i, \mathbf{w}_i)|x_i = x]$ *is*

Lipschitz continuous on $\mathcal{X}$, and $\mathbb{E}[|\psi(y_i, \eta(\theta_0(x_i, \mathbf{w}_i)))|^\nu | x_i = x, \mathbf{w}_i = \mathbf{w}]$ is uniformly bounded over $\mathcal{X} \times \mathcal{W}$ for some $\nu > 2$. $\mathbb{E}[\psi(y_i, \eta)|x_i = x, \mathbf{w}_i = \mathbf{w}]$ is twice continuously differentiable with respect to η.

(iv) For $\Upsilon(x, \mathbf{w}) := \frac{\partial}{\partial \eta}\mathbb{E}[\psi(y_i, \eta)|x_i = x, \mathbf{w}_i = \mathbf{w}]|_{\eta=\eta(\theta_0(x,\mathbf{w}))}$, $\Upsilon(x, \mathbf{w})\eta^{(1)}(\theta_0(x, \mathbf{w}))^2$ is bounded away from zero uniformly over $\mathcal{X} \times \mathcal{W}$ and $\mathbb{E}[\Upsilon(x_i, \mathbf{w}_i)\eta^{(1)}(\theta_0(x_i, \mathbf{w}_i))^2 | x_i = x]$ is Lipschitz continuous on $\mathcal{X}$.

(v) $\mu_0(\cdot)$ is ς-times continuously differentiable for some $\varsigma \geq p + 1$.

This assumption imposes regularity conditions on the statistical model in (2.1), particularly on the loss function and resulting parameters of interest. The complexity of part (i) reflects the breadth of the class of models and parameters we cover. When y_i is continuous the loss function can have points of nondifferentiability, but for discrete outcomes the loss must be smoother. To illustrate, consider first Example 4 in Section 2: the loss function for quantile regression is continuous but not differentiable everywhere, which is covered by our assumptions with $\psi(y, \eta) = \mathbb{1}(y - \eta < 0) - \tau$, where $\psi^\dagger(y - \eta) = \mathbb{1}(y - \eta < 0) - \tau$ exhibits a discontinuity point, and $\psi^\ddagger(\eta) = 1$ is smooth. Alternatively, for logistic regression (Example 2) $y_i \in \{0, 1\}$ and we have $\psi(y, \eta) = (y - \eta)[\eta(1 - \eta)]^{-1}$, which exactly matches the required structure of $\psi^\dagger(y - \eta)\psi^\ddagger(\eta)$. Both functions are clearly as smooth as required and the definition of $\eta(\theta)$ ensures that $\psi^\ddagger > 0$. The rest of the assumption gives standard moment and boundedness conditions to ensure that the parameters and their estimators are well-defined; those regularity conditions are also satisfied in all examples of interest. Finally, the nonparametric object $\mu_0(\cdot)$ is assumed to be smooth, as is standard in the nonparametric inference literature.

We next give several high-level conditions on the estimation procedure. These conditions ensure that the partitioning scheme is sufficiently regular and that the evaluation point for the control variables $\mathbf{w}_i$ and the Gram matrix can be estimated sufficiently well.

Assumption 3 (High-Level Estimation Conditions)**.**

(i) The partition $\widehat{\Delta}$ is independent of $\{y_i\}_{i=1}^n$ given $\{x_i, \mathbf{w}_i\}_{i=1}^n$ and, with probability approaching 1, $\max_{1\leq j\leq J} |\widehat{\tau}_j - \widehat{\tau}_{j-1}| \leq C \min_{1\leq j\leq J} |\widehat{\tau}_j - \widehat{\tau}_{j-1}|$, *for an absolute constant* $C > 0$.

(ii) $\|\widehat{\mathsf{w}} - \mathsf{w}\| = o_{\mathbb{P}}(1)$ *and* $\|\widehat{\boldsymbol{\gamma}} - \boldsymbol{\gamma}_0\| = o_{\mathbb{P}}(\sqrt{J/n} + J^{-p-1})$, *where* $\|\cdot\|$ *is the Euclidean norm.*

(iii) For the infeasible Gram matrix $\bar{\mathbf{Q}}_p := n^{-1}\sum_{i=1}^n \widehat{\mathbf{b}}_p(x_i)\widehat{\mathbf{b}}_p(x_i)'\Upsilon(x_i, \mathbf{w}_i)\eta^{(1)}(\theta_0(x_i, \mathbf{w}_i))^2$, *there is an estimator* $\widehat{\Upsilon}(x_i, \mathbf{w}_i)$ *such that* $\|\bar{\mathbf{Q}}_p - \widehat{\mathbf{Q}}_p\| = O_{\mathbb{P}}\big(J^{-p-1} + \big(\frac{J\log n}{n^{1-2/\nu}}\big)^{1/2}\big)$, *where* $\widehat{\mathbf{Q}}_p := n^{-1}\sum_{i=1}^n \widehat{\mathbf{b}}_p(x_i)\widehat{\mathbf{b}}_p(x_i)'\widehat{\Upsilon}(x_i, \mathbf{w}_i)\eta^{(1)}(\widehat{\theta}_p(x_i, \mathbf{w}_i))^2$, *and* $\|\cdot\|$ *is the operator norm.*

The requirement that the partition intervals are not too dissimilar in length is satisfied for evenly spaced partitioning, trivially, and is shown to hold for quantile spacing in the SA (Lemma SA-5.2). For other data-driven methods this condition must be checked. This assumed property of the random bining structure is often called quasi-uniformity (Cattaneo et al., 2020; Huang, 2003), and is important for controlling the approximation bias and, when combined with the assumptions on the density of x_i, for ensuring that each bin contains sufficient data to control the variance. Part (ii) requires that the desired evaluation point of $\mathbf{w}_i$ (such as the mean) can be estimated consistently and that the coefficient vector $\boldsymbol{\gamma}_0$ can be estimated sufficiently accurately. Generally neither is restrictive, as the nonparametric estimation of $\mu_0(x)$ is the most statistically difficult estimation in this setting. Finally, part (iii) ensures that we have a feasible estimator of the Gram matrix that converges rapidly enough. The infeasible Gram matrix $\bar{\mathbf{Q}}_p$ defined above is not a population object, but rather retains the randomness of the estimated basis. This will be key in our results and is discussed following Theorem 1. See Section SA-4.1 for examples of $\widehat{\Upsilon}(x_i, \mathbf{w}_i)$ for different models.

Our first theoretical result is a uniform (in x) Bahadur representation for $\widehat{\vartheta}_p(x, \widehat{\mathsf{w}})$ as defined in (2.4).

Theorem 1 (Bahadur Representation). *Suppose that Assumptions 1, 2, and 3 hold, and* *that* $J^{\frac{\nu}{\nu-2}}\log n + J(\log n)^{7/3} + (J^2(\log n))^{\frac{\nu}{\nu-1}} = o(n)$ *and* $\log n = o(J)$. *Then,*

$$\sup_{x\in\mathcal{X}}\left|\widehat{\vartheta}_p(x,\widehat{\mathsf{w}}) - \vartheta_0(x,\mathsf{w}) - \widehat{\mathsf{L}}_p(x,\mathsf{w})\right| = O_{\mathbb{P}}(r_n),$$

where

$$\widehat{\mathsf{L}}_p(x,\mathsf{w}) := \eta^{(1)}(\theta_0(x,\mathsf{w}))\widehat{\mathbf{b}}_p(x)'\bar{\mathbf{Q}}_p^{-1}\frac{1}{n}\sum_{i=1}^n \widehat{\mathbf{b}}_p(x_i)\eta^{(1)}(\theta_0(x_i,\mathbf{w}_i))\psi(y_i,\eta(\theta_0(x_i,\mathbf{w}_i))),$$

and $r_n := \big(\frac{J\log n}{n}\big)^{3/4}\log n + J^{-\frac{p}{2}}\big(\frac{\log^2 n}{n}\big)^{1/2} + J^{-p-1} + \|\widehat{\boldsymbol{\gamma}} - \boldsymbol{\gamma}_0\| + \|\widehat{\mathsf{w}} - \mathsf{w}\|$.

This result is essentially a stochastic linearization of the estimator, and yields important consequences including the mean squared error expansion used to choose J and the asymptotic variance formula for inference. The form of the variance is reminiscent of its parametric counterpart (e.g., for generalized linear models), but estimation is more complicated. Herein we maintain general high-level conditions justifying several alternatives commonly used in practice. These are discussed in Section SA-4.1. The analogous Bahadur representations for $\widehat{\mu}_p^{(v)}(x)$ and $\widehat{\zeta}_p(x,\widehat{\mathsf{w}})$, under the same assumptions, are given in Theorem SA-3.1. The "linear" term is slightly different to account for the different structure of the three estimands and the remainder rate for derivative estimation is slower.

With the Bahadur representation in place, we can develop tools for inference. Our main result is a strong approximation for the (Studentized) t-statistic process for each of the three estimators, allowing us to obtain a feasible asymptotic distributional approximation. Again we give the details only for $\widehat{\vartheta}_p(x,\widehat{\mathsf{w}})$ and defer the others to the SA. The variance is an immediate consequence of the expansion in Theorem 1, and is made feasible by replacing unknown objects by their estimators. For a given p, define the statistic

$$T_{\vartheta,p}(x) = \frac{\widehat{\vartheta}_p(x,\widehat{\mathsf{w}}) - \vartheta_0(x,\mathsf{w})}{\sqrt{\widehat{\Omega}_{\vartheta,p}(x)/n}}, \qquad \widehat{\Omega}_{\vartheta,p}(x) := \eta^{(1)}(\widehat{\theta}_p(x,\widehat{\mathsf{w}}))^2\widehat{\mathbf{b}}_p(x)'\widehat{\mathbf{Q}}_p^{-1}\widehat{\boldsymbol{\Sigma}}_p\widehat{\mathbf{Q}}_p^{-1}\widehat{\mathbf{b}}_p(x), \quad (3.1)$$

where $\widehat{\mathbf{\Sigma}}_p := n^{-1}\sum_{i=1}^n \widehat{\mathbf{b}}_p(x_i)\widehat{\mathbf{b}}_p(x_i)'\psi(y_i,\eta(\widehat{\theta}_p(x_i,\mathbf{w}_i)))^2\eta^{(1)}(\widehat{\theta}_p(x_i,\mathbf{w}_i))^2$ and $\widehat{\mathbf{Q}}_p$ is defined in Assumption 3. The t-statistics $T_{\mu^{(v)},p}(x)$ and $T_{\zeta,p}(x)$, for $\mu_0^{(v)}(x)$ and $\zeta_0(x,\mathsf{w})$ respectively, are entirely analogous, and are defined in Section SA-3.3.

Our inference results will follow from the next key theorem.

Theorem 2 (Strong Approximation)**.** *Suppose that Assumptions 1, 2, and 3 hold, and let $(a_n : n \geq 1)$ be a sequence of non-vanishing constants such that J and $\widehat{\mathsf{w}}$ obey*

$$\frac{J(\log n)^2}{n^{1-\frac{2}{\nu}}} + \Big(\frac{J(\log n)^7}{n}\Big)^{1/2} + nJ^{-2p-3} + \frac{(\log n)^2}{J^{p+1}} + nJ^{-1}\|\widehat{\boldsymbol{\gamma}} - \boldsymbol{\gamma}_0\|^2 = o(a_n^{-2}),$$

$\|\widehat{\mathsf{w}} - \mathsf{w}\| = o_{\mathbb{P}}(a_n^{-1}\sqrt{J/n})$, and $(J^2\log(n))^{\nu/(\nu-1)} = o(n)$. Then, on a properly enriched probability space, there exists a $(J+p)$-dimensional standard Normal random vector $\mathbf{N}$ such that for any $\xi > 0$,

$$\mathbb{P}\Big(\sup_{x\in\mathcal{X}}|T_{\vartheta,p}(x) - \bar{Z}_{\vartheta,p}(x)| > \xi a_n^{-1}\Big) = o(1), \qquad \bar{Z}_{\vartheta,p}(x) = \frac{\widehat{\mathbf{b}}_p(x)'\eta^{(1)}(\theta_0(x,\mathsf{w}))\bar{\mathbf{Q}}_p^{-1}\bar{\mathbf{\Sigma}}_p^{1/2}}{\sqrt{\bar{\Omega}_{\vartheta,p}(x)}}\mathbf{N},$$

where $\bar{\mathbf{\Sigma}}_p$ and $\bar{\Omega}_{\vartheta,p}(x)$ are shown in Section SA-3. On a further enriched space, there exists a conformable standard Normal vector $\mathbf{N}^\star$, independent of $\{(y_i, x_i, \mathbf{w}_i')'\}_{i=1}^n$ and $\widehat{\Delta}$, such that for any $\xi > 0$,

$$\mathbb{P}\Big(\sup_{x\in\mathcal{X}}|\bar{Z}_{\vartheta,p}(x) - \widehat{Z}_{\vartheta,p}(x)| > \xi a_n^{-1}\Big|\{(y_i, x_i, \mathbf{w}_i')'\}_{i=1}^n, \widehat{\Delta}\Big) = o_{\mathbb{P}}(1),$$
$$\widehat{Z}_{\vartheta,p}(x) = \frac{\widehat{\mathbf{b}}_p(x)'\eta^{(1)}(\widehat{\theta}_p(x,\widehat{\mathsf{w}}))\widehat{\mathbf{Q}}_p^{-1}\widehat{\mathbf{\Sigma}}_p^{1/2}}{\sqrt{\widehat{\Omega}_{\vartheta,p}(x)}}\mathbf{N}^\star.$$

The approximating process, $\bar{Z}_{\vartheta,p}(\cdot)$, is a Gaussian process conditional on $\{x_i,\mathbf{w}_i\}_{i=1}^n$ and $\widehat{\Delta}$ by construction, and the elements of $\bar{\mathbf{\Sigma}}_p$, $\bar{\Omega}_{\vartheta,p}(x)$ and $\widehat{\mathbf{b}}_p(x)$ reflect this conditioning. This process is infeasible but the second result shows that all the unknown quantities in in $\bar{Z}_{\vartheta,p}(\cdot)$ can be replaced by their sample analogues to obtain a feasible approximation. Theorems SA-3.5 and SA-3.6 give the corresponding results for $T_{\mu^{(v)},p}(x)$ and $T_{\zeta,p}(x)$, under the same assumptions. Pointwise inference results are also given in the SA for completeness.

Theorems 1 and 2 substantially generalize the least squares results in Cattaneo et al. (2024b), under essentially the same rate restrictions, with an error of approximation that is optimal up to $\log(n)$ terms. Our results are on par with, or improve upon, prior theory for kernel estimators of nonlinear models (Kong et al., 2010) and series estimation for quantile regression (Belloni et al., 2019). There are several key improvements. First, having sharp rate conditions allows us to accommodate $p = 0$, which is generally excluded by the prior literature but necessary for binned scatter plots. Note that these theorems give approximations for the entire t-statistic process, and not just functionals thereof, under such weak conditions. Prior work has obtained such sharp results only for the supremum of the process. Further, we allow for random partitioning (i.e. series estimation with data-dependent basis functions), which represents a major technical hurdle, and also allow for additional control variables.

In fact, beyond being ruled out by prior work, the randomness in the basis functions requires a novel theoretical approach. The key motivation behind this approach is that the basis functions $\widehat{\mathbf{b}}_p^{(v)}(x)$ do not converge uniformly to a nonrandom counterpart, due to the sharp discontinuity of the (random) indicator functions. It is not possible to obtain the uniform results of Theorem 1 (or the strong approximations below) by expanding around a nonrandom limit. Thus $\widehat{\mathbf{b}}_p^{(v)}(x)$ is left as random in the Bahadur representation, including in the matrices $\bar{\mathbf{Q}}_p$ and $\bar{\mathbf{\Sigma}}_p$. This further separates our results from prior literature. The more general theorems in the SA are followed by remarks detailing how our work improves on the relevant literature in each case.

4 Tuning Parameter Selection

We can use our theoretical results from the previous section to directly inform implementation in empirical applications. Our first task is selecting the number of bins. The choice of J determines both the visual and statistical properties of the estimator. Consistent estimation and valid inference is possible for a range of diverging sequences of J, but this does not provide sufficiently precise guidance for implementation. Thus, our first methodological result is a Nagar-type integrated mean squared error expansion that enables an optimal choice of J in empirical applications.

To obtain the result, we need one further assumption to characterize the leading terms of the expansion. Intuitively, we require that the random partition $\widehat{\Delta}$ "converges" to a fixed one which obeys the same restrictions as in Assumption 3. This assumption is not needed for any other results.

Assumption 4. *There exists a non-random partition* $\Delta_0 = \{\mathcal{B}_1, \cdots, \mathcal{B}_J\}$ *with* $\mathcal{B}_j = [\tau_{j-1}, \tau_j)$ *for* $j \leq J-1$ *and* $\mathcal{B}_J = [\tau_{J-1}, \tau_J]$ *such that* $\max_{1\leq j\leq J} |\tau_j - \tau_{j-1}| \leq C \min_{1\leq j\leq J} |\tau_j - \tau_{j-1}|$, *for an absolute constant* $C > 0$ *and* $\max_{1\leq j\leq J} |\hat{\tau}_j - \tau_j| = o_{\mathbb{P}}(J^{-1})$.

This condition trivially holds for well-behaved nonrandom partitions, but also holds for the leading case of quantile-spacing, since sample quantiles converge to their population counterparts. In more general cases with data-driven partitions this condition could fail. However, all our other results remain valid, and furthermore, even if this condition fails a rule-of-thumb choice of J is available, which has the optimal rate but suboptimal constants. See Section SA-4.2 for discussion.

Our IMSE result for $\widehat{\vartheta}_p(x, \widehat{\mathsf{w}})$ is given by the following. The corresponding results for $\widehat{\mu}_p^{(v)}(x)$ and $\widehat{\zeta}_p(x, \widehat{\mathsf{w}})$ are stated in Theorem SA-3.4.

Theorem 3. *Set* $\omega(x)$ *to be a continuous weighting function over* $\mathcal{X}$ *bounded away from zero. Suppose that Assumptions 1, 2, 3, and 4 hold, and let* $J^{\frac{\nu}{\nu-2}} \log n + J^{\frac{2\nu}{\nu-1}} (\log n)^{\frac{\nu}{\nu-1}} + J(\log n)^7 = o(n)$ *and* $\log n = o(\sqrt{J})$, *and* $\|\widehat{\mathsf{w}} - \mathsf{w}\| = o_{\mathbb{P}}(\sqrt{J/n} + J^{-p-1})$. *Then,*

$$\int_{\mathcal{X}} \Big(\widehat{\vartheta}_p(x, \widehat{\mathsf{w}}) - \vartheta_0(x, \mathsf{w})\Big)^2 \omega(x) dx = \mathtt{AISE}_{\vartheta} + o_{\mathbb{P}}\Big(\frac{J}{n} + J^{-2(p+1)}\Big)$$

where $\texttt{AISE}_\vartheta$ *obeys*

$$\mathbb{E}[\texttt{AISE}_\vartheta|\{(x_i,\mathbf{w}_i')'\}_{i=1}^n,\widehat{\Delta}]=\frac{J}{n}\mathscr{V}_n(p)+J^{-2(p+1)}\mathscr{B}_n(p)+o_{\mathbb{P}}\Big(\frac{J}{n}+J^{-2(p+1)}\Big),$$

for nonrandom terms $\mathscr{V}_n(p)$ *and* $\mathscr{B}_n(p)$ *shown in Theorem SA-3.4 that are bounded and nonzero in general.*

This result is stated in terms of J, as it is the tuning parameter, but the rates and constants depend on the fixed polynomial order p (recall that we set $p = s$, see Section 2.1). An L_2 convergence rate immediately follows from this result. An L_∞ convergence rate is also available in the SA (Corollary SA-3.1). This theorem, and the L_2 and L_∞ rates, are new to the literature, even in the case of non-random partitioning and without covariate adjustment, for nonlinear series estimators and binscatter methods.

The practical consequence of Theorem 3 is that we can balance the (squared) bias and variance to obtain an IMSE-optimal choice of J, which is given by

$$J_{\texttt{IMSE}}(p):=\left(\frac{2(p+1)\mathscr{B}_n(p)}{\mathscr{V}_n(p)}\right)^{\frac{1}{2p+3}}n^{\frac{1}{2p+3}}. \tag{4.1}$$

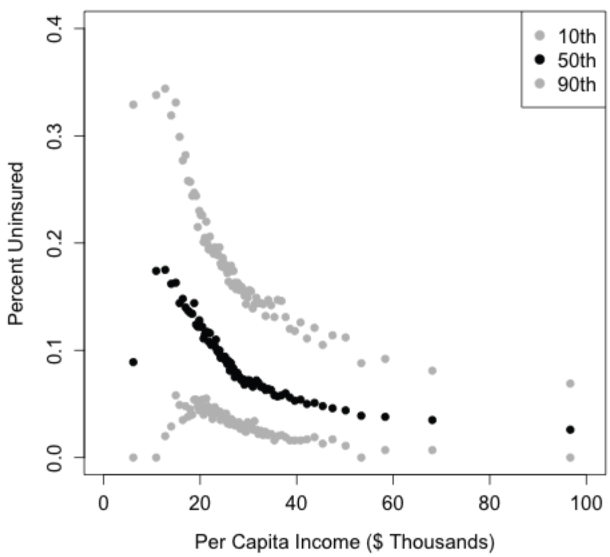
Implementing binscatter with this J is optimal in the sense of providing the IMSE-optimal estimate of the unknown function $\vartheta_0(\cdot,\mathtt{w})$. In the next section we discuss the use of $J_{\texttt{IMSE}}(p)$ for inference, where, as is typical, a bias correction must be applied. A feasible version of $J_{\texttt{IMSE}}(p)$ is described in Section SA-4 and implemented in the `binsreg` package (Cattaneo et al., 2024a). Section SA-4.3 discusses binned scatter plots with a fixed choice of J, which can be visually appealing and interpretable in some applications.

Figure 2 demonstrates the use of the optimal J for quantile estimation (see Figure 1(d) for the mean). Quantile regression can be used to visualize the spread of the conditional distribution. Observe that Figure 2 restores the visualization of the variability in the data that is present in Figure 1(a) but hidden by the averaging in Figure 1(c). Figure 2(a) shows that there is much larger variance in the fraction insured in lower income areas, but Figure

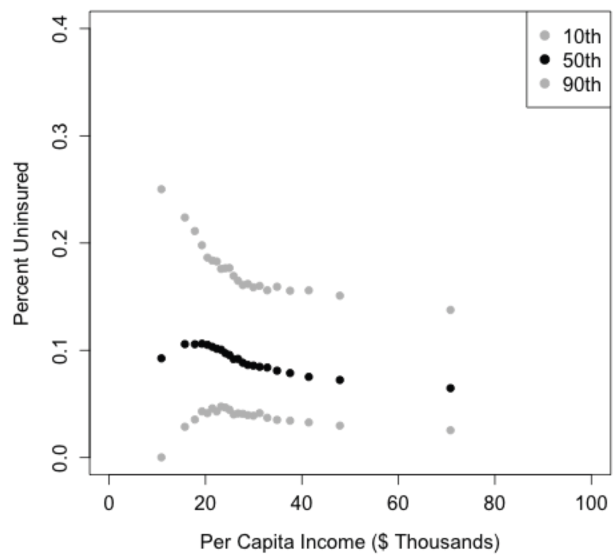
2(b) shows that this gap narrows when controlling for demographics ($\widehat{\mathsf{w}}$ is set to the sample mean). In this case we control for (i) percentage of residents with a high school degree, (ii) percentage of residents with a bachelor's degree, (iii) median age of residents, and (iv) the local unemployment rate.

Figure 2: **Conditional Quantiles.** This figure illustrates the use of quantile regression to visualize the spread in the ACS data (see Example 4). As the link function is the identity, panel (a) shows estimates of $\vartheta_0(x, \mathsf{w})$ for $\tau = 0.1, 0.5, 0.9$, while panel (b) shows the same quantiles including additional covariates controlling for demographics: (i) percentage of residents with a high school degree, (ii) percentage of residents with a bachelor's degree, (iii) median age of residents, and (iv) the local unemployment rate.

(a) *Quantile Regression w/o Controls*



(b) *Quantile Regression w/ Controls*



5 Uniform Inference

We now turn to uniform inference for the three functions defined in Section 2. Uniform inference is required to make statistical statements about the functions $\vartheta_0(x, \mathsf{w})$, $\mu_0^{(v)}(x)$, and $\zeta_0(x, \mathsf{w})$, rather than about their values at a specific point x. Pointwise inference methods (e.g. confidence intervals) will not suffice for our main applications of interest, including treatment effect heterogeneity and continuous treatment effects, as well as shape restrictions and functional forms. For completeness, pointwise inference results are given in the SA and implemented in the `binsreg` package, but omitted here to save space.

A key element of our uniform inference results—from a practical point of view—is the pairing of a feasible tuning parameter choice with valid inference. To give the most accurate estimate, and therefore also visualization, of $\vartheta_0(x, \mathsf{w})$ we prefer to use $J_{\mathtt{IMSE}}(p)$ of (4.1). However, as is typical for nonparametric problems, the (I)MSE-optimal tuning parameter choice delivers invalid inference, as it fails to eliminate a first-order bias. We therefore use robust bias correction to ensure that $J_{\mathtt{IMSE}}(p)$ remains a valid choice across all uses, delivering optimal estimation and valid inference. With this eye toward practicality, we state all results below specifying $J = J_{\mathtt{IMSE}}(p)$, but the SA gives the more general results under mild rate restrictions on J.

Bias correction involves estimating, and removing, the leading smoothing bias term, and is made "robust" by correcting the standard errors to account for the additional sampling variability that has been introduced. Robust bias correction has theoretically superior higher-order inference properties (Calonico et al., 2018, 2022), performs well in simulations, and has been empirically validated in specific contexts (Hyytinen et al., 2018). Robust bias correction is operationalized in the present context by (i) selecting a degree p and creating a partition $\widehat{\Delta}$ based on $J_{\mathtt{IMSE}}(p)$ to form the optimal point estimate of $\widehat{\vartheta}_p(x, \widehat{\mathsf{w}})$ and then (ii) conducting inference using $T_{\vartheta,p+1}(x)$ (or its feasible analogue $\widehat{Z}_{\vartheta,p+1}(x)$), i.e. the statistic formed using a higher-degree polynomial but the partitioning scheme based on p in (i): $\widehat{\Delta} = \widehat{\Delta}(J_{\mathtt{IMSE}}(p))$. Any higher-degree polynomial may be used, but $p+1$ is simple and robust. Cattaneo et al. (2020) give further discussion of robust bias correction in the context of partition regression, including alternative strategies.

5.1 Confidence Bands

Our first uniform inference result delivers confidence bands for the functions $\vartheta_0(x, \mathsf{w})$, $\mu_0^{(v)}(x)$, and $\zeta_0(x, \mathsf{w})$. Confidence bands are similar in spirit as the more familiar concept of confidence intervals, but instead cover the entire function (uniformly over $x \in \mathcal{X}$) with a prespecified probability. Confidence bands are the appropriate tool for visualizing the uncertainty around the estimated function. The size of the band also changes to reflect the presence of heteroskedasticity in the data. These bands can be used directly to identify interesting or important features of the function, for example, regions where it is statistically indistinguishable from zero or from a constant function. Bands are also useful for assessing the functional form or shape, such as regions of linearity or monotonicity, and therefore visually complement the formal hypothesis tests we introduce below.

The confidence bands are built from Theorem 2, coupled with robust bias correction, as discussed above. Confidence bands are defined as the area between an upper and lower bounding function. Recall that we employ robust bias correction, so that $\widehat{\vartheta}_{p+1}(x, \widehat{\mathsf{w}})$ is the bias-corrected version of $\widehat{\vartheta}_p(x, \widehat{\mathsf{w}})$, and is thus the "center" of the confidence band, and using $\widehat{\Omega}_{\vartheta,p+1}(x)$ in the standard error accounts for the additional variability. The robust bias-corrected confidence band for $\vartheta_0(x, \mathsf{w})$ is given by

$$\widehat{I}_{\vartheta,p+1}(x) = \left[\widehat{\vartheta}_{p+1}(x, \widehat{\mathsf{w}}) \pm \mathfrak{c}_\vartheta \sqrt{\widehat{\Omega}_{\vartheta,p+1}(x)/n}\right], \tag{5.1}$$

where the critical value is determined by

$$\mathfrak{c}_\vartheta = \inf\left\{c \in \mathbb{R}_+ : \mathbb{P}\left[\sup_{x\in\mathcal{X}} |\widehat{Z}_{\vartheta,p+1}(x)| \le c \;\middle|\; \{(y_i, x_i, \mathbf{w}_i')'\}_{i=1}^n, \widehat{\Delta}\right] \ge 1-\alpha\right\}. \tag{5.2}$$

The asymptotic validity of this confidence band follows from Theorem 2, which allows us to approximate the distribution of the supremum of $T_{\vartheta,p+1}(x)$ by applying the same functional to $\widehat{Z}_{\vartheta,p+1}(x)$. This yields the following result. Here we assume directly that the optimal J is used. Theorem SA-3.8 states a more general result, valid for a range of J, as well as inference results for $\mu_0^{(v)}(x)$ and $\zeta_0(x, \mathsf{w})$.

Theorem 4. *Set $J = J_{\mathtt{IMSE}}(p)$. Suppose that Assumptions 1, 2, and 3 hold, with $p+1$ in place of p and $\nu > 3$, and let $\|\widehat{\mathsf{w}} - \mathsf{w}\| = o_{\mathbb{P}}(\sqrt{J/(n \log J)})$ and $\|\widehat{\boldsymbol{\gamma}} - \boldsymbol{\gamma}_0\| = o_{\mathbb{P}}(\sqrt{J/(n \log J)})$.*

Then, for $\widehat{I}_{\vartheta,p+1}(x)$ *defined in* (5.1),

$$\mathbb{P}\Big[\vartheta_0(x,\mathsf{w}) \in \widehat{I}_{\vartheta,p+1}(x),\ \textit{for all } x \in \mathcal{X}\Big] = 1 - \alpha + o(1).$$

This result establishes valid confidence bands for generalized, covariate-adjusted binscatters. We can use this result to visually assess uncertainty about the form and shape of the regression function. One can visually "test" hypotheses of interest, though formal testing (Section 5.2) is recommended. Plotting both the estimate $\widehat{\vartheta}_p(x,\widehat{\mathsf{w}})$ and the band $\widehat{I}_{\vartheta,p+1}(x)$ is advisable in applications because doing so presents both the IMSE-optimal point estimate and a valid measure of uncertainty (and one that uses the same bins).

In nonlinear models, particularly in social sciences, partial effects are often the preferred way of summarizing the relationship (causal or not) of x_i to y_i, controlling for $\mathbf{w}_i$. In our setting this corresponds to the estimate of $\zeta_0(x,\mathsf{w})$ and its associated confidence band. When x_i is a treatment variable, $\zeta_0(x,\mathsf{w})$ captures the effect of increasing the treatment dosage, and the band can help identify regions of $\mathcal{X}$ with the largest effects, or any other noteworthy shape.

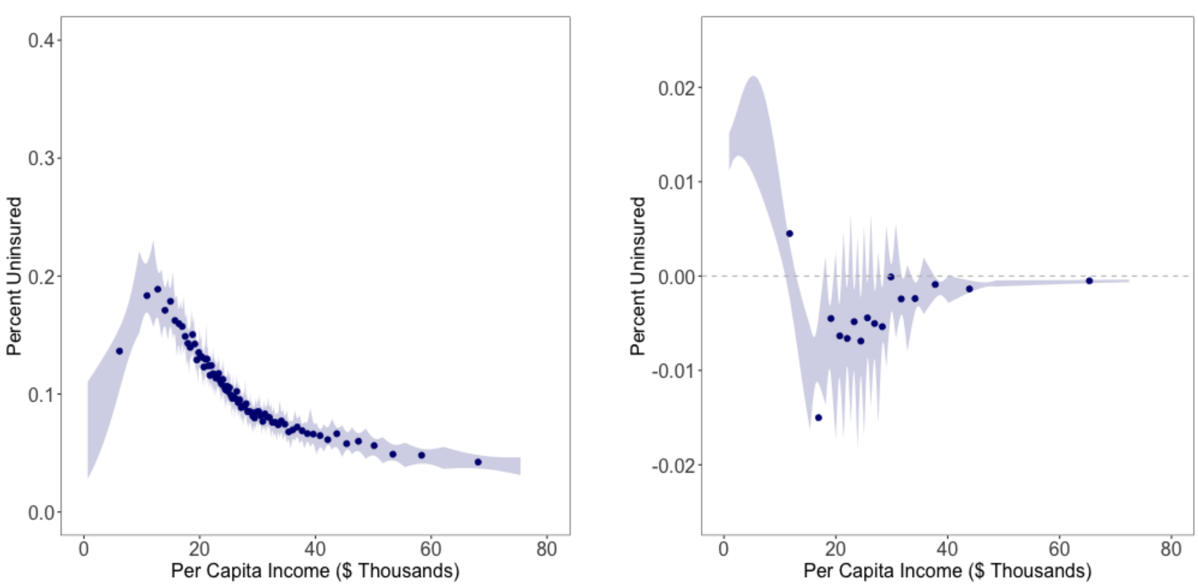
Figure 3 shows examples of confidence bands using our running empirical application. The confidence band in Figure 3(a) displays the uncertainty surrounding the estimate first shown in Figure 1(d). The presence of the Medicaid program is clearly delineated by the shape of the band at lower income levels. From the band we can immediately conclude that the relationship is nonmonotonic. This is further emphasized in Figure 3(b), showing the marginal effect. Using the bands, we can reject the null hypothesis of monotonicity as the band lies completely on either side of zero at low and high income levels.

There are two features of our confidence bands that warrant mention. First, the user-selected point of evaluation w can impact the shape, placement, and size, of the confidence band. One might expect that since the additional controls are modeled as additively linear, the evaluation point w (and the coefficient $\boldsymbol{\gamma}_0$) should not impact conclusions about the nonparametric relationship between y and x. But this intuition overlooks the fact that the function $\mu_0(x)$ is only defined relative to how $\mathbf{w}_i$ is coded. For example, if $\mathbf{w}_i$ contains a binary variable indicating groups of substantially different sizes, then the estimation uncertainty will be different between the two groups. This can cause a level shift in $\widehat{\mu}_p^{(v)}(x)$ and alter the uncertainty around the estimate. For $\vartheta_0(x, \mathsf{w})$ and $\zeta_0(x, \mathsf{w})$, the shape may also change. This impacts all aspects of inference, both visual and the formal tests below. This is not particular to our method; it is always present in analyses of models like (2.1).

Second, the bias correction may result in the point estimate lying outside the confidence band. This occurs in regions of high bias. This is formally correct but can be visually unappealing. Figure 3(b) shows an example of this phenomenon. This can arise in any application of bias correction methods, and is not necessarily a failing: the point estimate remains IMSE-optimal and inference remains valid.

Figure 3: **Confidence Bands.** This figure illustrates confidence bands in a nonlinear binned scatter plot using the ACS data. Panel (a) shows the point estimate (dots) and robust bias corrected confidence band (shaded region) for the conditional mean function with no controls, i.e., $\vartheta_0(x) = \eta(\mu_0(x))$, while panel (b) shows the corresponding point estimate and confidence band for the marginal effect, $\zeta_0(x)$. Shaded regions denote 95% confidence bands and are based on 50,000 random draws.

(a) *Conditional Mean* (b) *Marginal Effect*



5.2 Hypothesis Testing

We also provide formal hypothesis tests for substantive questions including functional form or shape restrictions for $\vartheta_0(x, \mathsf{w})$, $\mu_0(x)$, and $\zeta_0(x, \mathsf{w})$. Our discussion here is brief. Full details are given in the SA.

A leading case is testing a parametric functional form for $\mu_0(x)$. This is a two-sided testing problem where under the null there exists some finite-dimensional parameter $\boldsymbol{\theta}$ such that $\mu_0(x) = m(x; \boldsymbol{\theta})$, uniformly in $x \in \mathcal{X}$ (we can also test any derivative of $\mu_0(x)$). The testing problem is

$$\begin{aligned}
\dot{\mathsf{H}}_0^{\mu}: \quad & \sup_{x\in\mathcal{X}} \Big|\mu_0(x) - m(x; \boldsymbol{\theta})\Big| = 0, \qquad \text{for some } \boldsymbol{\theta}, \qquad vs. \\
\dot{\mathsf{H}}_{\mathrm{A}}^{\mu}: \quad & \sup_{x\in\mathcal{X}} \Big|\mu_0(x) - m(x; \boldsymbol{\theta})\Big| > 0, \qquad \text{for all } \boldsymbol{\theta}.
\end{aligned}$$

This test formalizes the notion of visual inspection in plots like Figure 1(c) and (d), beyond what is already done by adding a confidence band. We test this hypothesis using the statistic

$$\dot{T}_{\mu,p+1}(x) := \frac{\widehat{\mu}_{p+1}(x) - m(x; \widetilde{\boldsymbol{\theta}})}{\sqrt{\widehat{\Omega}_{\mu,p+1}(x)/n}},$$

where $\widetilde{\boldsymbol{\theta}}$ and $\widetilde{\boldsymbol{\gamma}}$ are estimators of $\boldsymbol{\theta}$ and $\boldsymbol{\gamma}_0$ that are consistent under $\dot{\mathsf{H}}_0^{\mu}$ where $\theta_0(x, \mathsf{w}) = m(x; \boldsymbol{\theta}) + \mathsf{w}'\boldsymbol{\gamma}_0$. Theorem 2 again provides the tools to obtain the appropriate critical value. Theorem SA-3.9 gives the formal result showing size control and consistency of the test, as well as the corresponding tests for $\vartheta_0(x, \mathsf{w})$ and $\zeta_0(x, \mathsf{w})$. The tests can be performed using any L_q norm for $q \geq 1$, instead of L_∞ as shown above. Last, we also provide for testing shape restrictions, which are conceptually similar but are generally one-sided testing problems. A leading example would be testing monotonicity of $\vartheta_0(x, \mathsf{w})$ or $\zeta_0(x, \mathsf{w})$.

Table 1 shows several testing examples using the L_∞ norm. Consider first the left column of results. We test against the linear specification model, formalizing the visual comparison in Figure 1(d). We also test against a cubic (in x) logistic quasi-likelihood model for added flexibility. Both parametric specifications are rejected, and moreover, also rejected when including other controls. This highlights the need for nonparametric modeling in this application. Finally we test the substantive null hypothesis that the uninsuredness rate is monotonically decreasing with income. This null is also strongly rejected due to the existence of Medicaid. This motivates the right column of results, where we repeat the analysis after restricting the sample to zip codes with per capita income above 138% of the 2013–2017 average federal poverty line for a single-person household ($16,248). This is the cutoff for expanded Medicaid eligibility based only on income. When we restrict to this sample, which diminishes the influence of the Medicaid program, we fail to reject the null hypothesis of a monotonic decline, but still reject the parametric specifications. This aligns with the need for flexible estimation and matches the conclusions we draw from the shape of the confidence bands shown in Figure 3.

Table 1: **Specification and Shape Testing**

	Full Sample			**Above Income Cutoff**		
	Test Stat.	p-value	$\hat{J}_{\texttt{IMSE}}$	Test Stat.	p-value	$\hat{J}_{\texttt{IMSE}}$
Test of Linear Fit						
No Covariates	3113.083	0.000	80	4315.983	0.000	40
Covariates, $\widehat{\mathbf{w}} = \bar{\mathbf{w}}$	1979.468	0.000	22	2908.763	0.000	12
Test of Cubic Fit						
No Covariates	2245.499	0.000	80	14814.862	0.000	40
Covariates, $\widehat{\mathbf{w}} = \bar{\mathbf{w}}$	1981.105	0.000	22	3587.529	0.000	12
Test of Monotonic Decline						
No Covariates	23.991	0.000	16	0.644	0.998	10
Covariates, $\widehat{\mathbf{w}} = \bar{\mathbf{w}}$	6.997	0.000	13	-0.016	1.000	8

Notes. This table reports the test statistics and associated p-values along with the IMSE-optimal choice of J from hypothesis tests of parametric specifications and shape restrictions using the ACS data. The first and second panels report test results for the null hypotheses of linear and cubic (in x) logistic quasi-likelihood models, respectively, while the third panel reports results for the null hypothesis of a monotonic decline in the level (i.e., negative derivative). All tests are performed with and without control variables (control variables are same as in Figure 2). The left panel ("Full Sample") reports results for the full sample whereas the right panel ("Above Income Cutoff") restricts to the sample of zip codes with per capita income above $16,248. All p-values are based on 50,000 random draws.

5.3 Multi-Sample Comparisons

Our results extend to comparisons between different samples, or groups, within the data. This is a common goal in program evaluation and causal inference settings. With discrete (e.g., binary) treatments, the groups are defined by treatment arms and the differences define heterogeneous (in x_i) effects. In the continuous case, the grouping is the dimension of heterogeneity and x_i is the treatment. Our results extend naturally to this setting. For a grouping indicator $t_i = 0, \ldots, L$, we replace the scalar index in the model (2.1) with $\theta_0(x_i, \mathbf{w}_i, t_i) := \sum_{t=0}^{L} \mathbb{1}\{t_i = t\}\theta_{0,t}(x_i, \mathbf{w}_i)$, where each $\theta_{0,t}(x_i, \mathbf{w}_i) = \mu_{0,t}(x_i) + \mathbf{w}_i'\boldsymbol{\gamma}_{0,t}$. The level and marginal effect can then be defined groupwise, as $\vartheta_{0,t}(x, \mathsf{w}) = \eta(\theta_{0,t}(x, \mathsf{w}))$ and $\zeta_{0,t}(x, \mathsf{w}) = \eta^{(1)}(\theta_{0,t}(x, \mathsf{w}))\mu_{0,t}^{(1)}(x)$ for some evaluation point w of control variables.

For example, in a randomized experiment $\vartheta_{0,1}(x, \mathsf{w}) - \vartheta_{0,0}(x, \mathsf{w})$ is the conditional average treatment effect (CATE) function, and the binscatter naturally captures treatment effect heterogeneity along the x_i dimension holding fixed $\mathbf{w}_i = \mathsf{w}$. The rate of change in this heterogeneity is $\zeta_{0,1}(x, \mathsf{w}) - \zeta_{0,0}(x, \mathsf{w})$. Our methods can be used to formally test the null hypothesis that $\vartheta_{0,1}(x, \mathsf{w}) = \vartheta_{0,0}(x, \mathsf{w})$ for all $x \in \mathcal{X}$, which captures the idea of no (heterogeneous) treatment effect. As a second example, our theory can be used to quantify uncertainty for the largest heterogeneous treatment effect:

$$\widehat{x}^{\star} = \underset{x \in \mathcal{X}}{\arg\sup} \left| \widehat{\vartheta}_{0,1}(x, \mathsf{w}) - \widehat{\vartheta}_{0,0}(x, \mathsf{w}) \right|.$$

These and many other problems of interest in applied microeconometrics concern the uniform discrepancy of two or more binscatter function estimators, which can be analyzed using our strong approximation and related theoretical results in the supplemental appendix. We do not provide further details here to conserve space, but our software implements several multi-sample estimation, uncertainty quantification, and hypothesis testing procedures.

Figure 4 shows an example of this type of analysis. We divide states into two groups based on their population density, with low and high density states as those with population densities below or above 100 people per square mile, respectively. Density is defined as the average population per square mile, and the data is available from the Census Bureau. Panel (a) shows $\widehat{\vartheta}_t(x)$ for each group, i.e. without controls, while panel (b) adds controls and shows $\widehat{\vartheta}_t(x, \widehat{\mathsf{w}})$, with $\widehat{\mathsf{w}}$ set to the sample mean. The point estimates show higher uninsured rates in zip codes in low population density states as compared to high density states. Without controls, there is generally overlap in the confidence bands except for very low incomes. In contrast, when covariates are added, there is a much clearer delineation between the two groups at all but the lowest of income levels. This is made clear in panels (c) and (d), which plot the point estimate of the difference (the CATE) and the associated confidence band. The null hypothesis that $\vartheta_{0,1}(x, \mathsf{w}) = \vartheta_{0,0}(x, \mathsf{w})$ for all $x \in \mathcal{X}$ is rejected in both cases, with test statistics of 7.719 and 8.308, respectively, and negligible p-values. Multi-sample comparisons share the same sensitivity to the chosen evaluation point as discussed above. These issues are unavoidable; researchers must be mindful when implementing the tests and interpreting the results.

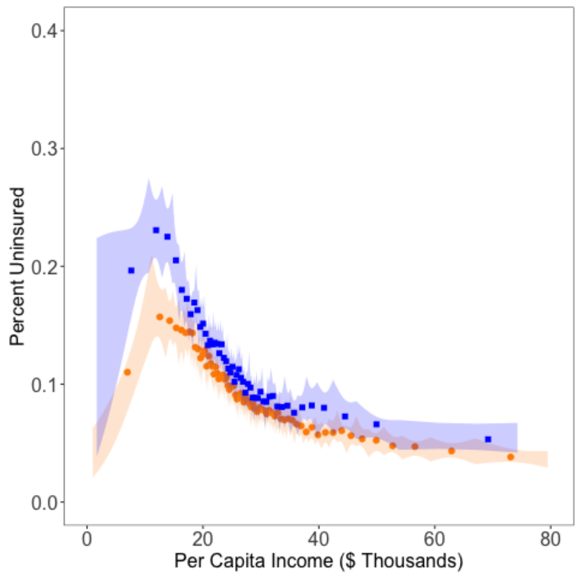
6 Conclusion

With the rise of large data sets, new visualization tools, such as binned scatter plots, have emerged and gained in popularity. This paper has thoroughly studied binned scatter plots in nonlinear, nonsmooth regression models. Our main contributions are to propose novel nonlinear binscatter methods, together with IMSE-optimal tuning parameter selection and uniform inference methods, including valid confidence bands and functional testing. Our companion `binsreg` software package makes these tools available for applications.

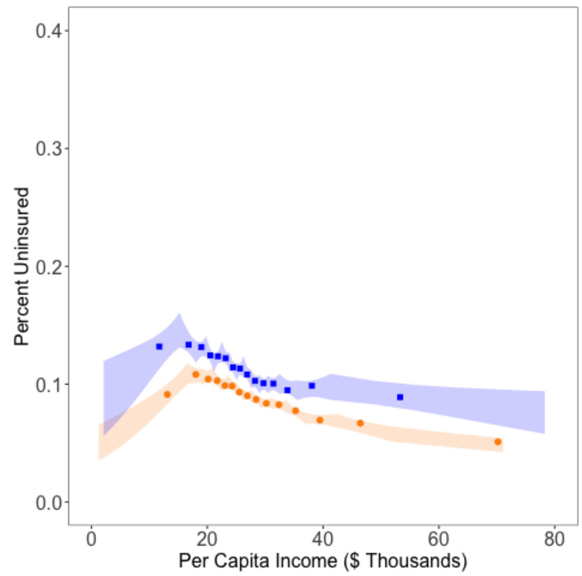
One avenue for future work would be to generalize the analysis beyond a scalar covariate of interest. For example, in two dimensions such an approach would produce "heat maps" which are the bivariate extension of binned scatter plots. Extending our results to that case would be a valuable addition to the practitioner's toolkit.

Figure 4: **Two Sample Comparison.** This figure uses the same ACS data to compare areas in low density states (blue) and high density states (orange). Low density states are defined as those with average population per square mile below 100. Panels (a) and (b) show the point estimate (squares or dots) and robust bias corrected confidence band (shaded region) for each group, first without control variables and then with controls added (the same controls as in Figure 2). Panels (c) and (d) show the estimated difference (evaluated using the binning of the low density states) and the associated confidence bands. Shaded regions denote 95% confidence bands and are based on 50,000 random draws.

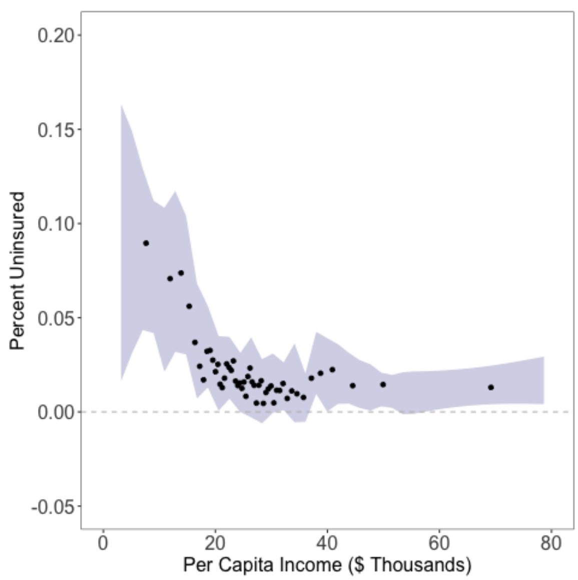
(a) *Conditional means w/o controls*



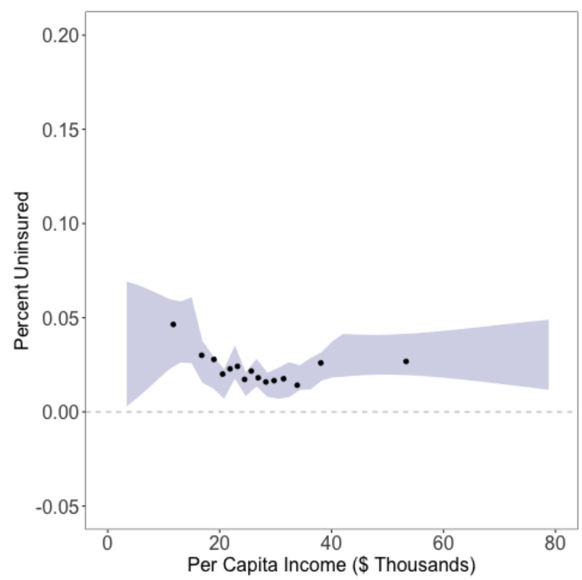
(b) *Conditional means w/ controls*



(c) *Difference of means w/o controls*



(d) *Difference of means w/ controls*



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Nonlinear Binscatter Methods
Supplemental Appendix*

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July 21, 2024

Abstract

This supplement collects all technical proofs for more general theoretical results than those reported in the main paper. Several of our new theoretical results for nonlinear partitioning-based series estimation may be of independent interest. More details on methodological aspects of nonlinear binscatter are also provided. Companion general-purpose software and replication files are available at https://nppackages.github.io/binsreg/.

*Cattaneo gratefully acknowledges financial support from the National Science Foundation through grants SES-1947805, SES-2019432, and SES-2241575. Feng gratefully acknowledges financial support from the National Natural Science Foundation of China (NSFC) through grants 72203122, 72133002, and 72250064. The views expressed in this paper are those of the authors and do not necessarily reflect the position of the Federal Reserve Bank of New York or the Federal Reserve System.

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Contents

SA-1 Introduction

This supplemental appendix is a comprehensive collection of all our new theoretical results for nonlinear binscatter estimators with semi-linear covariate-adjustment and random partitioning. Many of our results contribute to the broader literature on nonparametric estimation and inference, particularly when using series estimators, and are thus of independent interest outside of binned scatter plots. To help place our results in the literature, we include a remark labelled "Improvements over literature" at the end of each technical subsection that discusses in detail the technical improvements presented in that subsection and gives related references.

Here we give a brief summary of this appendix, including pointers to some of the major new results. We proceed as follows. The next subsection lists notation used throughout; further notation is defined throughout Section SA-2 and at the outset of Section SA-3. Then, Section SA-2 describes the setup for nonlinear binscatter methods, including the statistical model, parameters of interest, and assumptions, as well as the (random) partitioning and estimation details. Specifically, Assumption SA-DGP imposes some basic conditions on the data generating process. Assumption SA-SM imposes some technical conditions that characterizes and restricts the statistical model of interest. The loss function specified there is general enough to cover many practically important examples such as mean regression, quantile regression, logit/probit estimation, and Huber regression. Assumption SA-HLE imposes some mild high-level conditions on the estimation procedure. Assumption SA-RP summarizes the key conditions on the partitioning scheme used in our theory. We allow for a large class of random partitions. Importantly, the "convergence" of the random partition (Assumption SA-RP(ii)) is not necessary for most of our main theoretical results, thereby allowing for flexible data-driven partitioning methods, including certain recursive adaptive partitioning methods: see Devroye et al. (2013), Zhang and Singer (2010), and references therein.

Section SA-3.1 presents some preliminary technical lemmas for analyzing nonlinear binscatter (and thus also partitioning-based estimators more broadly). New results include precise non-asymptotic concentration results related to Gram matrices (Lemma SA-3.1), asymptotic variances (Lemmas SA-3.2 and SA-3.3), approximation errors (Lemma SA-3.4), and uniform convergence (Lemma SA-3.5). Sharp control of these objects is a crucial ingredient for obtaining results under weak conditions, as below.

Section SA-3.2 presents a tight uniform (in x) Bahadur representation for nonlinear binscatter (Theorem SA-3.1). This is our first main result. We allow for random partitions and much weaker rate restrictions (on the tuning parameter J) than previously imposed in the literature, in addition to additional controls. The data-dependent partitioning means our series estimator uses random basis functions, and this is entirely new. In terms of tuning parameter rate restrictions, previous results required $J^4/n \to 0$ (up to $\log(n)$ terms) or something stricter, while our restriction is that $J^{\frac{2\nu}{\nu-1}}/n \to 0$ (up to $\log(n)$ terms), with $\nu > 2$ denoting the number of finite moments of the "score", and thus may be substantially weaker. Note that our class of models is often broader than prior work also. Importantly, our results can now allow for piecewise constant binscatters, i.e., with degree $p = 0$, which is excluded by prior results in the literature (i.e., for previous technical results there was no sequence $J \to \infty$ such that the bias and variance are simultaneously controlled). In addition, employing our novel uniform Bahadur representation, we can establish the uniform convergence rates of nonlinear binscatter (Corollary SA-3.1) and variance estimators (Theorem SA-3.2) under similarly weak restrictions.

Section SA-3.3 studies the pointwise distributional approximation for nonlinear binscatter estimators. These results are omitted from the main paper to save space, but are standard properties of interest in the nonparametrics literature and thus are included for completeness. The main result is Theorem SA-3.3, which establishes pointwise asymptotic Normality for our point estimators, again allowing for random (and possibly "non-convergent") partitions, and under mild rate restrictions similar to those for the (uniform) Bahadur representation.

Section SA-3.4 presents a new Nagar-type approximate IMSE expansion for nonlinear binscatter estimators with semi-linear covariate-adjustment and random partitions (Theorem SA-3.4), which has no antecedent in the literature. Our results can be used to design data-driven procedures for selecting IMSE-optimal choices of tuning parameters for nonlinear binscatter. Again, these results are novel in their breadth, the weakness of the assumptions, and the conditions on the partitioning. Here we do require an extra assumption on the partitioning in order to characterize the leading terms in the expansion: intuitively, the random partitioning must "settle" to a population partition so that the leading constants of the expansion can be expressed. For example, sample quantiles converge to population quantiles, so this assumption is satisfied.

Uniform inference is dealt with in the next several sections of this appendix. First, Section SA-3.5

establishes a uniform (in x) distributional approximation for nonlinear binscatter estimators. The two main results, which are combined into one in the main text (Theorem 2), are the (conditional) strong approximation in Theorem SA-3.5 and the feasible implementation thereof in Theorem SA-3.6. Again, We allow for a large class of random partitions, a broad class of (possibly) nonlinear and nonsmooth models, and additional controls. Here the partitions do not need to be "convergent" in any sense. These results are obtained under weak assumptions, including in particular mild rate restrictions, as in the case of the uniform Bahadur representation, all of which improves on the literature in various directions as explained in the text below. Finally, Theorem SA-3.7 shows a distributional approximation for the suprema of the t-statistic processes in the case of the convergent partition (as in the previous paragraph).

Sections SA-3.6–SA-3.8 employ the strong approximation results to study uniform inference for various parameters of interest in the specific context of nonlinear binscatter. These results rely on, and inherit the novelty of, Theorems SA-3.5 and SA-3.6. New results include valid uniform confidence bands (Theorem SA-3.8), consistent hypothesis tests about parametric specification (Theorem SA-3.9) and tests for shape restrictions (Theorem SA-3.10). All these results explicitly account for the possibly random partitioning scheme and semi-linear covariate-adjustment with random evaluation points.

Section SA-4 discusses implementation details for nonlinear binscatter, including standard error computation, feasible data-driven number of bins selector, and choices of polynomial orders given a fixed number of bins. For a more explicit treatment of the package `binsreg` per se, see Cattaneo et al. (2024a) and `https://nppackages.github.io/binsreg/`.

Finally, Section SA-5 contains the proofs for all the technical results in Section SA-3.

SA-1.1 Notation

See van der Vaart and Wellner (1996), Bhatia (2013), Giné and Nickl (2016), and references therein, for background definitions.

Matrices and Norms. For (column) vectors, $\|\cdot\|$ denotes the Euclidean norm, $\|\cdot\|_1$ denotes the L_1 norm, $\|\cdot\|_\infty$ denotes the sup-norm, and $\|\cdot\|_0$ denotes the number of nonzeros. For matrices, $\|\cdot\|$ is the operator matrix norm induced by the L_2 norm, and $\|\cdot\|_\infty$ is the matrix norm induced by the supremum norm, i.e., the maximum absolute row sum of a matrix. For a square matrix

$\mathbf{A}$, $\lambda_{\max}(\mathbf{A})$ and $\lambda_{\min}(\mathbf{A})$ are the maximum and minimum eigenvalues of $\mathbf{A}$, respectively. $[\mathbf{A}]_{ij}$ denotes the (i,j)th entry of a generic matrix $\mathbf{A}$. We will use $\mathcal{S}^L$ to denote the unit circle in $\mathbb{R}^L$, i.e., $\|\mathbf{a}\| = 1$ for any $\mathbf{a} \in \mathcal{S}^L$. For a real-valued function $g(\cdot)$ defined on a measure space $\mathcal{Z}$, let $\|g\|_{\mathbb{Q},2} := (\int_{\mathcal{Z}} |g|^2 d\mathbb{Q})^{1/2}$ be its L_2-norm with respect to the measure $\mathbb{Q}$. In addition, let $\|g\|_\infty = \sup_{z\in\mathcal{Z}} |g(z)|$ be L_∞-norm of $g(\cdot)$, and if g is a univariate function, let $g^{(v)}(z) = d^v g(z)/dz^v$ be the vth derivative for $v \geq 0$.

Asymptotics. For sequences of numbers or random variables, we use $l_n \lesssim m_n$ to denote that $\limsup_n |l_n/m_n|$ is finite, $l_n \lesssim_{\mathbb{P}} m_n$ or $l_n = O_{\mathbb{P}}(m_n)$ to denote $\limsup_{\varepsilon\to\infty} \limsup_n \mathbb{P}[|l_n/m_n| \geq \varepsilon] = 0$, $l_n = o(m_n)$ implies $l_n/m_n \to 0$, and $l_n = o_{\mathbb{P}}(m_n)$ implies that $l_n/m_n \to_{\mathbb{P}} 0$, where $\to_{\mathbb{P}}$ denotes convergence in probability. Accordingly, we write $l_n \gtrsim m_n$ if $m_n \lesssim l_n$, and $l_n \gtrsim_{\mathbb{P}} m_n$ if $m_n \lesssim_{\mathbb{P}} l_n$. $l_n \asymp m_n$ implies that $l_n \lesssim m_n$ and $m_n \lesssim l_n$.

Empirical Process. We employ standard empirical process notation: $\mathbb{E}_n[g(\mathbf{v}_i)] = \frac{1}{n}\sum_{i=1}^n g(\mathbf{v}_i)$, and $\mathbb{G}_n[g(\mathbf{v}_i)] = \frac{1}{\sqrt{n}}\sum_{i=1}^n (g(\mathbf{v}_i) - \mathbb{E}[g(\mathbf{v}_i)])$ for a sequence of random variables $\{\mathbf{v}_i\}_{i=1}^n$. In addition, we employ the notion of covering number extensively in the proofs. Specifically, given a measurable space $(A, \mathcal{A})$ and a suitably measurable class of functions $\mathcal{G}$ mapping A to $\mathbb{R}$ equipped with a measurable envelop function $\bar{G}(z) \geq \sup_{g\in\mathcal{G}} |g(z)|$, the *covering number* of $N(\mathcal{G}, L_2(\mathbb{Q}), \varepsilon)$ is the minimal number of $L_2(\mathbb{Q})$-balls of radius ε needed to cover $\mathcal{G}$ for a measure $\mathbb{Q}$. The covering number of $\mathcal{G}$ relative to the envelope is denoted as $N(\mathcal{G}, L_2(\mathbb{Q}), \varepsilon\|\bar{G}\|_{\mathbb{Q},2})$.

Other. $\lceil z \rceil$ outputs the smallest integer no less than z and $a \wedge b = \min\{a, b\}$. "w.p.a. 1" means "with probability approaching one".

SA-2 Setup

Suppose that $(y_i, x_i, \mathbf{w}_i')$, $1 \leq i \leq n$, is a random sample where $y_i \in \mathcal{Y}$ is a scalar response variable, $x_i \in \mathcal{X}$ is a scalar covariate, and $\mathbf{w}_i \in \mathcal{W}$ is a vector of additional control variables of dimension d. Let $\mathbf{D} = [(y_i, x_i, \mathbf{w}_i')' : i = 1, 2, \ldots, n]$.

For a loss function $\rho(\cdot;\cdot)$ and a strictly monotonic transformation function $\eta(\cdot)$, define

$$(\mu_0(\cdot), \boldsymbol{\gamma}_0) = \underset{\mu\in\mathcal{M}, \boldsymbol{\gamma}\in\mathbb{R}^d}{\arg\min}\ \mathbb{E}\Big[\rho\Big(y_i; \eta(\mu(x_i) + \mathbf{w}_i'\boldsymbol{\gamma})\Big)\Big], \tag{SA-2.1}$$

where $\mathcal{M}$ is a space of functions satisfying certain smoothness conditions to be specified later.

This setup is general. For example, consider $\boldsymbol{\gamma}_0 = \mathbf{0}$. If $\rho(\cdot;\cdot)$ is a squared loss and $\eta(\cdot)$ is the identity function, $\mu_0(x)$ is the conditional expectation of y_i given $x_i = x$. Let $\mathbb{1}(\cdot)$ denote the indicator function. If $\rho(y;\eta) = (q - \mathbb{1}(y < \eta))(y - \eta)$ for some $0 < q < 1$ and $\eta(\cdot)$ is an identity function, then $\mu_0(x)$ is the qth conditional quantile of y_i given $x_i = x$. Introducing a transformation function $\eta(\cdot)$ is useful. For instance, it may accommodate logistic regression for binary responses. When $\boldsymbol{\gamma}_0 \neq \mathbf{0}$, the parametric and the nonparametric components are additively separable, and thus (SA-2.1) becomes a generalized partially linear model.

Binscatter estimators are typically constructed based on a (possibly random) partition of the support of the covariate x_i. Specifically, the relevant support of x_i is partitioned into J disjoint intervals, leading to the partitioning scheme $\widehat{\Delta} = \{\widehat{\mathcal{B}}_1, \widehat{\mathcal{B}}_2, \dots, \widehat{\mathcal{B}}_J\}$, where

$$\widehat{\mathcal{B}}_j = \begin{cases} [\widehat{\tau}_{j-1}, \widehat{\tau}_j) & \text{if } j = 1, \cdots, J-1 \\ [\widehat{\tau}_{J-1}, \widehat{\tau}_J] & \text{if } j = J \end{cases},$$

One popular choice in binscatter applications is the quantile-based partition: $\widehat{\tau}_j = \widehat{F}_X^{-1}(j/J)$ with $\widehat{F}_X(u) = n^{-1}\sum_{i=1}^n \mathbb{1}(x_i \leq u)$ the empirical cumulative distribution function and $\widehat{F}_X^{-1}$ its generalized inverse. Our theory is general enough to cover other partitioning schemes satisfying certain regularity conditions specified below. An innovation herein is accounting for the additional randomness from the partition $\widehat{\Delta}$. The number of bins J plays the role of the tuning parameter for the binscatter method, and is assumed to diverge: $J \to \infty$ as $n \to \infty$ throughout the supplement, unless explicitly stated otherwise.

The piecewise polynomial basis of degree p, for some choice of $p = 0, 1, 2, \dots,$ is defined as

$$\begin{bmatrix} \mathbb{1}_{\widehat{\mathcal{B}}_1}(x) & \mathbb{1}_{\widehat{\mathcal{B}}_2}(x) & \cdots & \mathbb{1}_{\widehat{\mathcal{B}}_J}(x) \end{bmatrix}' \otimes \begin{bmatrix} 1 & x & \cdots & x^p \end{bmatrix}',$$

where $\mathbb{1}_{\mathcal{A}}(x) = \mathbb{1}(x \in \mathcal{A})$ and $\otimes$ is the Kronecker product operator. For convenience of later analysis, we use $\widehat{\mathbf{b}}_{p,0}(x)$ to denote a *standardized rotated* basis, the jth element of which is given by

$$\sqrt{J} \times \mathbb{1}_{\widehat{\mathcal{B}}_{\bar{j}}}(x) \times \Big(\frac{x - \widehat{\tau}_{\bar{j}-1}}{\hat{h}_{\bar{j}}}\Big)^{j-1-(\bar{j}-1)(p+1)}, \quad j = 1, \cdots, (p+1)J,$$

where $\bar{j} = \lceil j/(p+1) \rceil$, $\lceil \cdot \rceil$ is the ceiling operator, and $\hat{h}_{\bar{j}} = \widehat{\tau}_{\bar{j}} - \widehat{\tau}_{\bar{j}-1}$. Thus, each local polynomial is centered at the start of each bin and scaled by the length of the bin. $\sqrt{J}$ is an additional scaling factor which helps simplify some expressions of our results. The standardized rotated basis $\widehat{\mathbf{b}}_{p,0}(x)$ is equivalent to the original piecewise polynomial basis in the sense that they represent the same (linear) function space.

To impose the restriction that the estimated function is $(s-1)$-times continuously differentiable for $1 \leq s \leq p$, we introduce the following basis

$$\widehat{\mathbf{b}}_{p,s}(x) = \Big(\widehat{b}_{p,s,1}(x), \dots, \widehat{b}_{p,s,K_{p,s}}(x)\Big)' = \widehat{\mathbf{T}}_s \widehat{\mathbf{b}}_{p,0}(x), \qquad K_{p,s} = (p+1)J - s(J-1),$$

where $\widehat{\mathbf{T}}_s := \widehat{\mathbf{T}}_s(\widehat{\Delta})$ is a $K_{p,s} \times (p+1)J$ matrix depending on $\widehat{\Delta}$, which transforms a piecewise polynomial basis into a smoothed binscatter basis. Some useful properties of $\widehat{\mathbf{T}}_s$ are given in Lemma SA-5.3 in Section SA-5, and the explicit representation of $\widehat{\mathbf{T}}_s$ is available in the proof of Lemma SA-3.2 in Cattaneo, Crump, Farrell and Feng (2024b). When $s = 0$, we let $\widehat{\mathbf{T}}_0 = \mathbf{I}_{(p+1)J}$, the identity matrix of dimension $(p+1)J$. When $s = p$, $\widehat{\mathbf{b}}_{p,s}(x)$ is the well-known B-spline basis of order $p+1$ with simple knots, which is $(p-1)$-times continuously differentiable. When $0 < s < p$, they can be defined similarly as B-splines with knots of certain multiplicities. See Definition 4.1 in Section 4 of Schumaker (2007) for more details about spline functions. We require $s \leq p$, since if $s = p+1$, $\widehat{\mathbf{b}}_{p,s}(x)$ reduces to a global polynomial basis of degree p.

Given a choice of basis, we consider the following generalized binscatter estimator:

$$\widehat{\mu}^{(v)}_{p,s}(x) := \widehat{\mathbf{b}}^{(v)}_{p,s}(x)'\widehat{\boldsymbol{\beta}}, \qquad \begin{bmatrix} \widehat{\boldsymbol{\beta}} \\ \widehat{\boldsymbol{\gamma}} \end{bmatrix} = \operatorname*{arg\,min}_{\boldsymbol{\beta},\boldsymbol{\gamma}} \sum_{i=1}^{n} \rho\Big(y_i;\ \eta\big(\widehat{\mathbf{b}}_{p,s}(x_i)'\boldsymbol{\beta} + \mathbf{w}_i'\boldsymbol{\gamma}\big)\Big), \tag{SA-2.2}$$

where $\widehat{\mathbf{b}}^{(v)}_{p,s}(x) = \frac{d^v}{dx^v}\widehat{\mathbf{b}}_{p,s}(x)$ for some $v \in \mathbb{Z}_+$ such that $v \leq p$. This estimator can be written as:

$$\widehat{\mu}^{(v)}_{p,s}(x) = \widehat{\mathbf{b}}^{(v)}_{p,s}(x)'\widehat{\boldsymbol{\beta}}, \quad \widehat{\boldsymbol{\beta}} := \widehat{\boldsymbol{\beta}}(\widehat{\boldsymbol{\gamma}}) := \operatorname*{arg\,min}_{\boldsymbol{\beta} \in \mathbb{R}^{K_{p,s}}} \sum_{i=1}^{n} \rho\Big(y_i;\ \eta\big(\widehat{\mathbf{b}}_{p,s}(x_i)'\boldsymbol{\beta} + \mathbf{w}_i'\widehat{\boldsymbol{\gamma}}\big)\Big). \tag{SA-2.3}$$

The representation (SA-2.3) allows us to be more general and agnostic about the estimation of $\boldsymbol{\gamma}_0$, and also simplifies some of the proofs. More specifically, our theory requires only a sufficiently fast convergence rate of $\widehat{\boldsymbol{\gamma}}$ (see Assumption SA-HLE below), which in nonlinear estimation models cases can be justified in different ways, e.g., joint estimation, backfitting, profiling, and split-sampling, among other possibilities. Our software implementation (Cattaneo, Crump, Farrell and Feng, 2024a) relies on joint estimation, as done by the default base estimation packages in `Python`, `R`, and `Stata`.

In this supplement, we focus on estimation and inference of the following three parameters:

(i) the nonparametric component $\mu_0^{(v)}(x)$ for any $v \geq 0$,

(ii) the level function $\vartheta_0(x, \mathsf{w}) = \eta(\mu_0(x) + \mathsf{w}'\boldsymbol{\gamma}_0)$, and

(iii) the marginal effect $\zeta_0(x, \mathsf{w}) = \frac{\partial}{\partial x}\eta(\mu_0(x) + \mathsf{w}'\boldsymbol{\gamma}_0)$,

where w is a user-chosen evaluation point of the control variables, and thus these parameters are viewed as functions of x only in our theory. Nevertheless, all our results are readily applied to other linear or nonlinear transformations of $\mu_0(x)$, such as the higher-order derivatives $\frac{\partial^v}{\partial x^v}\eta(\mu_0(x)+\mathsf{w}'\boldsymbol{\gamma}_0)$. Given the binscatter estimates $\widehat{\mu}_{p,s}(x)$ and $\widehat{\boldsymbol{\gamma}}$ in (SA-2.2), the estimators of the three parameters defined above are given by

$$\widehat{\mu}^{(v)}_{p,s}(x), \quad \widehat{\vartheta}_{p,s}(x, \widehat{\mathsf{w}}) = \eta(\widehat{\mu}_{p,s}(x) + \widehat{\mathsf{w}}'\widehat{\boldsymbol{\gamma}}), \quad \text{and} \quad \widehat{\zeta}_{p,s}(x, \widehat{\mathsf{w}}) = \eta^{(1)}(\widehat{\mu}_{p,s}(x) + \widehat{\mathsf{w}}'\widehat{\boldsymbol{\gamma}})\widehat{\mu}^{(1)}_{p,s}(x)$$

respectively, for some consistent estimate $\widehat{\mathsf{w}}$ (non-random or generated based on $\{\mathbf{w}_i\}_{i=1}^n$) of the evaluation point w. As a reminder, we need to require $p \geq v$ to get $\widehat{\mu}^{(v)}_{p,s}(x)$, $p \geq 0$ to get $\widehat{\vartheta}_{p,s}(x, \widehat{\mathsf{w}})$, and $p \geq 1$ to get $\widehat{\zeta}_{p,s}(x, \widehat{\mathsf{w}})$.

Recall that in the main text we always set $s = p$ and omit the dependence of estimators on s. Thus, $\widehat{\mu}^{(v)}_p(x) = \widehat{\mu}^{(v)}_{p,p}(x)$, $\widehat{\vartheta}_p(x, \widehat{\mathsf{w}}) = \widehat{\vartheta}_{p,p}(x, \widehat{\mathsf{w}})$, and $\widehat{\zeta}_p(x, \widehat{\mathsf{w}}) = \widehat{\zeta}_{p,p}(x, \widehat{\mathsf{w}})$. In this supplement, however, all our results hold for a general choice of the degree and the smoothness of the basis. For ease of notation, the subscripts p and s of the above estimators are dropped hereafter:

$$\widehat{\mu}^{(v)}(x) := \widehat{\mu}^{(v)}_{p,s}(x), \quad \widehat{\vartheta}(x, \widehat{\mathsf{w}}) := \widehat{\vartheta}_{p,s}(x, \widehat{\mathsf{w}}), \quad \text{and} \quad \widehat{\zeta}(x, \widehat{\mathsf{w}}) := \widehat{\zeta}_{p,s}(x, \widehat{\mathsf{w}}).$$

Remark SA-2.1 (Smoothness and Bias Correction)**.** This supplemental appendix presents *all* results under general choices of the number of bins J, the degree of the basis p, and the smoothness of the basis s. By contrast, for simplicity, the main paper employs the basis with the maximum smoothness, i.e. choosing $s = p$, and considers the special case in which J is taken to be the IMSE-optimal choice for a fixed p (see Theorem SA-3.4), and inference is conducted based on the binscatter basis of degree $(p+1)$. Such a choice of J guarantees that the smoothing bias of the binscatter estimator is negligible in inference under mild conditions and thus can be viewed as a bias correction strategy. ⌟

We first assume the following basic conditions on the data generating process.

Assumption SA-DGP (Data Generating Process)**.**

(i) $\{(y_i, x_i, \mathbf{w}_i') : 1 \leq i \leq n\}$ *are i.i.d. random vectors satisfying* (SA-2.1) *and supported on* $\mathcal{Y} \times \mathcal{X} \times \mathcal{W}$*, where* $\mathcal{X}$ *is a compact interval and* $\mathcal{W}$ *is a compact set.*

(ii) $F_X(x) := \mathbb{P}[x_i \leq x]$ *has a Lipschitz continuous (Lebesgue) density* $f_X(x)$ *bounded away from zero on* $\mathcal{X}$*.*

(iii) $F_{Y|XW}(y|x_i, \mathbf{w}_i) := \mathbb{P}[y_i \leq y|x_i, \mathbf{w}_i]$ *has a (conditional) density* $f_{Y|XW}(y|x_i, \mathbf{w}_i)$ *supported on* $\mathcal{Y}_{x\mathbf{w}}$ *with respect to some sigma-finite measure, and* $\sup_{x\in\mathcal{X},\mathbf{w}\in\mathcal{W}} \sup_{y\in\mathcal{Y}_{x\mathbf{w}}} f_{Y|XW}(y|x, \mathbf{w}) \lesssim 1$.

Next, we impose several technical conditions related to the statistical model of interest.

Assumption SA-SM (Statistical Model)**.**

(i) $\rho(y; \eta)$ *is absolutely continuous with respect to* $\eta \in \mathbb{R}$ *and admits a derivative* $\psi(y, \eta) := \psi^\dagger(y - \eta)\psi^\ddagger(\eta)$ *almost everywhere.* $\psi^\ddagger(\cdot)$ *is continuously differentiable and strictly positive or negative.* $\psi^\dagger(\cdot)$ *is Lipschitz continuous if* $F_{Y|XW}(y|x_i, \mathbf{w}_i)$ *does not have a Lebesgue density, or piecewise Lipschitz with finitely many discontinuity points otherwise.*

(ii) $\rho(y; \eta(\theta))$ *is convex with respect to* θ*.* $\eta(\cdot)$ *is strictly monotonic and three-times continuously differentiable.*

(iii) $\mathbb{E}[\psi(y_i, \eta(\mu_0(x_i)+\mathbf{w}_i'\gamma_0))|x_i, \mathbf{w}_i] = 0$*. For* $\sigma^2(x, \mathbf{w}) := \mathbb{E}[\psi(y_i, \eta(\mu_0(x_i)+\mathbf{w}_i'\gamma_0))^2|x_i = x, \mathbf{w}_i = \mathbf{w}]$*,* $\inf_{x\in\mathcal{X},\mathbf{w}\in\mathcal{W}} \sigma^2(x, \mathbf{w}) \gtrsim 1$*.* $\mathbb{E}[\eta^{(1)}(\mu_0(x_i)+\mathbf{w}_i'\gamma_0)^2\sigma^2(x_i, \mathbf{w}_i)|x_i = x]$ *is Lipschitz continuous on* $\mathcal{X}$*, and* $\sup_{x\in\mathcal{X},\mathbf{w}\in\mathcal{W}} \mathbb{E}[|\psi(y_i, \eta(\mu_0(x_i) + \mathbf{w}_i'\gamma_0))|^\nu|x_i = x, \mathbf{w}_i = \mathbf{w}] \lesssim 1$ *for some* $\nu > 2$*.* $\mathbb{E}[\psi(y_i, \eta)|x_i = x, \mathbf{w}_i = \mathbf{w}]$ *is twice continuously differentiable with respect to* η*.*

(iv) $\inf_{x\in\mathcal{X},\mathbf{w}\in\mathcal{W}} \varkappa(x,\mathbf{w}) \gtrsim 1$ *and* $\mathbb{E}[\varkappa(x_i,\mathbf{w}_i)|x_i=x]$ *is Lipschitz continuous on* $\mathcal{X}$ *where* $\varkappa(x,\mathbf{w}) := \Psi_1(x,\mathbf{w};\eta(\mu_0(x)+\mathbf{w}'\boldsymbol{\gamma}_0))(\eta^{(1)}(\mu_0(x)+\mathbf{w}'\boldsymbol{\gamma}_0))^2$, $\Psi_1(x,\mathbf{w};\eta) := \frac{\partial}{\partial\eta}\Psi(x,\mathbf{w};\eta)$, *and* $\Psi(x,\mathbf{w};\eta) := \mathbb{E}[\psi(y_i,\eta)|x_i=x,\mathbf{w}_i=\mathbf{w}]$.

(v) $\mu_0(\cdot)$ *is* ς*-times continuously differentiable for some* $\varsigma \geq p+1$.

Our next assumption imposes mild high-level conditions on the estimator $\widehat{\boldsymbol{\gamma}}$ of the coefficient vector $\boldsymbol{\gamma}_0$, the estimator $\widehat{\mathsf{w}}$ of the evaluation point w for control variables, and the estimator of the function Ψ_1 defined previously in Assumption SA-SM(iv).

Assumption SA-HLE (High-Level Estimation Conditions)**.**

(i) $\|\widehat{\boldsymbol{\gamma}}-\boldsymbol{\gamma}_0\| \lesssim_{\mathbb{P}} \mathfrak{r}_\gamma$ *for* $\mathfrak{r}_\gamma = o(\sqrt{J/n}+J^{-p-1})$, *and* $\|\widehat{\mathsf{w}}-\mathsf{w}\| = o_{\mathbb{P}}(1)$.

(ii) For some estimator $\widehat{\Psi}_1$ *of* Ψ_1, $\|\mathbb{E}_n[\widehat{\mathbf{b}}_{p,s}(x_i)\widehat{\mathbf{b}}_{p,s}(x_i)'(\widehat{\varkappa}(x_i,\mathbf{w}_i)-\varkappa(x_i,\mathbf{w}_i))\| \lesssim_{\mathbb{P}} J^{-p-1} + \left(\frac{J\log n}{n^{1-2/\nu}}\right)^{1/2}$ *where* $\widehat{\varkappa}(x_i,\mathbf{w}_i) = \widehat{\Psi}_1(x_i,\mathbf{w}_i;\eta(\widehat{\mu}(x_i)+\mathbf{w}_i'\widehat{\boldsymbol{\gamma}}))(\eta^{(1)}(\widehat{\mu}(x_i)+\mathbf{w}_i'\widehat{\boldsymbol{\gamma}}))^2$.

Note that $\Upsilon(x,\mathbf{w}) = \Psi_1(x,\mathbf{w};\eta(\mu_0(x)+\mathbf{w}'\boldsymbol{\gamma}_0))$ in the main paper to streamline the presentation. Part (i) is a mild condition on the convergence of $\widehat{\boldsymbol{\gamma}}$ and $\widehat{\mathsf{w}}$. Part (ii) is a high-level condition that ensures we have a valid feasible estimator of the Gram matrix ($\bar{\mathbf{Q}}$ or $\mathbf{Q}_0$ defined at the outset of Section SA-3 below). Note that the convergence rate of $\eta^{(1)}(\widehat{\mu}(x_i)+\mathbf{w}_i'\widehat{\boldsymbol{\gamma}})$ can be deduced from Corollary SA-3.1 below. Thus, part (ii) can be largely viewed as a restriction on $\widehat{\Psi}_1$ only. Note that $\widehat{\Psi}_1$ does not have to be consistent for Ψ_1 in any sense; it suffices that the estimator $\mathbb{E}_n[\widehat{\mathbf{b}}_{p,s}(x_i)\widehat{\mathbf{b}}_{p,s}(x_i)'\widehat{\varkappa}(x_i,\mathbf{w}_i)]$ based on $\widehat{\Psi}_1$ as a whole is consistent. See Section SA-4 for several examples of the estimator $\widehat{\Psi}_1$.

SA-2.1 Partitions

We need some regularity conditions on the partitioning scheme, which can be verified in a case-by-case basis. We first define a family of "quasi-uniform" partitions for some absolute constant $C>0$:

$$\Pi_C = \left\{\Delta : \frac{\max_{1\leq j\leq J} h_j(\Delta)}{\min_{1\leq j\leq J} h_j(\Delta)} \leq C\right\}, \tag{SA-2.4}$$

where $h_j(\Delta)$ denotes the length of the jth bin in the partition Δ. Roughly speaking, (SA-2.4) says that the bins in any $\Delta \in \Pi_C$ do not differ too much in length. Also, let $\mathbf{X} = [x_1,\ldots,x_n]'$, $\mathbf{W} = [\mathbf{w}_1,\cdots,\mathbf{w}_n]'$ and $\mathbf{Y} = [y_1,\cdots,y_n]'$.

Assumption SA-RP (Random Partition)**.**

(i) $\widehat{\Delta} \perp\!\!\!\perp \mathbf{Y}|(\mathbf{X},\mathbf{W})$ *and* $\widehat{\Delta} \in \Pi_C$ *w.p.a.* 1 *for some absolute constant* $C > 0$.

(ii) There exists a non-random partition $\Delta_0 = \{\mathcal{B}_1, \cdots, \mathcal{B}_J\}$ *with* $\mathcal{B}_j = [\tau_{j-1}, \tau_j)$ *for* $j \leq J-1$ *and* $\mathcal{B}_J = [\tau_{J-1}, \tau_J]$ *such that* $\frac{\max_{1\leq j\leq J} h_j}{\min_{1\leq j\leq J} h_j} \leq c_{\texttt{QU}}$ *for some absolute constant* $c_{\texttt{QU}} > 0$, *and* $\max_{1\leq j\leq J} |\hat{h}_j - h_j| \lesssim_{\mathbb{P}} J^{-1}\mathfrak{r}_{\texttt{RP}}$ *for* $\mathfrak{r}_{\texttt{RP}} = o(1)$.

Part (i) is the key condition for our main results and will be imposed throughout. First, it requires that the possibly random partition $\widehat{\Delta}$ be independent of the outcome $\mathbf{Y}$ given the covariates $(\mathbf{X},\mathbf{W})$. This conditional independence assumption is trivially satisfied if $\widehat{\Delta}$ is deterministic (e.g., equally-spaced partition) or depends on $\mathbf{X}$ and $\mathbf{W}$ only (e.g., quantile-spaced partition based on $\mathbf{X}$). It also holds if a sample splitting scheme is used: a subsample (including the information about the outcome) is used for constructing the partition, and the other is employed to construct the binscatter estimator (so that $\widehat{\Delta}$ is independent of the data $(\mathbf{X},\mathbf{W},\mathbf{Y})$). Second, $\widehat{\Delta}$ is required to be "quasi-uniform" with large probability. It is trivially true for equally-spaced partitions and can be verified for quantile-spaced partitions under the mild conditions on the covariates density imposed before (see Lemma SA-5.2). However, this condition may be too restrictive for other modern machine-learning-based partitioning methods, in which case some additional regularization may be necessary to recover the quasi-uniformity property.

Part (ii) requires that the random partition $\widehat{\Delta}$ "stabilizes" to a fixed one in large samples. This is true if the partition is non-deterministic or generated by sample quantiles (since sample quantiles converge to population quantiles), but more generally, it is not always possible. Fortunately, this "convergence" requirement is not necessary for most of our main results (except Theorem SA-3.4 and Theorem SA-3.7). Thus, we will always make it clear if part (ii) of Assumption SA-RP is imposed.

Given the random partition $\widehat{\Delta}$, we use the notation $\mathbb{E}_{\widehat{\Delta}}[\cdot]$ to denote the expectation operator with the partition $\widehat{\Delta}$ viewed as fixed. To further simplify notation, let $\hat{h}_j = \hat{\tau}_j - \hat{\tau}_{j-1}$ be the width of the jth bin $\widehat{\mathcal{B}}_j$, and when the "limiting" partition $\Delta_0 = \{\mathcal{B}_1, \cdots, \mathcal{B}_J\}$ is defined (Assumption SA-RP(ii) holds), let h_j be the width of $\mathcal{B}_j$. Analogously to $\widehat{\mathbf{b}}_{p,s}(x)$, $\mathbf{b}_{p,s}(x)$ denotes the binscatter basis of degree p that is $(s-1)$-times continuously differentiable and is constructed based on the *nonrandom* partition Δ_0. We sometimes write $\mathbf{b}_{p,s}(x;\Delta) = (b_{p,s,1}(x;\Delta), \ldots, b_{p,s,K_{p,s}}(x;\Delta))'$ to emphasize a binscatter basis is constructed based on a particular partition Δ. Therefore, $\widehat{\mathbf{b}}_{p,s}(x) = \mathbf{b}_{p,s}(x; \widehat{\Delta})$ and $\mathbf{b}_{p,s}(x) = \mathbf{b}_{p,s}(x; \Delta_0)$. Accordingly, we use $\mathbf{T}_s$ to denote the transformation matrix based on the non-random partition Δ_0 (which transforms $\mathbf{b}_{p,0}(x)$ to $\mathbf{b}_{p,s}(x)$).

SA-3 Main Results

We introduce the following quantities that will be extensively used throughout the supplement:

$$
\begin{aligned}
&\eta_i = \eta(\mu_0(x_i) + \mathbf{w}_i'\boldsymbol{\gamma}_0), \qquad \widehat{\eta}_i = \eta(\widehat{\mu}(x_i) + \mathbf{w}_i'\widehat{\boldsymbol{\gamma}}),\\
&\eta_{i,1} = \eta^{(1)}(\mu_0(x_i) + \mathbf{w}_i'\boldsymbol{\gamma}_0), \qquad \widehat{\eta}_{i,1} = \eta^{(1)}(\widehat{\mu}(x_i) + \mathbf{w}_i'\widehat{\boldsymbol{\gamma}}),\\
&\eta_{0,1}(x, \mathsf{w}) = \eta^{(1)}(\mu_0(x) + \mathsf{w}'\boldsymbol{\gamma}_0), \qquad \widehat{\eta}_{0,1}(x, \widehat{\mathsf{w}}) = \eta^{(1)}(\widehat{\mu}(x) + \widehat{\mathsf{w}}'\widehat{\boldsymbol{\gamma}}),\\
&\widehat{\mu}(x_i) = \widehat{\mathbf{b}}_{p,s}(x_i)'\widehat{\boldsymbol{\beta}}, \qquad \epsilon_i = y_i - \eta_i, \qquad \widehat{\epsilon}_i = y_i - \widehat{\eta}_i,\\
&\widehat{\mathbf{Q}}_{p,s} := \widehat{\mathbf{Q}}_{p,s}(\widehat{\Delta}) := \mathbb{E}_n[\widehat{\mathbf{b}}_{p,s}(x_i)\widehat{\mathbf{b}}_{p,s}(x_i)'\Psi_1(x_i, \mathbf{w}_i; \widehat{\eta}_i)\widehat{\eta}_{i,1}^2],\\
&\bar{\mathbf{Q}}_{p,s} := \bar{\mathbf{Q}}_{p,s}(\widehat{\Delta}) := \mathbb{E}_n[\widehat{\mathbf{b}}_{p,s}(x_i)\widehat{\mathbf{b}}_{p,s}(x_i)'\Psi_1(x_i, \mathbf{w}_i; \eta_i)\eta_{i,1}^2],\\
&\mathbf{Q}_{0,p,s} := \mathbf{Q}_{p,s}(\Delta_0) := \mathbb{E}[\mathbf{b}_{p,s}(x_i)\mathbf{b}_{p,s}(x_i)'\Psi_1(x_i, \mathbf{w}_i; \eta_i)\eta_{i,1}^2],\\
&\widehat{\boldsymbol{\Sigma}}_{p,s} := \widehat{\boldsymbol{\Sigma}}_{p,s}(\widehat{\Delta}) := \mathbb{E}_n[\widehat{\mathbf{b}}_{p,s}(x_i)\widehat{\mathbf{b}}_{p,s}(x_i)'\psi(y_i, \widehat{\eta}_i)^2\widehat{\eta}_{i,1}^2],\\
&\bar{\boldsymbol{\Sigma}}_{p,s} := \bar{\boldsymbol{\Sigma}}_{p,s}(\widehat{\Delta}) := \mathbb{E}_n\Big[\mathbb{E}\Big[\widehat{\mathbf{b}}_{p,s}(x_i)\widehat{\mathbf{b}}_{p,s}(x_i)'\psi(y_i, \eta_i)^2\eta_{i,1}^2\Big|\mathbf{X}, \mathbf{W}\Big]\Big],\\
&\boldsymbol{\Sigma}_{0,p,s} := \boldsymbol{\Sigma}_{p,s}(\Delta_0) := \mathbb{E}\Big[\mathbf{b}_{p,s}(x_i)\mathbf{b}_{p,s}(x_i)'\psi(y_i, \eta_i)^2\eta_{i,1}^2\Big],\\
&\widehat{\Omega}_{\mu^{(v)},p,s}(x) := \widehat{\Omega}_{\mu^{(v)},p,s}(x; \widehat{\Delta}) := \widehat{\mathbf{b}}_{p,s}^{(v)}(x)'\widehat{\mathbf{Q}}_{p,s}^{-1}\widehat{\boldsymbol{\Sigma}}_{p,s}\widehat{\mathbf{Q}}_{p,s}^{-1}\widehat{\mathbf{b}}_{p,s}^{(v)}(x),\\
&\bar{\Omega}_{\mu^{(v)},p,s}(x) := \bar{\Omega}_{\mu^{(v)},p,s}(x; \widehat{\Delta}) := \widehat{\mathbf{b}}_{p,s}^{(v)}(x)'\bar{\mathbf{Q}}_{p,s}^{-1}\bar{\boldsymbol{\Sigma}}_{p,s}\bar{\mathbf{Q}}_{p,s}^{-1}\widehat{\mathbf{b}}_{p,s}^{(v)}(x),\\
&\Omega_{\mu^{(v)},p,s}(x) := \Omega_{\mu^{(v)},p,s}(x; \widehat{\Delta}) := \widehat{\mathbf{b}}_{p,s}^{(v)}(x)'\mathbf{Q}_{0,p,s}^{-1}\boldsymbol{\Sigma}_{0,p,s}\mathbf{Q}_{0,p,s}^{-1}\widehat{\mathbf{b}}_{p,s}^{(v)}(x),\\
&\widehat{\Omega}_{\vartheta,p,s}(x) := \widehat{\Omega}_{\vartheta,p,s}(x; \widehat{\Delta}) := [\eta^{(1)}(\widehat{\mu}(x) + \widehat{\mathsf{w}}'\widehat{\boldsymbol{\gamma}})]^2\widehat{\mathbf{b}}_{p,s}(x)'\widehat{\mathbf{Q}}_{p,s}^{-1}\widehat{\boldsymbol{\Sigma}}_{p,s}\widehat{\mathbf{Q}}_{p,s}^{-1}\widehat{\mathbf{b}}_{p,s}(x),\\
&\bar{\Omega}_{\vartheta,p,s}(x) := \bar{\Omega}_{\vartheta,p,s}(x; \widehat{\Delta}) := [\eta^{(1)}(\mu_0(x) + \mathsf{w}'\boldsymbol{\gamma}_0)]^2\widehat{\mathbf{b}}_{p,s}(x)'\bar{\mathbf{Q}}_{p,s}^{-1}\bar{\boldsymbol{\Sigma}}_{p,s}\bar{\mathbf{Q}}_{p,s}^{-1}\widehat{\mathbf{b}}_{p,s}(x),\\
&\Omega_{\vartheta,p,s}(x) := \Omega_{\vartheta,p,s}(x; \widehat{\Delta}) := [\eta^{(1)}(\mu(x) + \mathsf{w}'\boldsymbol{\gamma}_0)]^2\widehat{\mathbf{b}}_{p,s}(x)'\mathbf{Q}_{0,p,s}^{-1}\boldsymbol{\Sigma}_{0,p,s}\mathbf{Q}_{0,p,s}^{-1}\widehat{\mathbf{b}}_{p,s}(x),\\
&\widehat{\Omega}_{\zeta,p,s}(x) := \widehat{\Omega}_{\zeta,p,s}(x; \widehat{\Delta}) := [\eta^{(1)}(\widehat{\mu}(x) + \widehat{\mathsf{w}}'\widehat{\boldsymbol{\gamma}})]^2\widehat{\mathbf{b}}_{p,s}^{(1)}(x)'\widehat{\mathbf{Q}}_{p,s}^{-1}\widehat{\boldsymbol{\Sigma}}_{p,s}\widehat{\mathbf{Q}}_{p,s}^{-1}\widehat{\mathbf{b}}_{p,s}^{(1)}(x),\\
&\bar{\Omega}_{\zeta,p,s}(x) := \bar{\Omega}_{\zeta,p,s}(x; \widehat{\Delta}) := [\eta^{(1)}(\mu_0(x) + \mathsf{w}'\boldsymbol{\gamma}_0)]^2\widehat{\mathbf{b}}_{p,s}^{(1)}(x)'\bar{\mathbf{Q}}_{p,s}^{-1}\bar{\boldsymbol{\Sigma}}_{p,s}\bar{\mathbf{Q}}_{p,s}^{-1}\widehat{\mathbf{b}}_{p,s}^{(1)}(x), \quad \text{and}\\
&\Omega_{\zeta,p,s}(x) := \Omega_{\zeta,p,s}(x; \widehat{\Delta}) := [\eta^{(1)}(\mu_0(x) + \mathsf{w}'\boldsymbol{\gamma}_0)]^2\widehat{\mathbf{b}}_{p,s}^{(1)}(x)'\mathbf{Q}_{0,p,s}^{-1}\boldsymbol{\Sigma}_{0,p,s}\mathbf{Q}_{0,p,s}^{-1}\widehat{\mathbf{b}}_{p,s}^{(1)}(x).
\end{aligned}
$$

Recall that in the main text we always set $s = p$ and omit the dependence on s whenever there is no confusion. Thus,

$$
\begin{aligned}
&\widehat{\mathbf{Q}}_p = \widehat{\mathbf{Q}}_{p,p}, \quad \bar{\mathbf{Q}}_p = \bar{\mathbf{Q}}_{p,p}, \quad \mathbf{Q}_{0,p} = \mathbf{Q}_{0,p,p},\\
&\widehat{\mathbf{\Sigma}}_p = \widehat{\mathbf{\Sigma}}_{p,p}, \quad \bar{\mathbf{\Sigma}}_p = \bar{\mathbf{\Sigma}}_{p,p}, \quad \mathbf{\Sigma}_{0,p} = \mathbf{\Sigma}_{0,p,p},\\
&\widehat{\Omega}_{\mu^{(v)},p}(x) = \widehat{\Omega}_{\mu^{(v)},p,p}(x), \quad \bar{\Omega}_{\mu^{(v)},p}(x) = \bar{\Omega}_{\mu^{(v)},p,p}(x), \quad \Omega_{\mu^{(v)},p}(x) = \Omega_{\mu^{(v)},p,p}(x),\\
&\widehat{\Omega}_{\vartheta,p}(x) = \widehat{\Omega}_{\vartheta,p,p}(x), \quad \bar{\Omega}_{\vartheta,p}(x) = \bar{\Omega}_{\vartheta,p,p}(x), \quad \Omega_{\vartheta,p}(x) = \Omega_{\vartheta,p,p}(x),\\
&\widehat{\Omega}_{\zeta,p}(x) = \widehat{\Omega}_{\zeta,p,p}(x), \quad \bar{\Omega}_{\zeta,p}(x) = \bar{\Omega}_{\zeta,p,p}(x), \quad \text{and} \quad \Omega_{\zeta,p}(x) = \Omega_{\zeta,p,p}(x).
\end{aligned}
$$

In this supplement, however, all our results hold for a general choice of the degree and the smoothness of the basis. For ease of notation, the subscripts p and s of the above quantities are dropped hereafter:

$$
\begin{aligned}
&\widehat{\mathbf{Q}} = \widehat{\mathbf{Q}}_{p,s}, \quad \bar{\mathbf{Q}} = \bar{\mathbf{Q}}_{p,s}, \quad \mathbf{Q}_{0} = \mathbf{Q}_{0,p,s},\\
&\widehat{\mathbf{\Sigma}} = \widehat{\mathbf{\Sigma}}_{p,s}, \quad \bar{\mathbf{\Sigma}} = \bar{\mathbf{\Sigma}}_{p,s}, \quad \mathbf{\Sigma}_{0} = \mathbf{\Sigma}_{0,p,s},\\
&\widehat{\Omega}_{\mu^{(v)}}(x) = \widehat{\Omega}_{\mu^{(v)},p,s}(x), \quad \bar{\Omega}_{\mu^{(v)}}(x) = \bar{\Omega}_{\mu^{(v)},p,s}(x), \quad \Omega_{\mu^{(v)}}(x) = \Omega_{\mu^{(v)},p,s}(x),\\
&\widehat{\Omega}_{\vartheta}(x) = \widehat{\Omega}_{\vartheta,p,s}(x), \quad \bar{\Omega}_{\vartheta}(x) = \bar{\Omega}_{\vartheta,p,s}(x), \quad \Omega_{\vartheta}(x) = \Omega_{\vartheta,p,s}(x),\\
&\widehat{\Omega}_{\zeta}(x) = \widehat{\Omega}_{\zeta,p,s}(x), \quad \bar{\Omega}_{\zeta}(x) = \bar{\Omega}_{\zeta,p,s}(x), \quad \text{and} \quad \Omega_{\zeta}(x) = \Omega_{\zeta,p,s}(x).
\end{aligned}
$$

In addition, given the family Π_C of the quasi-uniform partitions defined in (SA-2.4), for any $\Delta \in \Pi$, we let $\boldsymbol{\beta}_0(\Delta) \in \mathbb{R}^{K_{p,s}}$ be any vector such that for every $v \leq p$,

$$
\sup_{x \in \mathcal{X}} \left| \mu_0^{(v)}(x) - \mathbf{b}_{p,s}^{(v)}(x;\Delta)'\boldsymbol{\beta}_0(\Delta) \right| \lesssim J^{-p-1+v}.
$$

Let $r_{0,v}(x;\Delta) = \mu_0^{(v)}(x) - \mathbf{b}_{p,s}^{(v)}(x;\Delta)'\boldsymbol{\beta}_0(\Delta)$ denote the corresponding approximation error. Accordingly, given the random partition $\widehat{\Delta}$, we let $\widehat{\boldsymbol{\beta}}_0 := \boldsymbol{\beta}_0(\widehat{\Delta})$, and $\widehat{r}_{0,v}(x) = \mu_0^{(v)}(x) - \widehat{\mathbf{b}}_{p,s}^{(v)}(x)'\widehat{\boldsymbol{\beta}}_0$ denote the corresponding approximation error. The existence of such vectors is guaranteed by Assumptions SA-DGP and SA-SM(v), and is verified in Lemma SA-5.5 in Section SA-5.

SA-3.1 Preliminary Lemmas

Lemma SA-3.1 (Gram)**.** *Suppose that Assumptions SA-DGP, SA-SM, SA-HLE and SA-RP(i) hold. If* $\frac{J \log J}{n} = o(1)$*, then*

$$1 \lesssim \lambda_{\min}(\bar{\mathbf{Q}}) \leq \lambda_{\max}(\bar{\mathbf{Q}}) \lesssim 1, \quad [\bar{\mathbf{Q}}^{-1}]_{ij} \lesssim \varrho^{|i-j|} \quad \text{w.p.a. } 1, \quad \text{and} \quad \|\bar{\mathbf{Q}}^{-1}\|_\infty \lesssim_{\mathbb{P}} 1,$$

where $\varrho \in (0,1)$ *is some absolute constant.*

If, in addition, Assumption SA-RP(ii) holds. Then,

$$1 \lesssim \lambda_{\min}(\mathbf{Q}_0) \leq \lambda_{\max}(\mathbf{Q}_0) \lesssim 1,$$
$$\|\bar{\mathbf{Q}} - \mathbf{Q}_0\| \lesssim_{\mathbb{P}} \left(\frac{J \log J}{n}\right)^{1/2} + \mathfrak{r}_{\texttt{RP}}, \quad \text{and} \quad \|\bar{\mathbf{Q}}^{-1} - \mathbf{Q}_0^{-1}\|_\infty \lesssim_{\mathbb{P}} \left(\frac{J \log J}{n}\right)^{1/2} + \mathfrak{r}_{\texttt{RP}}.$$

The next lemma shows that the limiting variance is bounded from above and below.

Lemma SA-3.2 (Asymptotic Variance)**.** *Suppose that Assumptions SA-DGP, SA-SM, SA-HLE and SA-RP(i) hold. If* $\frac{J \log J}{n} = o(1)$*, then w.p.a. 1,*

$$J^{1+2v} \lesssim \inf_{x \in \mathcal{X}} \bar{\Omega}_{\mu^{(v)}}(x) \leq \sup_{x \in \mathcal{X}} \bar{\Omega}_{\mu^{(v)}}(x) \lesssim J^{1+2v},$$
$$J \lesssim \inf_{x \in \mathcal{X}} \bar{\Omega}_{\vartheta}(x) \leq \sup_{x \in \mathcal{X}} \bar{\Omega}_{\vartheta}(x) \lesssim J,$$
$$J^3 \lesssim \inf_{x \in \mathcal{X}} \bar{\Omega}_{\zeta}(x) \leq \sup_{x \in \mathcal{X}} \bar{\Omega}_{\zeta}(x) \lesssim J^3.$$

If, in addition, Assumption SA-RP(ii) holds, then w.p.a. 1,

$$J^{1+2v} \lesssim \inf_{x \in \mathcal{X}} \Omega_{\mu^{(v)}}(x) \leq \sup_{x \in \mathcal{X}} \Omega_{\mu^{(v)}}(x) \lesssim J^{1+2v},$$
$$J \lesssim \inf_{x \in \mathcal{X}} \Omega_{\vartheta}(x) \leq \sup_{x \in \mathcal{X}} \Omega_{\vartheta}(x) \lesssim J,$$
$$J^3 \lesssim \inf_{x \in \mathcal{X}} \Omega_{\zeta}(x) \leq \sup_{x \in \mathcal{X}} \Omega_{\zeta}(x) \lesssim J^3.$$

The next lemma gives a bound on the variance component of the nonlinear binscatter estimator.

Lemma SA-3.3 (Uniform Convergence: Variance)**.** *Suppose that Assumptions SA-DGP, SA-SM,* *SA-HLE and SA-RP(i) hold. If* $\frac{J^{\frac{\nu}{\nu-2}}\log J}{n}=o(1)$*, then*

$$\sup_{x\in\mathcal{X}}\left|\widehat{\mathbf{b}}^{(v)}_{p,s}(x)'\bar{\mathbf{Q}}^{-1}\mathbb{E}_n[\widehat{\mathbf{b}}_{p,s}(x_i)\eta_{i,1}\psi(y_i,\eta_i)]\right|\lesssim_{\mathbb{P}} J^v\Big(\frac{J\log J}{n}\Big)^{1/2}.$$

Lemma SA-3.4 (Projection of Approximation Error)**.** *Suppose that Assumptions SA-DGP, SA-SM, SA-HLE and SA-RP(i) hold. If* $\frac{J^{\frac{\nu}{\nu-2}}\log J}{n}=o(1)$*, then*

$$\begin{aligned}&\sup_{x\in\mathcal{X}}\left|\widehat{\mathbf{b}}^{(v)}_{p,s}(x)'\bar{\mathbf{Q}}^{-1}\mathbb{E}_n\Big[\widehat{\mathbf{b}}_{p,s}(x_i)\Big(\eta_{i,1}\psi(y_i,\eta_i)-\eta^{(1)}(\widehat{\mathbf{b}}_{p,s}(x_i)'\widehat{\boldsymbol{\beta}}_0+\mathbf{w}_i'\boldsymbol{\gamma}_0)\psi(y_i,\eta(\widehat{\mathbf{b}}_{p,s}(x_i)'\widehat{\boldsymbol{\beta}}_0+\mathbf{w}_i'\boldsymbol{\gamma}_0))\Big)\Big]\right|\\&\lesssim_{\mathbb{P}} J^{-p-1+v}+J^{\frac{2v-p-1}{2}}\Big(\frac{J\log J}{n}\Big)^{1/2}+\frac{J^{1+v}\log J}{n}.\end{aligned}$$

Lemma SA-3.5 (Uniform Consistency)**.** *Suppose that Assumptions SA-DGP, SA-SM, SA-HLE and SA-RP(i) hold. If* $\frac{J^{\frac{2\nu}{\nu-1}}(\log J)^{\frac{\nu}{\nu-1}}}{n}=o(1)$*, then*

$$\|\widehat{\boldsymbol{\beta}}-\widehat{\boldsymbol{\beta}}_0\|_\infty=o_{\mathbb{P}}(J^{-1/2})\quad\textit{and}\quad\sup_{x\in\mathcal{X}}|\widehat{\mu}(x)-\mu_0(x)|=o_{\mathbb{P}}(1).$$

Remark SA-3.1 (Side rate conditions)**.** When $\nu\to\infty$, the rate restriction $\frac{J^{\frac{2\nu}{\nu-1}}(\log J)^{\frac{\nu}{\nu-1}}}{n}=o(1)$ tends to be $\frac{J^2\log J}{n}=o(1)$. We conjecture this rate restriction is stronger than needed. In fact, for piecewise polynomials (i.e., $s=0$), we can show that $\frac{J^{\frac{\nu}{\nu-1}}(\log J)^{\frac{\nu}{\nu-1}}}{n}=o(1)$ suffices to establish the uniform consistency of $\widehat{\boldsymbol{\beta}}$, and this restriction is redundant in our main theorems in view of the condition $\frac{J^{\frac{\nu}{\nu-2}}(\log n)^{\frac{\nu}{\nu-2}}}{n}=o(1)$ imposed below. In other words, in this special case ($s=0$), the condition $\frac{J^{\frac{2\nu}{\nu-1}}(\log J)^{\frac{\nu}{\nu-1}}}{n}=o(1)$ in all theorems below can be dropped. ⌟

Our result holds without imposing any smoothness restrictions on the estimation space. Specifically, the estimation procedure (SA-2.3) searches for solutions in $\mathbb{R}^{K_{p,s}}$, leading to an estimation space $\{\widehat{\mathbf{b}}_{p,s}(x)'\boldsymbol{\beta}:\boldsymbol{\beta}\in\mathbb{R}^{K_{p,s}}\}$. In contrast, many studies of series (or sieve) methods restrict the functions in the estimation space to satisfy certain smoothness conditions, e.g., Lipschitz continuity, to derive the uniform consistency. See, for example, Chernozhukov, Imbens and Newey (2007) and references therein.

Remark SA-3.2 (Improvements over literature)**.** Most of the results in this subsection are new to the literature, even in the case of non-random partitioning and without covariate-adjustments, because they take advantage of the specific binscatter structure (i.e., locally bounded series basis).

The closest antecedent in the literature is Belloni, Chernozhukov, Chetverikov and Fernandez-Val (2019), while it focuses on series-based quantile regression only. Furthermore, relative to prior work, our results allow for random partitioning schemes, formally taking into account both the potential randomness of the partition and the semi-linear regression estimation structure. Importantly, we highlight the key conditions on the possibly random partition (Assumptions SA-RP(i) and SA-RP(ii)) used to derive various properties of the Gram matrix, asymptotic variance and other quantities. ⌟

SA-3.2 Bahadur Representation

Theorem SA-3.1 (Bahadur Representation)**.** *Suppose that Assumptions SA-DGP, SA-SM, SA-HLE and SA-RP(i) hold and* $\frac{J^{\frac{\nu}{\nu-2}}\log n}{n}+\frac{J(\log n)^{7/3}}{n}+\frac{J^{\frac{2\nu}{\nu-1}}(\log n)^{\frac{\nu}{\nu-1}}}{n}+\frac{\log n}{J}=o(1)$. *Then,*

(i) $\widehat{\mu}^{(v)}(x)$ *satisfies that*

$$\sup_{x\in\mathcal{X}}\Big|\widehat{\mu}^{(v)}(x)-\mu_0^{(v)}(x)+\widehat{\mathbf{b}}^{(v)}_{p,s}(x)'\bar{\mathbf{Q}}^{-1}\mathbb{E}_n[\widehat{\mathbf{b}}_{p,s}(x_i)\eta_{i,1}\psi(y_i,\eta_i)]\Big|$$
$$\lesssim_{\mathbb{P}} J^{v}\Big\{\Big(\frac{J\log n}{n}\Big)^{3/4}\log n+J^{-\frac{p+1}{2}}\Big(\frac{J\log^2 n}{n}\Big)^{1/2}+J^{-p-1}+\mathfrak{r}_\gamma\Big\}.$$

(ii) $\widehat{\vartheta}(x,\widehat{\mathsf{w}})$ *satisfies that*

$$\sup_{x\in\mathcal{X}}\Big|\widehat{\vartheta}(x,\widehat{\mathsf{w}})-\vartheta_0(x,\mathsf{w})+\eta^{(1)}(\mu_0(x)+\mathsf{w}'\boldsymbol{\gamma}_0)\widehat{\mathbf{b}}_{p,s}(x)'\bar{\mathbf{Q}}^{-1}\mathbb{E}_n[\widehat{\mathbf{b}}_{p,s}(x_i)\eta_{i,1}\psi(y_i,\eta_i)]\Big|$$
$$\lesssim_{\mathbb{P}}\Big(\frac{J\log n}{n}\Big)^{3/4}\log n+J^{-\frac{p+1}{2}}\Big(\frac{J\log^2 n}{n}\Big)^{1/2}+J^{-p-1}+\mathfrak{r}_\gamma+\|\widehat{\mathsf{w}}-\mathsf{w}\|.$$

(iii) $\widehat{\zeta}(x,\widehat{\mathsf{w}})$ *satisfies that*

$$\sup_{x\in\mathcal{X}}\Big|\widehat{\zeta}(x,\widehat{\mathsf{w}})-\zeta_0(x,\mathsf{w})+\eta^{(1)}(\mu_0(x)+\mathsf{w}'\boldsymbol{\gamma}_0)\widehat{\mathbf{b}}^{(1)}_{p,s}(x)'\bar{\mathbf{Q}}^{-1}\mathbb{E}_n[\widehat{\mathbf{b}}_{p,s}(x_i)\eta_{i,1}\psi(y_i,\eta_i)]\Big|$$
$$\lesssim_{\mathbb{P}}\Big(\frac{J\log n}{n}\Big)^{1/2}+J\Big\{\Big(\frac{J\log n}{n}\Big)^{3/4}\log n+J^{-\frac{p+1}{2}}\Big(\frac{J\log^2 n}{n}\Big)^{1/2}+J^{-p-1}+\mathfrak{r}_\gamma\Big\}$$
$$+\|\widehat{\mathsf{w}}-\mathsf{w}\|\Big(1+J\Big(\frac{J\log n}{n}\Big)^{1/2}\Big).$$

The following corollary is an immediate result of Lemma SA-3.3 and Theorem SA-3.1, and hence its proof is omitted.

Corollary SA-3.1 (Uniform Convergence)**.** *Suppose that the conditions of Theorem SA-3.1 hold and* $\frac{J(\log n)^5}{n} \lesssim 1$. *Then*

$$\sup_{x\in\mathcal{X}} |\widehat{\mu}^{(v)}(x) - \mu_0^{(v)}(x)| \lesssim_{\mathbb{P}} J^v\Big(\Big(\frac{J\log n}{n}\Big)^{1/2} + J^{-p-1}\Big).$$

If, in addition, $\|\widehat{\mathsf{w}} - \mathsf{w}\| \lesssim_{\mathbb{P}} \Big(\frac{J\log n}{n}\Big)^{1/2} + J^{-p-1}$, *then*

$$\sup_{x\in\mathcal{X}} |\widehat{\vartheta}(x,\widehat{\mathsf{w}}) - \vartheta_0(x,\mathsf{w})| \lesssim_{\mathbb{P}} \Big(\frac{J\log n}{n}\Big)^{1/2} + J^{-p-1} \quad \textit{and}$$
$$\sup_{x\in\mathcal{X}} |\widehat{\zeta}(x,\widehat{\mathsf{w}}) - \zeta_0(x,\mathsf{w})| \lesssim_{\mathbb{P}} J\Big(\Big(\frac{J\log n}{n}\Big)^{1/2} + J^{-p-1}\Big).$$

The next theorem shows that the proposed variance estimator is consistent.

Theorem SA-3.2 (Variance Estimate)**.** *Suppose that Assumptions SA-DGP, SA-SM, SA-HLE and SA-RP(i) hold. If* $\frac{J^{\frac{\nu}{\nu-2}}(\log n)^{\frac{\nu}{\nu-2}}}{n} + \frac{J^{\frac{2\nu}{\nu-1}}(\log n)^{\frac{\nu}{\nu-1}}}{n} + \frac{J(\log n)^5}{n} + \frac{\log n}{J} = o(1)$ *and* $\|\widehat{\mathsf{w}} - \mathsf{w}\| \lesssim_{\mathbb{P}} \Big(\frac{J\log n}{n}\Big)^{1/2} + J^{-p-1}$, *then*

$$\Big\|\widehat{\boldsymbol{\Sigma}} - \bar{\boldsymbol{\Sigma}}\Big\| \lesssim_{\mathbb{P}} J^{-p-1} + \Big(\frac{J\log n}{n^{1-\frac{2}{\nu}}}\Big)^{1/2},$$
$$\sup_{x\in\mathcal{X}} \Big|\widehat{\Omega}_{\mu^{(v)}}(x) - \bar{\Omega}_{\mu^{(v)}}(x)\Big| \lesssim_{\mathbb{P}} J^{1+2v}\Big(J^{-p-1} + \Big(\frac{J\log n}{n^{1-\frac{2}{\nu}}}\Big)^{1/2}\Big),$$
$$\sup_{x\in\mathcal{X}} \Big|\widehat{\Omega}_{\vartheta}(x) - \bar{\Omega}_{\vartheta}(x)\Big| \lesssim_{\mathbb{P}} J\Big(J^{-p-1} + \Big(\frac{J\log n}{n^{1-\frac{2}{\nu}}}\Big)^{1/2}\Big), \quad \textit{and}$$
$$\sup_{x\in\mathcal{X}} \Big|\widehat{\Omega}_{\zeta}(x) - \bar{\Omega}_{\zeta}(x)\Big| \lesssim_{\mathbb{P}} J^3\Big(J^{-p-1} + \Big(\frac{J\log n}{n^{1-\frac{2}{\nu}}}\Big)^{1/2}\Big).$$

If, in addition, Assumption SA-RP(ii) holds, then

$$\Big\|\widehat{\boldsymbol{\Sigma}} - \boldsymbol{\Sigma}_0\Big\| \lesssim_{\mathbb{P}} J^{-p-1} + \Big(\frac{J\log n}{n^{1-\frac{2}{\nu}}}\Big)^{1/2} + \mathfrak{r}_{\mathtt{RP}},$$
$$\sup_{x\in\mathcal{X}} \Big|\widehat{\Omega}_{\mu^{(v)}}(x) - \Omega_{\mu^{(v)}}(x)\Big| \lesssim_{\mathbb{P}} J^{1+2v}\Big(J^{-p-1} + \Big(\frac{J\log n}{n^{1-\frac{2}{\nu}}}\Big)^{1/2} + \mathfrak{r}_{\mathtt{RP}}\Big),$$
$$\sup_{x\in\mathcal{X}} \Big|\widehat{\Omega}_{\vartheta}(x) - \Omega_{\vartheta}(x)\Big| \lesssim_{\mathbb{P}} J\Big(J^{-p-1} + \Big(\frac{J\log n}{n^{1-\frac{2}{\nu}}}\Big)^{1/2} + \mathfrak{r}_{\mathtt{RP}}\Big), \quad \textit{and}$$
$$\sup_{x\in\mathcal{X}} \Big|\widehat{\Omega}_{\zeta}(x) - \Omega_{\zeta}(x)\Big| \lesssim_{\mathbb{P}} J^3\Big(J^{-p-1} + \Big(\frac{J\log n}{n^{1-\frac{2}{\nu}}}\Big)^{1/2} + \mathfrak{r}_{\mathtt{RP}}\Big).$$

Remark SA-3.3 (Improvements over literature)**.** Theorem SA-3.1 and Corollary SA-3.1 construct the Bahadur representation and uniform convergence of nonlinear binscatter-based M-estimators, which improve upon prior results in the literature in at least two aspects. First, our results allow for random partitioning schemes, and the key condition imposed on the partition is Assumption SA-RP(i), i.e., the conditional independence between the partition and the outcome and the quasi-uniformity of the partition. The "convergence" of the random partition (Assumption SA-RP(ii)) is not required, which implies that our results can accommodate more complex partitioning schemes other than evenly-spaced or empirical-quantile-spaced partitions.

Second, our results are established under weaker rate restrictions. Specifically, we require $J^{\frac{8}{3}}/n = o(1)$ up to $\log n$ terms when $\nu \geq 4$, thus accommodating IMSE-optimal piecewise constant binscatter estimators. In fact, for piecewise polynomials ($s = 0$), we can show that the Bahadur representation still holds under $J/n = o(1)$ up to $\log n$ terms when a subexponential moment restriction holds for the "score" $\psi(y_i, \eta_i)$, which is analogous to the result for kernel-based estimators in the literature (see, e.g., Kong et al., 2010). For series estimators, similar results were established for particular choices of loss functions under more stringent conditions in the literature. For example, Belloni, Chernozhukov, Chetverikov and Fernandez-Val (2019) considers series-based quantile regression, and Theorem 2 and Corollary 2 therein can be used to establish a Bahadur representation and uniform convergence of the resulting estimators under $J^4/n^{1-\varepsilon} = o(1)$ for some $\varepsilon > 0$.

The results in Belloni et al. (2019) are slightly stronger than our Theorem SA-3.1 in the sense that the expansion holds uniformly over both the evaluation point $x \in \mathcal{X}$ and the desired quantiles $u \in \mathcal{U}$ for a compact set of quantile indices $\mathcal{U} \subset (0, 1)$. Our results regarding Bahadur representation can be extended to achieve the same level of uniformity. In general, the parameter of interest (SA-2.1) and the estimator (SA-2.2) are defined for each particular choice of the loss function within a function class $\mathcal{F}$. For the class of check functions used in quantile regression or other function classes with low complexity, it can be shown that the Bahadur representation still holds uniformly over the evaluation point $x \in \mathcal{X}$ and the loss function $\rho \in \mathcal{F}$ under rate restrictions similar to those in Theorem SA-3.1, thereby providing an improvement over the literature. ⌟

SA-3.3 Pointwise Inference

Starting from this section, we consider statistical inference on $\mu_0^{(v)}(x)$, $\vartheta_0(x, \mathsf{w})$ and $\zeta_0(x, \mathsf{w})$ based on the following Studentized t-statistics:

$$
\begin{aligned}
T_{\mu^{(v)},p}(x) &= \frac{\widehat{\mu}^{(v)}(x) - \mu_0^{(v)}(x)}{\sqrt{\widehat{\Omega}_{\mu^{(v)}}(x)/n}}, \\
T_{\vartheta,p}(x) &= \frac{\widehat{\vartheta}(x, \widehat{\mathsf{w}}) - \vartheta_0(x, \mathsf{w})}{\sqrt{\widehat{\Omega}_{\vartheta}(x)/n}} \quad \text{and} \\
T_{\zeta,p}(x) &= \frac{\widehat{\zeta}(x, \widehat{\mathsf{w}}) - \zeta_0(x, \mathsf{w})}{\sqrt{\widehat{\Omega}_{\zeta}(x)/n}}.
\end{aligned}
$$

The next theorem shows the pointwise asymptotic normality of the binscatter estimators.

Theorem SA-3.3 (Pointwise Asymptotic Distribution)**.** *Suppose that Assumptions SA-DGP, SA-SM, SA-HLE and SA-RP(i) hold,* $\sup_{x\in\mathcal{X}} \mathbb{E}[|\psi(y_i, \eta_i)|^\nu | x_i = x] \lesssim 1$ *for some* $\nu \geq 3$*, and* $\frac{J^{\frac{\nu}{\nu-2}}(\log n)^{\frac{\nu}{\nu-2}}}{n} + \frac{J^{\frac{2\nu}{\nu-1}}(\log n)^{\frac{\nu}{\nu-1}}}{n} + nJ^{-2p-3} = o(1)$*. Then the following conclusions hold:*

(i) For $\widehat{\mu}^{(v)}(x)$*,*

$$\sup_{u\in\mathbb{R}} \Big|\mathbb{P}(T_{\mu^{(v)},p}(x) \leq u) - \Phi(u)\Big| = o(1), \quad \textit{for each } x \in \mathcal{X}.$$

(ii) For $\widehat{\vartheta}(x, \widehat{\mathsf{w}})$*, if, in addition,* $\|\widehat{\mathsf{w}} - \mathsf{w}\| = o_{\mathbb{P}}(\sqrt{J/n})$*, then*

$$\sup_{u\in\mathbb{R}} \Big|\mathbb{P}(T_{\vartheta,p}(x) \leq u) - \Phi(u)\Big| = o(1) \quad \textit{for each } x \in \mathcal{X}.$$

(iii) For $\widehat{\zeta}(x, \widehat{\mathsf{w}})$*, if, in addition,* $\|\widehat{\mathsf{w}} - \mathsf{w}\| = o_{\mathbb{P}}(\sqrt{J^3/n} + (\log n)^{-1/2})$*, then*

$$\sup_{u\in\mathbb{R}} \Big|\mathbb{P}(T_{\zeta,p}(x) \leq u) - \Phi(u)\Big| = o(1) \quad \textit{for each } x \in \mathcal{X}.$$

Remark SA-3.4 (Improvements over literature)**.** The result in this subsection is new to the literature, even in the case of non-random partitioning and without covariate adjustments, because it takes advantage of the specific binscatter structure (i.e., locally bounded series basis). The closest antecedent in the literature is Belloni et al. (2019), which focuses on series-based quantile regression only. Furthermore, relative to prior work, our results allow for more general partitioning schemes, formally take into account the potential randomness of the partition, and account for the semi-linear regression estimation structure. The key condition imposed on the partition for pointwise inference is Assumption SA-RP(i), and the "convergence" of the random partition is not required. ⌟

SA-3.4 Integrated Mean Squared Error

In this section we give a Nagar-type approximate IMSE expansion for each of the three estimators $\widehat{\mu}^{(v)}(x)$, $\widehat{\vartheta}(x,\widehat{\mathtt{w}})$ and $\widehat{\zeta}(x,\widehat{\mathtt{w}})$, with explicit characterization of the leading constants. Define

$$r^{\star}_{0,v}(x) = \frac{J^{-p-1+v}\mu_0^{(p+1)}(x)}{(p+1-v)!f_X(x)^{p+1-v}}\mathscr{E}_{p+1-v}\Big(\frac{x-\tau^{\mathtt{L}}_x}{h_x}\Big)$$

where $\mathscr{E}_m(\cdot)$ is the mth Bernoulli polynomial for each $m\in\mathbb{Z}_+$, $\tau^{\mathtt{L}}_x$ is the start of the interval in the non-random partition Δ_0 containing x and h_x denotes its length.

Theorem SA-3.4 (IMSE)**.** *Suppose that Assumptions SA-DGP, SA-SM, SA-HLE and SA-RP (including SA-RP(ii)) hold. Let $\omega(x)$ be a continuous weighting function over $\mathcal{X}$ bounded away from zero. Also, assume that $\frac{J^{\frac{\nu}{\nu-2}}\log n}{n}+\frac{J^{\frac{2\nu}{\nu-1}}(\log n)^{\frac{\nu}{\nu-1}}}{n}+\frac{J(\log n)^7}{n}+\frac{(\log n)^2}{J}=o(1)$.*

(i) For $\widehat{\mu}^{(v)}(x)$,

$$\int_{\mathcal{X}}\Big(\widehat{\mu}^{(v)}(x)-\mu_0^{(v)}(x)\Big)^2\omega(x)dx=\mathtt{AISE}_{\mu^{(v)}}+o_{\mathbb{P}}\Big(\frac{J^{1+2v}}{n}+J^{-2(p+1-v)}\Big)$$

where

$$\begin{aligned}
&\mathbb{E}[\mathtt{AISE}_{\mu^{(v)}}|\mathbf{X},\mathbf{W},\widehat{\Delta}]=\frac{J^{1+2v}}{n}\mathscr{V}_n(p,s,v)+J^{-2(p+1-v)}\mathscr{B}_n(p,s,v)+o_{\mathbb{P}}\Big(\frac{J^{1+2v}}{n}+J^{-2(p+1-v)}\Big),\\
&\mathscr{V}_n(p,s,v):=J^{-(1+2v)}\operatorname{trace}\Big(\mathbf{Q}_0^{-1}\mathbf{\Sigma}_0\mathbf{Q}_0^{-1}\int_{\mathcal{X}}\mathbf{b}^{(v)}_{p,s}(x)\mathbf{b}^{(v)}_{p,s}(x)'\omega(x)dx\Big)\asymp 1,\\
&\mathscr{B}_n(p,s,v):=J^{2p+2-2v}\int_{\mathcal{X}}\Big(r^{\star}_{0,v}(x)-\mathbf{b}^{(v)}_{p,s}(x)'\mathbf{Q}_0^{-1}\mathbb{E}[\mathbf{b}_{p,s}(x_i)\varkappa(x_i,\mathbf{w}_i)r^{\star}_{0,0}(x_i)]\Big)^2\omega(x)dx\lesssim 1.
\end{aligned}$$

(ii) For $\widehat{\vartheta}(x,\widehat{\mathtt{w}})$, if $\|\widehat{\mathtt{w}}-\mathtt{w}\|=o_{\mathbb{P}}(\sqrt{J/n}+J^{-p-1})$, then

$$\int_{\mathcal{X}}\Big(\widehat{\vartheta}(x,\widehat{\mathtt{w}})-\vartheta_0(x,\mathtt{w})\Big)^2\omega(x)dx=\mathtt{AISE}_{\vartheta}+o_{\mathbb{P}}\Big(\frac{J}{n}+J^{-2(p+1)}\Big)$$

where

$$
\begin{aligned}
&\mathbb{E}[\mathtt{AISE}_{\vartheta}|\mathbf{X},\mathbf{W},\widehat{\Delta}] = \frac{J}{n}\mathscr{V}_n(p,s) + J^{-2(p+1)}\mathscr{B}_n(p,s) + o_{\mathbb{P}}\Big(\frac{J}{n} + J^{-2(p+1)}\Big),\\
&\mathscr{V}_n(p,s) := J^{-1}\,\mathrm{trace}\left(\mathbf{Q}_0^{-1}\boldsymbol{\Sigma}_0\mathbf{Q}_0^{-1}\int_{\mathcal{X}}\eta_{0,1}(x,\mathsf{w})^2\mathbf{b}_{p,s}(x)\mathbf{b}_{p,s}(x)'\omega(x)dx\right) \asymp 1,\\
&\mathscr{B}_n(p,s) := J^{2p+2}\int_{\mathcal{X}}\Big[\eta_{0,1}(x,\mathsf{w})\Big(r^{\star}_{0,0}(x) - \mathbf{b}_{p,s}(x)'\mathbf{Q}_0^{-1}\mathbb{E}[\mathbf{b}_{p,s}(x_i)\varkappa(x_i,\mathbf{w}_i)r^{\star}_{0,0}(x_i)]\Big)\Big]^2\omega(x)dx \lesssim 1.
\end{aligned}
$$

(iii) For $\widehat{\zeta}(x,\widehat{\mathsf{w}})$*, if* $\|\widehat{\mathsf{w}}-\mathsf{w}\| = o_{\mathbb{P}}(\sqrt{J^3/n} + J^{-p} + (\log n)^{-1/2})$*, then*

$$
\int_{\mathcal{X}}\Big(\widehat{\zeta}(x,\widehat{\mathsf{w}}) - \zeta_0(x,\mathsf{w})\Big)^2\omega(x)dx = \mathtt{AISE}_{\zeta} + o_{\mathbb{P}}\Big(\frac{J^3}{n} + J^{-2p}\Big)
$$

where

$$
\begin{aligned}
&\mathbb{E}[\mathtt{AISE}_{\zeta}|\mathbf{X},\mathbf{W},\widehat{\Delta}] = \frac{J^3}{n}\mathscr{V}_n(p,s) + J^{-2p}\mathscr{B}_n(p,s) + o_{\mathbb{P}}\Big(\frac{J^3}{n} + J^{-2p}\Big),\\
&\mathscr{V}_n(p,s) := J^{-3}\,\mathrm{trace}\left(\mathbf{Q}_0^{-1}\boldsymbol{\Sigma}_0\mathbf{Q}_0^{-1}\int_{\mathcal{X}}\eta_{0,1}(x,\mathsf{w})^2\mathbf{b}^{(1)}_{p,s}(x)\mathbf{b}^{(1)}_{p,s}(x)'\omega(x)dx\right) \asymp 1,\\
&\mathscr{B}_n(p,s) := J^{2p}\int_{\mathcal{X}}\Big[\eta_{0,1}(x,\mathsf{w})\Big(r^{\star}_{0,1}(x) - \mathbf{b}^{(1)}_{p,s}(x)'\mathbf{Q}_0^{-1}\mathbb{E}[\mathbf{b}_{p,s}(x_i)\varkappa(x_i,\mathbf{w}_i)r^{\star}_{0,0}(x_i)]\Big)\Big]^2\omega(x)dx \lesssim 1.
\end{aligned}
$$

In general, $\mathscr{B}_n(p,s,v) \gtrsim 1$ (see Remark SA-3.7 in Cattaneo et al. (2024b)), and thus the above theorem implies that the (approximate) IMSE-optimal number of bins satisfies that $J_{\mathtt{AIMSE}} \asymp n^{\frac{1}{2p+3}}$. Relying on the IMSE expansion in Theorem SA-3.4, one may design a data-driven procedure to select the IMSE-optimal number of bins for nonlinear binscatter-based M-estimators.

Remark SA-3.5 (Improvements over literature)**.** The results in this subsection are new to the literature, even in the case of non-random partitioning and without covariate-adjustments, for both nonlinear series estimators and binscatter (piecewise polynomials and splines) nonlinear series estimators in particular. Furthermore, our results allow for random partitioning schemes, formally take into account the potential randomness of the partition, and account for the semi-linear regression estimation structure. We highlight the key conditions imposed on the partition (Assumption SA-RP) for the approximate IMSE expansion. The "convergence" of the random partition (Assumption SA-RP(ii)) is needed to derive the non-random variance and bias constants $\mathscr{V}_n(p,s)$ and $\mathscr{B}_n(p,s)$. ⌟

SA-3.5 Uniform Inference

Recall that $(a_n : n \geq 1)$ is a sequence of non-vanishing constants. We will first show that the (feasible) Studentized t-statistic processes $T_{\mu^{(v)},p}(\cdot)$, $T_{\vartheta,p}(\cdot)$ and $T_{\zeta,p}(\cdot)$ can be approximated by Gaussian processes in a proper sense at certain rate.

Theorem SA-3.5 (Strong Approximation)**.** *Suppose that Assumptions SA-DGP, SA-SM, SA-HLE and SA-RP(i) hold,*

$$\frac{J(\log n)^2}{n^{1-\frac{2}{\nu}}}+\Big(\frac{J(\log n)^7}{n}\Big)^{1/2}+nJ^{-2p-3}+\frac{(\log n)^2}{J^{p+1}}+nJ^{-1}\mathfrak{r}_\gamma^2=o(a_n^{-2})\quad \text{and}\quad \frac{J^{\frac{2\nu}{\nu-1}}(\log n)^{\frac{\nu}{\nu-1}}}{n}=o(1).$$

Then the following conclusions hold:

(i) On a properly enriched probability space, there exists some $K_{p,s}$-dimensional standard normal random vector $\mathbf{N}_{K_{p,s}}$ such that for any $\xi>0$,

$$\mathbb{P}\Big(\sup_{x\in\mathcal{X}}|T_{\mu^{(v)},p}(x)-\bar{Z}_{\mu^{(v)},p}(x)|>\xi a_n^{-1}\Big)=o(1),\quad \bar{Z}_{\mu^{(v)},p}(x)=\frac{\widehat{\mathbf{b}}_{p,0}^{(v)}(x)'\widehat{\mathbf{T}}_s'\bar{\mathbf{Q}}^{-1}\bar{\mathbf{\Sigma}}^{1/2}}{\sqrt{\bar{\Omega}_{\mu^{(v)}}(x)}}\mathbf{N}_{K_{p,s}}.$$

If Assumption SA-RP(ii) also holds with $\mathfrak{r}_{\mathtt{RP}}=o(a_n^{-1}(\log n)^{-1/2})$, then

$$\mathbb{P}\Big(\sup_{x\in\mathcal{X}}|T_{\mu^{(v)},p}(x)-Z_{\mu^{(v)},p}(x)|>\xi a_n^{-1}\Big)=o(1),\quad Z_{\mu^{(v)},p}(x)=\frac{\widehat{\mathbf{b}}_{p,0}^{(v)}(x)'\mathbf{T}_s'\mathbf{Q}_0^{-1}\mathbf{\Sigma}_0^{1/2}}{\sqrt{\Omega_{\mu^{(v)}}(x)}}\mathbf{N}_{K_{p,s}}.$$

(ii) If $\|\widehat{\mathsf{w}}-\mathsf{w}\|=o_{\mathbb{P}}(a_n^{-1}\sqrt{J/n})$, then on a properly enriched probability space there exists some $K_{p,s}$-dimensional standard normal random vector $\mathbf{N}_{K_{p,s}}$ such that for any $\xi>0$,

$$\mathbb{P}\Big(\sup_{x\in\mathcal{X}}|T_{\vartheta,p}(x)-\bar{Z}_{\vartheta,p}(x)|>\xi a_n^{-1}\Big)=o(1),\quad \bar{Z}_{\vartheta,p}(x)=\frac{\widehat{\mathbf{b}}_{p,0}(x)'\widehat{\mathbf{T}}_s'\eta_{0,1}(x,\mathsf{w})\bar{\mathbf{Q}}^{-1}}{\sqrt{\bar{\Omega}_\vartheta(x)}}\bar{\mathbf{\Sigma}}^{1/2}\mathbf{N}_{K_{p,s}}.$$

If Assumption SA-RP(ii) also holds with $\mathfrak{r}_{\mathtt{RP}}=o(a_n^{-1}(\log n)^{-1/2})$, then

$$\mathbb{P}\Big(\sup_{x\in\mathcal{X}}|T_{\vartheta,p}(x)-Z_{\vartheta,p}(x)|>\xi a_n^{-1}\Big)=o(1),\quad Z_{\vartheta,p}(x)=\frac{\widehat{\mathbf{b}}_{p,0}(x)'\mathbf{T}_s'\eta_{0,1}(x,\mathsf{w})\mathbf{Q}_0^{-1}}{\sqrt{\Omega_\vartheta(x)}}\mathbf{\Sigma}_0^{1/2}\mathbf{N}_{K_{p,s}}.$$

(iii) If $\|\widehat{\mathsf{w}}-\mathsf{w}\|=o_{\mathbb{P}}(a_n^{-1}(\sqrt{J^3/n}+(\log n)^{-1/2}))$, then on a properly enriched probability space

there exists some $K_{p,s}$-dimensional standard normal random vector $\mathbf{N}_{K_{p,s}}$ such that for any $\xi > 0$,

$$\mathbb{P}\Big(\sup_{x\in\mathcal{X}} |T_{\zeta,p}(x) - \bar{Z}_{\zeta,p}(x)| > \xi a_n^{-1}\Big) = o(1), \quad \bar{Z}_{\zeta,p}(x) = \frac{\widehat{\mathbf{b}}_{p,0}^{(1)}(x)'\widehat{\mathbf{T}}_s'\eta_{0,1}(x,\mathsf{w})\bar{\mathbf{Q}}^{-1}}{\sqrt{\bar{\Omega}_\zeta(x)}}\bar{\mathbf{\Sigma}}^{1/2}\mathbf{N}_{K_{p,s}}.$$

If Assumption SA-RP(ii) also holds with $\mathfrak{r}_{\mathtt{RP}} = o(a_n^{-1}(\log n)^{-1/2})$, then

$$\mathbb{P}\Big(\sup_{x\in\mathcal{X}} |T_{\zeta,p}(x) - Z_{\zeta,p}(x)| > \xi a_n^{-1}\Big) = o(1), \quad Z_{\zeta,p}(x) = \frac{\widehat{\mathbf{b}}_{p,0}^{(1)}(x)'\mathbf{T}_s'\eta_{0,1}(x,\mathsf{w})\mathbf{Q}_0^{-1}}{\sqrt{\Omega_\zeta(x)}}\mathbf{\Sigma}_0^{1/2}\mathbf{N}_{K_{p,s}}.$$

The approximating processes $\bar{Z}_{\mu^{(v)},p}(\cdot)$, $\bar{Z}_{\vartheta,p}(\cdot)$ and $\bar{Z}_{\zeta,p}(\cdot)$ are Gaussian processes conditional on $\mathbf{X}$, $\mathbf{W}$ and $\widehat{\Delta}$, and $Z_{\mu^{(v)},p}(\cdot)$, $Z_{\vartheta,p}(\cdot)$ and $Z_{\zeta,p}(\cdot)$ are Gaussian processes conditional on $\widehat{\Delta}$ by construction. In practice, one can replace all unknowns in $\bar{Z}_{\mu^{(v)},p}(\cdot)$, $\bar{Z}_{\vartheta,p}(\cdot)$ and $\bar{Z}_{\zeta,p}(\cdot)$ (or $Z_{\mu^{(v)},p}(\cdot)$, $Z_{\vartheta,p}(\cdot)$ and $Z_{\zeta,p}(\cdot)$) by their sample analogues, and then construct the following feasible (conditional) Gaussian processes:

$$\begin{aligned}
\widehat{Z}_{\mu^{(v)},p}(x) &= \frac{\widehat{\mathbf{b}}_{p,0}^{(v)}(x)'\widehat{\mathbf{T}}_s'\widehat{\mathbf{Q}}^{-1}\widehat{\mathbf{\Sigma}}^{1/2}}{\sqrt{\widehat{\Omega}_{\mu^{(v)}}(x)}}\mathbf{N}^\star_{K_{p,s}} = \frac{\widehat{\mathbf{b}}_{p,s}^{(v)}(x)'\widehat{\mathbf{Q}}^{-1}\widehat{\mathbf{\Sigma}}^{1/2}}{\sqrt{\widehat{\Omega}_{\mu^{(v)}}(x)}}\mathbf{N}^\star_{K_{p,s}},\\
\widehat{Z}_{\vartheta,p}(x) &= \frac{\widehat{\mathbf{b}}_{p,0}(x)'\widehat{\mathbf{T}}_s'\widehat{\eta}_{0,1}(x,\widehat{\mathsf{w}})\widehat{\mathbf{Q}}^{-1}\widehat{\mathbf{\Sigma}}^{1/2}}{\sqrt{\widehat{\Omega}_{\vartheta}(x)}}\mathbf{N}^\star_{K_{p,s}} = \frac{\widehat{\mathbf{b}}_{p,s}(x)'\widehat{\eta}_{0,1}(x,\widehat{\mathsf{w}})\widehat{\mathbf{Q}}^{-1}\widehat{\mathbf{\Sigma}}^{1/2}}{\sqrt{\widehat{\Omega}_{\vartheta}(x)}}\mathbf{N}^\star_{K_{p,s}},\\
\widehat{Z}_{\zeta,p}(x) &= \frac{\widehat{\mathbf{b}}_{p,0}^{(1)}(x)'\widehat{\mathbf{T}}_s'\widehat{\eta}_{0,1}(x,\widehat{\mathsf{w}})\widehat{\mathbf{Q}}^{-1}\widehat{\mathbf{\Sigma}}^{1/2}}{\sqrt{\widehat{\Omega}_{\zeta}(x)}}\mathbf{N}^\star_{K_{p,s}} = \frac{\widehat{\mathbf{b}}_{p,s}^{(1)}(x)'\widehat{\eta}_{0,1}(x,\widehat{\mathsf{w}})\widehat{\mathbf{Q}}^{-1}\widehat{\mathbf{\Sigma}}^{1/2}}{\sqrt{\widehat{\Omega}_{\zeta}(x)}}\mathbf{N}^\star_{K_{p,s}},
\end{aligned}$$

where $\mathbf{N}^\star_{K_{p,s}}$ denotes a $K_{p,s}$-dimensional standard normal vector independent of the data $\mathbf{D}$ and the partition $\widehat{\Delta}$.

For ease of presentation, we will always require a fast convergence rate of $\widehat{\mathsf{w}}$ hereafter: $\|\widehat{\mathsf{w}} - \mathsf{w}\| = o_{\mathbb{P}}(a_n^{-1}\sqrt{J/n})$. Nevertheless, note that as shown in Theorem SA-3.5, such a rate restriction on $\widehat{\mathsf{w}}$ can be different for inference of $\vartheta_0(x,\mathsf{w})$ and $\zeta_0(x,\mathsf{w})$ and are unnecessary for inference of $\mu_0^{(v)}(x)$.

Theorem SA-3.6 (Plug-in Approximation)**.** *Suppose that Assumptions SA-DGP, SA-SM, SA-HLE and SA-RP(i) hold,*

$$\tfrac{J(\log n)^2}{n^{1-\frac{2}{\nu}}} + \Big(\tfrac{J(\log n)^7}{n}\Big)^{1/2} + nJ^{-2p-3} + \tfrac{(\log n)^2}{J^{p+1}} + nJ^{-1}\mathfrak{r}_\gamma^2 = o(a_n^{-2}),$$

$$\frac{J^{\frac{2\nu}{\nu-1}}(\log n)^{\frac{\nu}{\nu-1}}}{n} = o(1), \quad \textit{and} \quad \|\widehat{\mathsf{w}} - \mathsf{w}\| = o_{\mathbb{P}}(a_n^{-1}\sqrt{J/n}).$$

Then on a properly enriched probability space, there exists a $K_{p,s}$*-dimensional standard normal random vector* $\mathbf{N}^{\star}_{K_{p,s}}$ *independent of* $\mathbf{D}$ *and* $\widehat{\Delta}$ *such that for any* $\xi > 0$*,*

(i) $\mathbb{P}\Big(\sup_{x\in\mathcal{X}} |\widehat{Z}_{\mu^{(v)},p}(x) - \bar{Z}_{\mu^{(v)},p}(x)| > \xi a_n^{-1}\Big|\mathbf{D}, \widehat{\Delta}\Big) = o_{\mathbb{P}}(1),$

(ii) $\mathbb{P}\Big(\sup_{x\in\mathcal{X}} |\widehat{Z}_{\vartheta,p}(x) - \bar{Z}_{\vartheta,p}(x)| > \xi a_n^{-1}\Big|\mathbf{D}, \widehat{\Delta}\Big) = o_{\mathbb{P}}(1),$

(iii) $\mathbb{P}\Big(\sup_{x\in\mathcal{X}} |\widehat{Z}_{\zeta,p}(x) - \bar{Z}_{\zeta,p}(x)| > \xi a_n^{-1}\Big|\mathbf{D}, \widehat{\Delta}\Big) = o_{\mathbb{P}}(1).$

If Assumption SA-RP(ii) also holds with $\mathfrak{r}_{\mathtt{RP}} = o(a_n^{-1}(\log n)^{-1/2})$*, then*

(iv) $\mathbb{P}\Big(\sup_{x\in\mathcal{X}} |\widehat{Z}_{\mu^{(v)},p}(x) - Z_{\mu^{(v)},p}(x)| > \xi a_n^{-1}\Big|\mathbf{D}, \widehat{\Delta}\Big) = o_{\mathbb{P}}(1),$

(v) $\mathbb{P}\Big(\sup_{x\in\mathcal{X}} |\widehat{Z}_{\vartheta,p}(x) - Z_{\vartheta,p}(x)| > \xi a_n^{-1}\Big|\mathbf{D}, \widehat{\Delta}\Big) = o_{\mathbb{P}}(1),$

(vi) $\mathbb{P}\Big(\sup_{x\in\mathcal{X}} |\widehat{Z}_{\zeta,p}(x) - Z_{\zeta,p}(x)| > \xi a_n^{-1}\Big|\mathbf{D}, \widehat{\Delta}\Big) = o_{\mathbb{P}}(1).$

Remark SA-3.6 (Improvements over literature)**.** Theorems SA-3.5 and SA-3.6 provide empirical researchers with powerful tools for uniform inference based on binscatter methods. Importantly, we allow for random partitioning schemes, formally take into account the potential randomness of the partition, and construct a novel strong approximation of nonlinear binscatter-based M-estimators under mild rate restrictions. For $a_n = \sqrt{\log n}$ and $\nu \geq 4$, we require $J^{\frac{8}{3}}/n = o(1)$, up to $\log n$ terms. In the literature, similar results were only available in some special cases under stringent rate restrictions. For instance, Belloni et al. (2019) considers strong approximations of more general series-based quantile regression estimators. For the binscatter basis considered in this paper, their Theorem 11 can be applied to construct strong approximation of the t-statistic process based on pivotal coupling that achieves the approximation rate $a_n = n^{-\varepsilon'}$ under $J^4/n^{1-\varepsilon} = o(1)$ for some constants $\varepsilon, \varepsilon' > 0$, whereas their Theorem 12 can be used to construct strong approximation based on Gaussian processes under $J^5/n^{1-\varepsilon} = o(1)$. It should be noted that their notion of strong approximation is stronger than ours in the sense that it holds uniformly over both the evaluation point $x \in \mathcal{X}$ and the desired quantile $u \in \mathcal{U}$ for a compact set of quantile indices $\mathcal{U} \subset (0, 1)$. On the other hand, our methods allow for other loss functions (e.g., Huber regression), a large class of random partitions, and semi-linear covariate adjustment, leading to new results that were previously unavailable in the literature. ⌟

Theorems SA-3.5 and SA-3.6 offer a way to approximate the distribution of the *whole* t-statistic process based on $\widehat{\mu}^{(v)}(\cdot)$, $\widehat{\vartheta}(\cdot, \widehat{\mathsf{w}})$ or $\widehat{\zeta}(\cdot, \widehat{\mathsf{w}})$. A direct application of these results is the distributional approximations to the suprema of these t-statistic processes.

Theorem SA-3.7 (Supremum Approximation)**.** *Suppose that Assumptions SA-DGP, SA-SM, SA-HLE and SA-RP (including SA-RP(ii)) hold,*

$$\frac{J(\log n)^2}{n^{1-\frac{2}{\nu}}} + nJ^{-2p-3} + nJ^{-1}\mathfrak{r}_\gamma^2 = o((\log J)^{-1}),$$
$$\frac{J^{\frac{2\nu}{\nu-1}}(\log n)^{\frac{\nu}{\nu-1}}}{n} = o(1), \quad \|\widehat{\mathsf{w}} - \mathsf{w}\| = o_{\mathbb{P}}\Big(\sqrt{\tfrac{J}{n\log J}}\Big), \quad \textit{and} \quad \mathfrak{r}_{\mathtt{RP}} = o\Big(\tfrac{1}{\sqrt{\log n}\,\log J}\Big).$$

Then,

$$\begin{aligned}
&\sup_{u\in\mathbb{R}}\Big|\mathbb{P}\Big(\sup_{x\in\mathcal{X}}|T_{\mu^{(v)},p}(x)| \le u\Big) - \mathbb{P}\Big(\sup_{x\in\mathcal{X}}|\widehat{Z}_{\mu^{(v)},p}(x)| \le u\Big|\mathbf{D}, \widehat{\Delta}\Big)\Big| = o_{\mathbb{P}}(1),\\
&\sup_{u\in\mathbb{R}}\Big|\mathbb{P}\Big(\sup_{x\in\mathcal{X}}|T_{\vartheta,p}(x)| \le u\Big) - \mathbb{P}\Big(\sup_{x\in\mathcal{X}}|\widehat{Z}_{\vartheta,p}(x)| \le u\Big|\mathbf{D}, \widehat{\Delta}\Big)\Big| = o_{\mathbb{P}}(1), \quad \textit{and}\\
&\sup_{u\in\mathbb{R}}\Big|\mathbb{P}\Big(\sup_{x\in\mathcal{X}}|T_{\zeta,p}(x)| \le u\Big) - \mathbb{P}\Big(\sup_{x\in\mathcal{X}}|\widehat{Z}_{\zeta,p}(x)| \le u\Big|\mathbf{D}, \widehat{\Delta}\Big)\Big| = o_{\mathbb{P}}(1).
\end{aligned}$$

SA-3.6 Confidence Bands

Let

$$\begin{aligned}
\widehat{I}_{\mu^{(v)},p}(x) &= \Big[\widehat{\mu}^{(v)}(x) \pm \mathfrak{c}_{\mu^{(v)}}\sqrt{\widehat{\Omega}_{\mu^{(v)}}(x)/n}\Big],\\
\widehat{I}_{\vartheta,p}(x,\mathsf{w}) &= \Big[\widehat{\vartheta}(x,\widehat{\mathsf{w}}) \pm \mathfrak{c}_{\vartheta}\sqrt{\widehat{\Omega}_{\vartheta}(x)/n}\Big] \quad \text{and}\\
\widehat{I}_{\zeta,p}(x,\mathsf{w}) &= \Big[\widehat{\zeta}(x,\widehat{\mathsf{w}}) \pm \mathfrak{c}_{\zeta}\sqrt{\widehat{\Omega}_{\zeta}(x)/n}\Big]
\end{aligned}$$

be confidence bands for $\mu_0^{(v)}(\cdot)$, $\vartheta_0(\cdot,\mathsf{w})$ and $\zeta_0(\cdot,\mathsf{w})$ respectively, where $\mathfrak{c}_{\mu^{(v)}}$, $\mathfrak{c}_\vartheta$ and $\mathfrak{c}_\zeta$ are corresponding critical values to be specified. Recall that w here is taken as a fixed evaluation point for the control variables, and these bands are constructed based on a certain choice of J and the pth-order binscatter basis. Using the previous results, we have the following theorem.

Theorem SA-3.8. *Suppose that Assumptions SA-DGP, SA-SM, SA-HLE and SA-RP(i) hold,*

$$\frac{J(\log n)^2}{n^{1-\frac{2}{\nu}}} + nJ^{-2p-3} + nJ^{-1}\mathfrak{r}_\gamma^2 = o((\log J)^{-1}),$$
$$\frac{J^{\frac{2\nu}{\nu-1}}(\log n)^{\frac{\nu}{\nu-1}}}{n} = o(1), \quad \text{and} \quad \|\widehat{\mathsf{w}} - \mathsf{w}\| = o_{\mathbb{P}}\Big(\sqrt{\frac{J}{n\log J}}\Big).$$

(i) If $\mathfrak{c}_{\mu^{(v)}} = \inf\Big\{c \in \mathbb{R}_+ : \mathbb{P}[\sup_{x\in\mathcal{X}} |\widehat{Z}_{\mu^{(v)},p}(x)| \leq c\,|\mathbf{D}, \widehat{\Delta}] \geq 1-\alpha\Big\}$*, then*

$$\mathbb{P}\Big[\mu_0^{(v)}(x) \in \widehat{I}_{\mu^{(v)},p}(x),\ \textit{for all } x \in \mathcal{X}\Big] = 1-\alpha+o(1).$$

(ii) If $\mathfrak{c}_{\vartheta} = \inf\Big\{c \in \mathbb{R}_+ : \mathbb{P}[\sup_{x\in\mathcal{X}} |\widehat{Z}_{\vartheta,p}(x)| \leq c\,|\mathbf{D}, \widehat{\Delta}] \geq 1-\alpha\Big\}$*, then*

$$\mathbb{P}\Big[\vartheta_0(x,\mathsf{w}) \in \widehat{I}_{\vartheta,p}(x,\mathsf{w}),\ \textit{for all } x \in \mathcal{X}\Big] = 1-\alpha+o(1).$$

(iii) If $\mathfrak{c}_{\zeta} = \inf\Big\{c \in \mathbb{R}_+ : \mathbb{P}[\sup_{x\in\mathcal{X}} |\widehat{Z}_{\zeta,p}(x)| \leq c\,|\mathbf{D}, \widehat{\Delta}] \geq 1-\alpha\Big\}$*, then*

$$\mathbb{P}\Big[\zeta_0(x,\mathsf{w}) \in \widehat{I}_{\zeta,p}(x,\mathsf{w}),\ \textit{for all } x \in \mathcal{X}\Big] = 1-\alpha+o(1).$$

Remark SA-3.7. The above results construct valid uniform confidence bands for nonlinear binscatter-based M-estimators under mild rate restrictions. Specifically, when $\nu \geq 4$, we require $J^{\frac{8}{3}}/n = o(1)$, up to $\log n$ terms. In contrast, Belloni et al. (2019) considers more general series-based quantile regression estimators, and Theorem 15 therein can be used to construct confidence bands for binscatter estimators via various resampling methods under $J^4/n^{1-\varepsilon} = o(1)$ for some $\varepsilon > 0$. Furthermore, our results allow for random partitioning schemes, formally taking its randomness and generic structure. The key condition imposed on the partition for the validity of confidence bands is Assumption SA-RP(i), but the "convergence" of the random partition (Assumption SA-RP(ii)) is not necessary. ⌟

SA-3.7 Parametric Specification Tests

As another application, we can test parametric specifications of $\mu_0^{(v)}(x)$, $\vartheta_0(x, \mathsf{w})$ and $\zeta_0(x, \mathsf{w})$. We introduce the following tests:

$$\dot{\mathsf{H}}_0^{\mu^{(v)}}: \quad \sup_{x\in\mathcal{X}} \left|\mu_0^{(v)}(x) - m^{(v)}(x;\boldsymbol{\theta})\right| = 0, \quad \text{for some } \boldsymbol{\theta}, \qquad vs.$$
$$\dot{\mathsf{H}}_{\mathrm{A}}^{\mu^{(v)}}: \quad \sup_{x\in\mathcal{X}} \left|\mu_0^{(v)}(x) - m^{(v)}(x;\boldsymbol{\theta})\right| > 0, \quad \text{for all } \boldsymbol{\theta}.$$

where $m(x;\boldsymbol{\theta})$ is some known function depending on some finite dimensional parameter $\boldsymbol{\theta}$. This testing problem can be viewed as a two-sided test where the equality between two functions holds *uniformly* over $x \in \mathcal{X}$. In this case, we introduce $\widetilde{\boldsymbol{\theta}}$ and $\widetilde{\boldsymbol{\gamma}}$ as consistent estimators of $\boldsymbol{\theta}$ and $\boldsymbol{\gamma}_0$ under $\dot{\mathsf{H}}_0^{\mu^{(v)}}$. Then we rely on the following test statistic:

$$\dot{T}_{\mu^{(v)},p}(x) := \frac{\widehat{\mu}^{(v)}(x) - m^{(v)}(x;\widetilde{\boldsymbol{\theta}})}{\sqrt{\widehat{\Omega}_{\mu^{(v)}}(x)/n}}.$$

The null hypothesis is rejected if $\sup_{x\in\mathcal{X}} |\dot{T}_{\mu^{(v)},p}(x)| > \mathfrak{c}_{\mu^{(v)}}$ for some critical value $\mathfrak{c}_{\mu^{(v)}}$.

Similarly, to test the specification of $\vartheta_0(x, \mathsf{w})$, we introduce

$$\dot{\mathsf{H}}_0^{\vartheta}: \quad \sup_{x\in\mathcal{X}} \left|\vartheta_0(x,\mathsf{w}) - M(x,\mathsf{w};\boldsymbol{\theta},\boldsymbol{\gamma}_0)\right| = 0, \quad \text{for some } \boldsymbol{\theta}, \qquad vs.$$
$$\dot{\mathsf{H}}_{\mathrm{A}}^{\vartheta}: \quad \sup_{x\in\mathcal{X}} \left|\vartheta_0(x,\mathsf{w}) - M(x,\mathsf{w};\boldsymbol{\theta},\boldsymbol{\gamma}_0)\right| > 0, \quad \text{for all } \boldsymbol{\theta}.$$

where $M(x, \mathsf{w}; \boldsymbol{\theta}, \boldsymbol{\gamma}_0) = \eta(m(x;\boldsymbol{\theta}) + \mathsf{w}'\boldsymbol{\gamma}_0)$. We rely on the following test statistic:

$$\dot{T}_{\vartheta,p}(x) := \frac{\widehat{\vartheta}(x,\widehat{\mathsf{w}}) - M(x,\widehat{\mathsf{w}};\widetilde{\boldsymbol{\theta}},\widetilde{\boldsymbol{\gamma}})}{\sqrt{\widehat{\Omega}_{\vartheta}(x)/n}}.$$

The null hypothesis is rejected if $\sup_{x\in\mathcal{X}} |\dot{T}_{\vartheta,p}(x)| > \mathfrak{c}_{\vartheta}$ for some critical value $\mathfrak{c}_{\vartheta}$.

To test the specification of $\zeta_0(x, \mathsf{w})$, we introduce

$$\dot{\mathsf{H}}_0^{\zeta}: \quad \sup_{x\in\mathcal{X}} \left|\zeta_0(x,\mathsf{w}) - M^{(1)}(x,\mathsf{w};\boldsymbol{\theta},\boldsymbol{\gamma}_0)\right| = 0, \quad \text{for some } \boldsymbol{\theta}, \qquad vs.$$
$$\dot{\mathsf{H}}_{\mathrm{A}}^{\zeta}: \quad \sup_{x\in\mathcal{X}} \left|\zeta_0(x,\mathsf{w}) - M^{(1)}(x,\mathsf{w};\boldsymbol{\theta},\boldsymbol{\gamma}_0)\right| > 0, \quad \text{for all } \boldsymbol{\theta}.$$

where $M^{(1)}(x, \mathsf{w}; \boldsymbol{\theta}, \boldsymbol{\gamma}_0) := \eta^{(1)}(m(x; \boldsymbol{\theta}) + \mathsf{w}'\boldsymbol{\gamma}_0)m^{(1)}(x; \boldsymbol{\theta})$. We rely on the following test statistic:

$$\dot{T}_{\zeta,p}(x) := \frac{\widehat{\zeta}(x, \widehat{\mathsf{w}}) - M^{(1)}(x, \widehat{\mathsf{w}}; \widetilde{\boldsymbol{\theta}}, \widetilde{\boldsymbol{\gamma}})}{\sqrt{\widehat{\Omega}_\zeta(x)/n}}.$$

The null hypothesis is rejected if $\sup_{x\in\mathcal{X}} |\dot{T}_{\zeta,p}(x)| > \mathfrak{c}_\zeta$ for some critical value $\mathfrak{c}_\zeta$.

Theorem SA-3.9 (Specification Tests)**.** *Suppose that the conditions in Theorem SA-3.8 hold.*

(i) *Let* $\mathfrak{c}_{\mu^{(v)}} = \inf\{c \in \mathbb{R}_+ : \mathbb{P}[\sup_{x\in\mathcal{X}} |\widehat{Z}_{\mu^{(v)},p}(x)| \leq c|\mathbf{D}, \widehat{\Delta}] \geq 1 - \alpha\}$.

Under $\dot{\mathsf{H}}_0^{\mu^{(v)}}$, *if* $\sup_{x\in\mathcal{X}} |\mu^{(v)}(x) - m^{(v)}(x; \widetilde{\boldsymbol{\theta}})| = o_\mathbb{P}\Big(\sqrt{\frac{J^{1+2v}}{n\log J}}\Big)$, *then*

$$\lim_{n\to\infty} \mathbb{P}\Big[\sup_{x\in\mathcal{X}} |\dot{T}_{\mu^{(v)},p}(x)| > \mathfrak{c}_{\mu^{(v)}}\Big] = \alpha.$$

Under $\dot{\mathsf{H}}_A^{\mu^{(v)}}$, *if there exist some fixed* $\bar{\boldsymbol{\theta}}$ *such that* $\sup_{x\in\mathcal{X}} |m^{(v)}(x; \widetilde{\boldsymbol{\theta}}) - m^{(v)}(x; \bar{\boldsymbol{\theta}})| = o_\mathbb{P}(1)$, *and* $J^v\Big(\frac{J\log J}{n}\Big)^{1/2} = o(1)$, *then*

$$\lim_{n\to\infty} \mathbb{P}\Big[\sup_{x\in\mathcal{X}} |\dot{T}_{\mu^{(v)},p}(x)| > \mathfrak{c}_{\mu^{(v)}}\Big] = 1.$$

(ii) *Let* $\mathfrak{c}_\vartheta = \inf\{c \in \mathbb{R}_+ : \mathbb{P}[\sup_{x\in\mathcal{X}} |\widehat{Z}_{\vartheta,p}(x)| \leq c|\mathbf{D}, \widehat{\Delta}] \geq 1 - \alpha\}$.

Under $\dot{\mathsf{H}}_0^{\vartheta}$, *if* $\sup_{x\in\mathcal{X}} |\vartheta_0(x, \mathsf{w}) - M(x, \widehat{\mathsf{w}}; \widetilde{\boldsymbol{\theta}}, \widetilde{\boldsymbol{\gamma}})| = o_\mathbb{P}\Big(\sqrt{\frac{J^{1+2v}}{n\log J}}\Big)$, *then*

$$\lim_{n\to\infty} \mathbb{P}\Big[\sup_{x\in\mathcal{X}} |\dot{T}_{\vartheta,p}(x)| > \mathfrak{c}\Big] = \alpha.$$

Under $\dot{\mathsf{H}}_A^{\vartheta}$, *if there exist some fixed* $\bar{\boldsymbol{\theta}}$ *and* $\bar{\boldsymbol{\gamma}}$ *such that* $\sup_{x\in\mathcal{X}} |M(x, \widehat{\mathsf{w}}; \widetilde{\boldsymbol{\theta}}, \widetilde{\boldsymbol{\gamma}}) - M(x, \mathsf{w}; \bar{\boldsymbol{\theta}}, \bar{\boldsymbol{\gamma}})| = o_\mathbb{P}(1)$, *and* $J^v\Big(\frac{J\log J}{n}\Big)^{1/2} = o(1)$, *then*

$$\lim_{n\to\infty} \mathbb{P}\Big[\sup_{x\in\mathcal{X}} |\dot{T}_{\vartheta,p}(x)| > \mathfrak{c}\Big] = 1.$$

(iii) *Let* $\mathfrak{c}_\zeta = \inf\{c \in \mathbb{R}_+ : \mathbb{P}[\sup_{x\in\mathcal{X}} |\widehat{Z}_{\zeta,p}(x)| \leq c|\mathbf{D}, \widehat{\Delta}] \geq 1 - \alpha\}$.

Under $\dot{\mathsf{H}}_0^{\zeta}$, *if* $\sup_{x\in\mathcal{X}} |\zeta_0(x,\mathsf{w}) - M^{(1)}(x,\widehat{\mathsf{w}};\widetilde{\boldsymbol{\theta}},\widetilde{\boldsymbol{\gamma}})| = o_{\mathbb{P}}\Big(\sqrt{\frac{J^{1+2v}}{n\log J}}\Big)$, *then*

$$\lim_{n\to\infty}\mathbb{P}\Big[\sup_{x\in\mathcal{X}}|\dot{T}_{\zeta,p}(x)| > \mathfrak{c}\Big] = \alpha.$$

Under $\dot{\mathsf{H}}_A^{\zeta}$, *if there exist some fixed* $\bar{\boldsymbol{\theta}}$ *and* $\bar{\boldsymbol{\gamma}}$ *such that* $\sup_{x\in\mathcal{X}} |M^{(1)}(x,\widehat{\mathsf{w}};\widetilde{\boldsymbol{\theta}},\widetilde{\boldsymbol{\gamma}}) - M^{(1)}(x,\mathsf{w};\bar{\boldsymbol{\theta}},\bar{\boldsymbol{\gamma}})| = o_{\mathbb{P}}(1)$, *and* $J^v\Big(\frac{J\log J}{n}\Big)^{1/2} = o(1)$, *then*

$$\lim_{n\to\infty}\mathbb{P}\Big[\sup_{x\in\mathcal{X}}|\dot{T}_{\zeta,p}(x)| > \mathfrak{c}\Big] = 1.$$

SA-3.8 Shape Restriction Tests

The third application of our results is to test certain shape restrictions on $\mu_0^{(v)}(x)$, $\vartheta_0(x,\mathsf{w})$ and $\zeta_0(x,\mathsf{w})$. To be specific, consider the following problem:

$$\begin{aligned}
\ddot{\mathsf{H}}_0^{\mu^{(v)}} &: \sup_{x\in\mathcal{X}}(\mu^{(v)}(x) - m^{(v)}(x;\bar{\boldsymbol{\theta}})) \le 0 \text{ for certain } \bar{\boldsymbol{\theta}} \text{ and } \bar{\boldsymbol{\gamma}} \quad \text{v.s.}\\
\ddot{\mathsf{H}}_{\mathrm{A}}^{\mu^{(v)}} &: \sup_{x\in\mathcal{X}}(\mu^{(v)}(x) - m^{(v)}(x;\bar{\boldsymbol{\theta}})) > 0 \text{ for } \bar{\boldsymbol{\theta}} \text{ and } \bar{\boldsymbol{\gamma}}.
\end{aligned}$$

This testing problem can be viewed as a one-sided test where the inequality holds *uniformly* over $x\in\mathcal{X}$. Importantly, it should be noted that under both $\ddot{\mathsf{H}}_0^{\mu^{(v)}}$ and $\ddot{\mathsf{H}}_{\mathrm{A}}^{\mu^{(v)}}$, we fix $\bar{\boldsymbol{\theta}}$ and $\bar{\boldsymbol{\gamma}}$ to be the same values in the parameter space. In such a case, we introduce $\widetilde{\boldsymbol{\theta}}$ and $\widetilde{\boldsymbol{\gamma}}$ as consistent estimators of $\bar{\boldsymbol{\theta}}$ and $\bar{\boldsymbol{\gamma}}$ under both $\ddot{\mathsf{H}}_0^{\mu^{(v)}}$ and $\ddot{\mathsf{H}}_{\mathrm{A}}^{\mu^{(v)}}$. Then we will rely on the following test statistic:

$$\ddot{T}_{\mu^{(v)},p}(x) := \frac{\widehat{\mu}^{(v)}(x) - m^{(v)}(x;\widetilde{\boldsymbol{\theta}})}{\sqrt{\widehat{\Omega}_{\mu^{(v)}}(x)/n}}.$$

The null hypothesis is rejected if $\sup_{x\in\mathcal{X}}\ddot{T}_{\mu^{(v)},p}(x) > \mathfrak{c}_{\mu^{(v)}}$ for some critical value $\mathfrak{c}_{\mu^{(v)}}$.

Similarly, define the test for the shape of $\vartheta_0(x,\mathsf{w})$:

$$\begin{aligned}
\ddot{\mathsf{H}}_0^{\vartheta} &: \sup_{x\in\mathcal{X}}(\vartheta_0(x,\mathsf{w}) - M(x,\mathsf{w};\bar{\boldsymbol{\theta}},\bar{\boldsymbol{\gamma}})) \le 0 \text{ for certain } \bar{\boldsymbol{\theta}} \text{ and } \bar{\boldsymbol{\gamma}} \quad \text{v.s.}\\
\ddot{\mathsf{H}}_{\mathrm{A}}^{\vartheta} &: \sup_{x\in\mathcal{X}}(\vartheta_0(x,\mathsf{w}) - M(x,\mathsf{w};\bar{\boldsymbol{\theta}},\bar{\boldsymbol{\gamma}})) > 0 \text{ for } \bar{\boldsymbol{\theta}} \text{ and } \bar{\boldsymbol{\gamma}}.
\end{aligned}$$

We will rely on the following test statistic:

$$\ddot{T}_{\vartheta,p}(x) := \frac{\widehat{\vartheta}(x,\widehat{\mathsf{w}}) - M(x,\widehat{\mathsf{w}};\widetilde{\boldsymbol{\theta}},\widetilde{\boldsymbol{\gamma}})}{\sqrt{\widehat{\Omega}_{\vartheta}(x)/n}}.$$

The null hypothesis is rejected if $\sup_{x\in\mathcal{X}}\ddot{T}_{\vartheta,p}(x) > \mathfrak{c}_{\vartheta}$ for some critical value $\mathfrak{c}_{\vartheta}$.

Also, define the test for the shape of $\zeta_0(x,\mathsf{w})$:

$$\begin{aligned}
\ddot{\mathsf{H}}_0^{\zeta}: &\sup_{x\in\mathcal{X}}\big(\zeta_0(x,\mathsf{w}) - M^{(1)}(x,\mathsf{w};\bar{\boldsymbol{\theta}},\bar{\boldsymbol{\gamma}})\big) \leq 0 \text{ for certain } \bar{\boldsymbol{\theta}} \text{ and } \bar{\boldsymbol{\gamma}} \quad \text{v.s.}\\
\ddot{\mathsf{H}}_{\mathrm{A}}^{\zeta}: &\sup_{x\in\mathcal{X}}\big(\zeta_0(x,\mathsf{w}) - M^{(1)}(x,\mathsf{w};\bar{\boldsymbol{\theta}},\bar{\boldsymbol{\gamma}})\big) > 0 \text{ for } \bar{\boldsymbol{\theta}} \text{ and } \bar{\boldsymbol{\gamma}}.
\end{aligned}$$

We will rely on the following test statistic:

$$\ddot{T}_{\zeta,p}(x) := \frac{\widehat{\zeta}(x,\widehat{\mathsf{w}}) - M^{(1)}(x,\widehat{\mathsf{w}};\widetilde{\boldsymbol{\theta}},\widetilde{\boldsymbol{\gamma}})}{\sqrt{\widehat{\Omega}_{\zeta}(x)/n}}.$$

The null hypothesis is rejected if $\sup_{x\in\mathcal{X}}\ddot{T}_{\zeta,p}(x) > \mathfrak{c}_{\zeta}$ for some critical value $\mathfrak{c}_{\zeta}$.

The following theorem characterizes the size and power of such tests.

Theorem SA-3.10 (Shape Restriction Tests)**.** *Suppose that the conditions in Theorem SA-3.8 hold.*

(i) Assume $\sup_{x\in\mathcal{X}}|m(x;\widetilde{\boldsymbol{\theta}}) - m(x;\bar{\boldsymbol{\theta}})| = o_{\mathbb{P}}\Big(\sqrt{\frac{J^{1+2v}}{n\log J}}\Big)$*. Let* $\mathfrak{c}_{\mu^{(v)}} = \inf\{c\in\mathbb{R}_+ : \mathbb{P}[\sup_{x\in\mathcal{X}}\widehat{Z}_{\mu^{(v)},p}(x) \leq c|\mathbf{D},\widehat{\Delta}] \geq 1-\alpha\}$.

Under $\ddot{\mathsf{H}}_0^{\mu^{(v)}}$,

$$\lim_{n\to\infty}\mathbb{P}\Big[\sup_{x\in\mathcal{X}}\ddot{T}_{\mu^{(v)},p}(x) > \mathfrak{c}_{\mu^{(v)}}\Big] \leq \alpha.$$

Under $\ddot{\mathsf{H}}_A^{\mu^{(v)}}$*, if* $J^v\Big(\frac{J\log J}{n}\Big)^{1/2} = o(1)$,

$$\lim_{n\to\infty}\mathbb{P}\Big[\sup_{x\in\mathcal{X}}\ddot{T}_{\mu^{(v)},p}(x) > \mathfrak{c}_{\mu^{(v)}}\Big] = 1.$$

(ii) Assume $\sup_{x\in\mathcal{X}}|M(x,\widehat{\mathsf{w}};\widetilde{\boldsymbol{\theta}},\widetilde{\boldsymbol{\gamma}}) - M(x,\mathsf{w};\bar{\boldsymbol{\theta}},\bar{\boldsymbol{\gamma}})| = o_{\mathbb{P}}\Big(\sqrt{\frac{J^{1+2v}}{n\log J}}\Big)$*. Let* $\mathfrak{c}_{\vartheta} = \inf\{c\in\mathbb{R}_+ : \mathbb{P}[\sup_{x\in\mathcal{X}}\widehat{Z}_{\vartheta,p}(x) \leq c|\mathbf{D},\widehat{\Delta}] \geq 1-\alpha\}$.

Under $\ddot{\mathsf{H}}_0^{\vartheta}$,

$$\lim_{n\to\infty} \mathbb{P}\Big[\sup_{x\in\mathcal{X}} \ddot{T}_{\vartheta,p}(x) > \mathfrak{c}_{\vartheta}\Big] \leq \alpha.$$

Under $\ddot{\mathsf{H}}_A^{\vartheta}$, *if* $J^v\Big(\frac{J\log J}{n}\Big)^{1/2} = o(1)$,

$$\lim_{n\to\infty} \mathbb{P}\Big[\sup_{x\in\mathcal{X}} \ddot{T}_{\vartheta,p}(x) > \mathfrak{c}_{\vartheta}\Big] = 1.$$

(iii) Assume $\sup_{x\in\mathcal{X}} |M^{(1)}(x,\widehat{\mathsf{w}};\widetilde{\boldsymbol{\theta}},\widetilde{\boldsymbol{\gamma}}) - M^{(1)}(x,\mathsf{w};\bar{\boldsymbol{\theta}},\bar{\boldsymbol{\gamma}})| = o_{\mathbb{P}}\Big(\sqrt{\frac{J^{1+2v}}{n\log J}}\Big)$. *Let* $\mathfrak{c}_{\zeta} = \inf\{c\in\mathbb{R}_+ : \mathbb{P}[\sup_{x\in\mathcal{X}} \widehat{Z}_{\zeta,p}(x) \leq c|\mathbf{D},\widehat{\Delta}] \geq 1-\alpha\}$.

Under $\ddot{\mathsf{H}}_0^{\zeta}$,

$$\lim_{n\to\infty} \mathbb{P}\Big[\sup_{x\in\mathcal{X}} \ddot{T}_{\zeta,p}(x) > \mathfrak{c}_{\zeta}\Big] \leq \alpha.$$

Under $\ddot{\mathsf{H}}_A^{\zeta}$, *if* $J^v\Big(\frac{J\log J}{n}\Big)^{1/2} = o(1)$,

$$\lim_{n\to\infty} \mathbb{P}\Big[\sup_{x\in\mathcal{X}} \ddot{T}_{\zeta,p}(x) > \mathfrak{c}_{\zeta}\Big] = 1.$$

Remark SA-3.8 (Improvements over literature)**.** The results in Sections SA-3.6–SA-3.8 are new to the literature, even in the case of non-random partitioning and without covariate-adjustments, because they take advantage of the specific binscatter structure (i.e., locally bounded series basis). Furthermore, relative to prior work, our results allow for a large class of random partitioning schemes, formally take into account the potential randomness of the partition, account for the generalized semi-linear structure, and consider an array of possibly nonlinear estimation and inference problems. In particular, the approach taken in Theorems SA-3.5 and SA-3.7 to establish strong approximation and related distributional approximations for nonlinear binscatter statistics may be of independent interest. The key condition imposed on the partition for uniform inference (confidence bands and hypothesis testing) is Assumption SA-RP(i), while "convergence" of the random partition (Assumption SA-RP(ii)) is not required. ⌟

SA-4 Implementation Details

SA-4.1 Standard Error Computation

In Section SA-3, we have given the variance formulas $\widehat{\Omega}_{\mu^{(v)}}(x)$, $\widehat{\Omega}_{\vartheta}(x)$ and $\widehat{\Omega}_{\zeta}(x)$ that can be used to obtain the standard errors of $\widehat{\mu}^{(v)}(x)$, $\widehat{\vartheta}(x, \widehat{\mathrm{w}})$ and $\widehat{\zeta}(x, \widehat{\mathrm{w}})$. Recall that the formula for the estimator $\widehat{\mathbf{\Sigma}}$ of $\mathbf{\Sigma}_0$ is

$$\widehat{\mathbf{\Sigma}} = \mathbb{E}_n\Big[\widehat{\mathbf{b}}_{p,s}(x_i)\widehat{\mathbf{b}}_{p,s}(x_i)'\psi(y_i, \widehat{\eta}_i)^2\eta^{(1)}(\widehat{\mu}(x_i) + \mathbf{w}_i'\widehat{\gamma})^2\Big].$$

It only relies on known or estimable quantities such as the derivative of the loss function $\psi(\cdot)$, the derivative of the inverse link function $\eta^{(1)}(\cdot)$, the residual $\widehat{\epsilon}_i$ and the binscatter estimates $\widehat{\mu}(\cdot)$ and $\widehat{\gamma}$. Thus, $\widehat{\mathbf{\Sigma}}$ and other types of heteroskedasticity-robust "meat" matrix estimators can be easily constructed using the data. Then, it remains to obtain an estimator $\widehat{\mathbf{Q}}$ of $\bar{\mathbf{Q}}$ (or $\mathbf{Q}_0$), which in general relies on an estimator $\widehat{\Psi}_1(\cdot)$ of $\Psi_1(\cdot)$ and can be constructed in a case-by-case basis. In the following we discuss several examples.

Example 1 (Least Squares Regression). For least squares regression, the loss function $\rho(y;\eta) = \frac{1}{2}(y-\eta)^2$ and the (inverse) link function $\eta(\theta) = \theta$. Therefore, $\psi(y_i, \eta_i) = -\epsilon_i$ and $\eta_{i,1} = 1$. Thus, the formula for $\widehat{\mathbf{Q}}$ given in Section SA-3 reduces to $\mathbb{E}_n[\widehat{\mathbf{b}}_{p,s}(x_i)\widehat{\mathbf{b}}_{p,s}(x_i)']$, which is immediately feasible in practice.

Example 2 (Logistic Regression). For logistic regression, the loss function is given by the corresponding likelihood function, i.e., $-\rho(y;\eta) = y\log\eta + (1-y)\log(1-\eta)$, and the inverse link is given by the logistic function $\eta(\theta) = \frac{e^\theta}{1+e^\theta}$. Accordingly, an estimator of $\mathbf{Q}_0$ is given by

$$\widehat{\mathbf{Q}} = \mathbb{E}_n\Big[\widehat{\mathbf{b}}_{p,s}(x_i)\widehat{\mathbf{b}}_{p,s}(x_i)'\widehat{\eta}_i(1-\widehat{\eta}_i)\Big], \quad \widehat{\eta}_i = \eta(\widehat{\mu}(x_i) + \mathbf{w}_i'\widehat{\gamma}).$$

Example 4 (Quantile Regression). For quantile regression, $\rho(y;\eta) = (q - \mathbb{1}(y<\eta))(y-\eta)$ for some $q \in (0,1)$ and $\eta(\theta) = \theta$. Accordingly, $\psi(y_i, \eta_i) = \mathbb{1}(\epsilon_i < 0) - q$, and one needs to estimate

$$\mathbf{Q}_0 = \mathbb{E}\Big[\mathbf{b}_{p,s}(x_i)\mathbf{b}_{p,s}(x_i)'f_{Y|XW}(\mu_0(x_i) + \mathbf{w}_i'\gamma_0|x_i, \mathbf{w}_i)\Big].$$

The key is to estimate the conditional density $f_{Y|XW}(\cdot|x_i, \mathbf{w}_i)$ evaluated at the conditional quantile of interest $(\mu_0(x_i) + \mathbf{w}_i'\gamma_0)$, whose reciprocal is termed "sparsity function" in the literature. Many different methods have been proposed. For example, the sparsity function is simply the derivative of the conditional quantile function with respect to the quantile, which can be estimated by using the difference quotient of the estimated conditional quantile function. Alternatively, $\mathbf{Q}_0$ can be viewed as a matrix-weighted density function, and one can construct a corresponding estimator based on kernel density estimation ideas. In addition, one can use bootstrapping methods to estimate the variance, avoiding the technical difficulty of estimating the sparsity function. See Section 3.4 and Section 3.9 of Koenker (2005) for more discussion of variance estimation for quantile regression.

SA-4.2 Number of Bins Selector

We discuss the implementation details for data-driven selection of the number of bins, based on the approximate integrated mean squared error expansion in Theorem SA-3.4.

We offer two procedures for estimating the bias and variance constants, and once these estimates ($\widehat{\mathscr{B}}_n(p,s,v)$ and $\widehat{\mathscr{V}}_n(p,s,v)$) are available, the estimated optimal J is

$$\widehat{J}_{\texttt{IMSE}} = \left\lceil \left(\frac{2(p-v+1)\widehat{\mathscr{B}}_n(p,s,v)}{(1+2v)\widehat{\mathscr{V}}_n(p,s,v)}\right)^{\frac{1}{2p+3}} n^{\frac{1}{2p+3}} \right\rceil.$$

We always let $\omega(x) = f_X(x)$ as weighting function for concreteness.

SA-4.2.1 Rule-of-thumb Selector

A rule-of-thumb choice of J can be obtained based on Corollary SA-3.2 in Cattaneo et al. (2024b), which gives an explicit characterization of the variance and bias constants for least squares binscatter using piecewise polynomials ($s = 0$).

Specifically, the variance constant $\mathscr{V}(p,0,v)$ is estimated by

$$\widehat{\mathscr{V}}(p,0,v) = \operatorname{trace}\left\{\left(\int_0^1 \boldsymbol{\varphi}(z)\boldsymbol{\varphi}(z)'dz\right)^{-1}\int_0^1 \boldsymbol{\varphi}^{(v)}(z)\boldsymbol{\varphi}^{(v)}(z)'dz\right\} \times \frac{1}{n}\sum_{i=1}^n \widehat{\sigma}^2(x_i,\mathbf{w}_i)\widehat{f}_X(x_i)^{2v}$$

where $\boldsymbol{\varphi}(z) = (1, z, \dots, z^p)'$, $\widehat{\sigma}^2(x_i,\mathbf{w}_i)$ is some estimate of the conditional variance $\mathbb{V}[y_i|x_i,\mathbf{w}_i]$ and $\widehat{f}_X(x_i)$ is some estimate of the density $f_X(x_i)$. On the other hand, the bias constant $\mathscr{B}(p,0,v)$ is estimated by

$$\widehat{\mathscr{B}}(p,0,v) = \frac{\int_0^1[\mathscr{B}_{p+1-v}(z)]^2dz}{((p+1-v)!)^2} \times \frac{1}{n}\sum_{i=1}^n \frac{[\widehat{\mu}^{(p+1)}(x_i)]^2}{\widehat{f}_X(x_i)^{2p+2-2v}}.$$

where $\mathscr{B}_p(z) = (-1)^p \sum_{k=0}^{p} \binom{p}{k}\binom{p+k}{k}(-z)^k / \binom{2p}{p}$ for each $p \in \mathbb{Z}_+$ and $\widehat{\mu}^{(p+1)}(x_i)$ is some preliminary estimate of $\mu_0^{(p+1)}(x_i)$. The details about getting the estimates $\widehat{\sigma}^2(x_i, \mathbf{w}_i)$, $\widehat{f}_X(x_i)$ and $\widehat{\mu}^{(p+1)}(x_i)$ can be found in Section SA-4.1 in Cattaneo et al. (2024b).

This procedure still yields a choice of J with the correct rate, though the constant approximations are inconsistent for general loss.

SA-4.2.2 Direct-plug-in Selector

The direct-plug-in selector is implemented based on nonlinear binscatter estimators, which applies to any user-specified p, s and v. It requires a preliminary choice of J, for which the rule-of-thumb selector previously described can be used.

More generally, suppose that a preliminary choice $J_{\texttt{pre}}$ is given, and then a binscatter basis $\widehat{\mathbf{b}}_{p,s}(x)$ (of order p) can be constructed immediately on the preliminary partition. Implementing a nonlinear binscatter estimation using this basis and partitioning, we can obtain the variance constant estimate using the variance matrix estimators discussed in Section SA-4.1.

Regarding the bias constant, the key unknown in the expression of the leading approximation error $r^{\star}_{0,v}(x)$ in Theorem SA-3.4 is $\mu_0^{(p+1)}(x)$, which can be estimated by implementing a nonlinear binscatter estimation of order $p+1$ (with the preliminary partition unchanged). Also, an estimate of $f_X(x_i)^{-1}$ in $r^{\star}_{0,v}(x)$ is $J\hat{h}_{x_i}$ where $\hat{h}_{x_i}$ denotes the length of the interval in $\widehat{\Delta}$ containing x_i. All other quantities in the expression of $\mathscr{B}(p, s, v)$ can be replaced by their sample analogues. Then, a bias constant estimate is available.

By this construction, the direct-plug-in selector employs the correct rate and consistent constant approximations for any nonlinear binscatter with any choice of p, s and v.

SA-4.3 Fixed J and choice of polynomial order

Our main theory treats J as diverging with the sample size. This reflects the standard approach wherein a researcher selects p and s in advance (often $s = p = 0$ or $s = p = 3$) and then selects J given the data. The partition must get finer to remove the nonparametric smoothing bias in estimating the function $\mu_0(x)$ (and along with it, $\vartheta_0(x, \mathsf{w})$ or $\zeta_0(x, \mathsf{w})$). Correct recovery (either for estimation or visualization) of these functions is the primary use of binscatter. However, researchers sometimes prefer to pre-specify a fixed $J = \mathtt{J}$, and we also discuss implementation and interpretation of binscatter in this case.

Instead of modeling J as diverging and searching for the optimal choice, a researcher may desire a fixed (often small and round) number of J, which we denote by $\mathtt{J}$. This is done either to make the estimate more visually appealing or because the results can be directly interpreted. In this case, instead of recovering the functions $\mu_0^{(v)}(x)$, $\vartheta_0(x, \mathsf{w})$, and $\zeta_0(x, \mathsf{w})$, the binscatter is interpreted as estimating their coarsened versions: the distribution of y_i conditional on x_i lying in a (fixed) bin, rather than at a single point. For some $\mathtt{J}$, this remains interpretable and all our inference results apply to this case. For example, in our application we can take $\mathtt{J} = 10$ and study the distribution of uninsured rate for each decile of income. The confidence bands then become pointwise confidence intervals that are simultaneously valid. For example, this could be used to examine inequality in health care access by asking if median uninsured rates are statistically different between the top and bottom decile.

A fixed $\mathtt{J}$ is also interpretable, and applicable, if x_i is discrete. Then each mass point is given its own bin and the results apply to the conditional distribution of y_i given $x_i = x$. Again, our theoretical results apply directly to this case and one obtains simultaneous inference over the set of points. Cattaneo et al. (2024b) give further discussion and examples.

As a practical compromise between the visual appeal and interpretation of a small, fixed $\mathtt{J}$ and the demand for consistent estimation, we propose a novel, albeit ad-hoc, adjustment to the estimator aimed at addressing the smoothing bias left by fixing $\mathtt{J}$ by adjusting the choice of polynomial order p. The standard approach fixes p in advance and selects J based on the data, but we can invert this and search for the value of p for which the fixed $\mathtt{J}$ would be IMSE-optimal. That is, we solve for

$$p_{\mathtt{IMSE}}(\mathtt{J}, v) = \underset{p \in \mathcal{P}}{\arg\min} \left| J_{\mathtt{IMSE}}(p, v) - \mathtt{J} \right|, \tag{SA-4.1}$$

where in principle the set $\mathcal{P}$ is all nonnegative integers, but in practice $\mathcal{P} = \{p_{\mathtt{min}}, p_{\mathtt{min}}+1, \ldots, p_{\mathtt{max}}-1, p_{\mathtt{max}}\}$, for some integers $0 \leq p_{\mathtt{min}} \leq p_{\mathtt{max}}$. The (in)flexibility of fixed $J = \mathtt{J}$ is offset by changing the polynomial approximation. This may have some practical appeal, but our theoretical results in the next section continue in the standard case of fixed p and diverging J.

To implement the data-driven choice $p_{\texttt{IMSE}}(\texttt{J}, v)$, users needs to specify the desired (often small) number of bins $\texttt{J}$, the derivative order v of interest, and a (finite) set $\mathcal{P}$ of acceptable polynomial orders. The size of $\mathcal{P}$ is usually small since in practice $p = 3$ or 4 often suffices to yield a small IMSE-optimal number of bins. Then, for each value of p in $\mathcal{P}$, we can implement the rule-of-thumb or direct plug-in procedure as described in Section SA-4.2 to obtain $J_{\texttt{IMSE}}(p, v)$. The "optimal" choice $p_{\texttt{IMSE}}(\texttt{J}, v)$ is the value of p with the resulting $J_{\texttt{IMSE}}(p, v)$ closest to $\texttt{J}$.

SA-5 Proofs

We begin with a subsection collecting some technical lemmas used in the proofs of our main results. We then collect all the proof of the results presented in this supplemental appendix, which are in several cases more general than those discussed in the main text. Some of our technical results may be of more broad independent interest in the nonlinear series estimation literature.

SA-5.1 Technical Lemmas

We first give several simple facts about $\widehat{\Delta}$ in the following lemma, which are immediate from Assumption SA-RP(ii).

Lemma SA-5.1 (Quasi-Uniformity). *Suppose that Assumption SA-RP(ii) holds. Then, (i)* $J^{-1} \lesssim \min_{1\leq j\leq J} h_j \leq \max_{1\leq j\leq J} h_j \lesssim J^{-1}$*, (ii)* $\max_{1\leq j\leq J} |\hat{\tau}_j - \tau_j| \lesssim_{\mathbb{P}} \mathfrak{r}_{\mathtt{RP}}$*, and (iii)* $\widehat{\Delta} \in \Pi_{3c_{\mathtt{QU}}}$ *w.p.a. 1.*

Proof. By Assumption SA-RP(ii), $\mathrm{len}(\mathcal{X}) = \sum_{j=1}^{J} h_j \geq J \min_{1\leq j\leq J} h_j \geq c_{\mathtt{QU}}^{-1} J \max_{1\leq j\leq J} h_j$ where $\mathrm{len}(\mathcal{X})$ denotes the length of $\mathcal{X}$ (which is a fixed number). On the other hand, $\mathrm{len}(\mathcal{X}) \leq J \max_{1\leq j\leq J} h_j \leq c_{\mathtt{QU}} J \min_{1\leq j\leq J} h_j$. Therefore, $c_{\mathtt{QU}}^{-1} J^{-1}\mathrm{len}(\mathcal{X}) \leq \min_{1\leq j\leq J} h_j \leq \max_{1\leq j\leq J} h_j \leq c_{\mathtt{QU}} J^{-1}\mathrm{len}(\mathcal{X})$.

Next, by Assumption SA-RP(ii), $\max_{1\leq j\leq J} |\hat{\tau}_j - \tau_j| = \max_{1\leq j\leq J} |\sum_{l=1}^{j} (\hat{h}_l - h_l)| \leq J \max_{1\leq l\leq J} |\hat{h}_l - h_l| \lesssim \mathfrak{r}_{\mathtt{RP}}$. In addition, $\max_{1\leq j\leq J} |\hat{h}_j - h_j| \leq \frac{1}{2} c_{\mathtt{QU}}^{-1} J^{-1}\mathrm{len}(\mathcal{X}) \leq \frac{1}{2}\min_{1\leq j\leq J} h_j$ w.p.a. 1, and thus

$$\frac{\max_{1\leq j\leq J} \hat{h}_j}{\min_{1\leq j\leq J} \hat{h}_j} = \frac{\max_{1\leq j\leq J} h_j + \max_{1\leq j\leq J} |\hat{h}_j - h_j|}{\min_{1\leq j\leq J} h_j - \max_{1\leq j\leq J} |\hat{h}_j - h_j|} \leq 3c_{\mathtt{QU}}, \quad \text{w.p.a.1.}$$

Then, the proof is complete. □

The next lemma then verifies Assumption SA-RP(ii) for the special case of quantile-spaced partitions. The proof is available in the supplemental appendix of Cattaneo et al. (2024b) (see Section SA-3.1 therein) and thus omitted here.

Lemma SA-5.2 (Quasi-Uniformity of Quantile-Spaced Partitions). *Suppose that Assumption SA-DGP(i) and SA-DGP(ii) holds and* $\widehat{\Delta}$ *is generated by sample quantiles, i.e.,* $\hat{\tau}_j = \hat{F}_X^{-1}(j/J)$*. If* $\frac{J\log J}{n} = o(1)$ *and* $\frac{\log n}{J} = o(1)$*, then Assumption SA-RP(ii) holds with* $\tau_j = F_X^{-1}(j/J)$ *and* $\mathfrak{r}_{\mathtt{RP}} = \left(\frac{J\log J}{n}\right)^{1/2}$.

The next three lemmas SA-5.3–SA-5.5 concern the properties of binscatter basis functions. Their proofs are the same as those for quantile-based partitions that are available in the supplemental appendix of Cattaneo et al. (2024b) (see Section SA-3.1 therein) and are omitted here to conserve space.

Lemma SA-5.3 (Transformation Matrix)**.** *Suppose that Assumption SA-RP(i) holds. Then* $\widehat{\mathbf{b}}_{p,s}(x) = \widehat{\mathbf{T}}_s \widehat{\mathbf{b}}_{p,0}(x)$ *with* $\|\widehat{\mathbf{T}}_s\|_\infty \lesssim_{\mathbb{P}} 1$ *and* $\|\widehat{\mathbf{T}}_s\| \lesssim_{\mathbb{P}} 1$*. If, in addition, Assumption SA-RP(ii) holds, then* $\|\widehat{\mathbf{T}}_s - \mathbf{T}_s\|_\infty \lesssim_{\mathbb{P}} \mathfrak{r}_{\mathtt{RP}}$ *and* $\|\widehat{\mathbf{T}}_s - \mathbf{T}_s\| \lesssim_{\mathbb{P}} \mathfrak{r}_{\mathtt{RP}}$*.*

Lemma SA-5.4 (Local Basis)**.** *Suppose that Assumption SA-RP(i) holds. Then* $\sup_{x\in\mathcal{X}} \|\widehat{\mathbf{b}}^{(v)}_{p,s}(x)\|_0 \leq (p+1)^2$ *and* $\sup_{x\in\mathcal{X}} \|\widehat{\mathbf{b}}^{(v)}_{p,s}(x)\| \lesssim_{\mathbb{P}} J^{\frac{1}{2}+v}$*.*

The following lemma provides a particular way to define $\boldsymbol{\beta}_0(\Delta)$ and $\widehat{\boldsymbol{\beta}}_0$ so that the required approximation rate is achieved. We define

$$\boldsymbol{\beta}_0^{\mathtt{LS}}(\Delta) := \underset{\boldsymbol{\beta}\in\mathbb{R}^{K_{p,s}}}{\arg\min}\ \mathbb{E}[(\mu_0(x_i) - \mathbf{b}_{p,s}(x_i;\Delta)'\boldsymbol{\beta})^2], \quad \widehat{\boldsymbol{\beta}}_0^{\mathtt{LS}} = \boldsymbol{\beta}_0^{\mathtt{LS}}(\widehat{\Delta}).$$

Lemma SA-5.5 (Approximation Error)**.** *Suppose that Assumptions SA-DGP(i)(ii), SA-SM(v) and SA-RP(i) hold. Then*

$$\sup_{\Delta\in\Pi}\sup_{x\in\mathcal{X}} |\mathbf{b}^{(v)}_{p,s}(x;\Delta)'\boldsymbol{\beta}_0^{\mathtt{LS}}(\Delta) - \mu_0^{(v)}(x)| \lesssim J^{-p-1+v} \quad \textit{and} \quad \sup_{x\in\mathcal{X}} |\widehat{\mathbf{b}}^{(v)}_{p,s}(x)'\widehat{\boldsymbol{\beta}}_0^{\mathtt{LS}} - \mu_0^{(v)}(x)| \lesssim_{\mathbb{P}} J^{-p-1+v}.$$

Next, the following maximal inequality is useful in our analysis. Its proof is available in Cattaneo et al. (2024c) and thus omitted here.

Lemma SA-5.6 (Maximal Inequality)**.** *Let* $Z_1, \cdots, Z_n$ *be independent but not necessarily identically distributed random variables taking values in a measurable space* $(\mathcal{S}; \mathscr{S})$*. Denote the joint distribution of* $Z_1, \cdots, Z_n$ *by* $\mathbb{P}$ *and the marginal distribution of* Z_i *by* $\mathbb{P}_i$*, and let* $\bar{\mathbb{P}} = \frac{1}{n}\sum_{i=1}^n \mathbb{P}_i$*. Let* $\mathcal{F}$ *be a class of Borel measurable functions from* $\mathcal{S}$ *to* $\mathbb{R}$ *which is pointwise measurable. Let* $\bar{F}$ *be a measurable envelope function for* $\mathcal{F}$*. Suppose that* $\|\bar{F}\|_{L_2(\bar{\mathbb{P}})} < \infty$*. Let* $\bar{\sigma} > 0$ *satisfy* $\sup_{f\in\mathcal{F}} \|f\|_{L_2(\bar{\mathbb{P}})} \leq \bar{\sigma} \leq \|\bar{F}\|_{L_2(\bar{\mathbb{P}})}$ *and define* $\bar{\bar{F}} = \max_{1\leq i\leq n} \bar{F}(Z_i)$*. Then, with* $\delta = \bar{\sigma}/\|\bar{F}\|_{L_2(\bar{\mathbb{P}})}$*,*

$$\mathbb{E}\Big[\sup_{f\in\mathcal{F}}\Big|\frac{1}{\sqrt{n}}\sum_{i=1}^n \Big(f(Z_i) - \mathbb{E}[f(Z_i)]\Big)\Big|\Big] \lesssim \|\bar{F}\|_{L_2(\bar{\mathbb{P}})} J(\delta,\mathcal{F},\bar{F}) + \frac{\|\bar{\bar{F}}\|_{L_2(\mathbb{P})} J(\delta,\mathcal{F},\bar{F})^2}{\delta^2\sqrt{n}},$$

where

$$J(\delta,\mathcal{F},\bar{F})=\int_0^\delta\sqrt{1+\sup_{\mathbb{Q}}\log N(\mathcal{F},L_2(\mathbb{Q}),\varepsilon\|\bar{F}\|_{L_2(\mathbb{Q})})}d\varepsilon.$$

SA-5.2 Proof of Lemma SA-3.1

Proof. We write $\Psi_{i,1}:=\Psi_1(x_i,\mathbf{w}_i;\eta_i)$.

(i) We first prove a convergence result for $\bar{\mathbf{Q}}$. In view of Lemma SA-5.3, it suffices to show the convergence for $s=0$. Let $\mathcal{A}_n$ denote the event on which $\widehat{\Delta}\in\Pi$. By Assumption SA-RP(i), $\mathbb{P}(\mathcal{A}_n^c)=o(1)$. On $\mathcal{A}_n$,

$$\begin{aligned}&\left\|\mathbb{E}_n[\widehat{\mathbf{b}}_{p,0}(x_i)\widehat{\mathbf{b}}_{p,0}(x_i)'\Psi_{i,1}\eta_{i,1}^2]-\mathbb{E}_{\widehat{\Delta}}[\widehat{\mathbf{b}}_{p,0}(x_i)\widehat{\mathbf{b}}_{p,0}(x_i)'\Psi_{i,1}\eta_{i,1}^2]\right\|\\ \leq&\sup_{\Delta\in\Pi}\|\mathbb{E}_n[\mathbf{b}_{p,0}(x_i;\Delta)\mathbf{b}_{p,0}(x_i;\Delta)'\Psi_{i,1}\eta_i^2]-\mathbb{E}[\mathbf{b}_{p,0}(x_i;\Delta)\mathbf{b}_{p,0}(x_i;\Delta)'\Psi_{i,1}\eta_i^2]\|_\infty.\end{aligned}$$

Let a_{kl} be a generic (k,l)th entry of the matrix inside the norm, i.e.,

$$|a_{kl}|=\Big|\mathbb{E}_n[b_{p,0,k}(x_i;\Delta)b_{p,0,l}(x_i;\Delta)'\Psi_{i,1}\eta_{i,1}^2]-\mathbb{E}[b_{p,0,k}(x_i;\Delta)b_{p,0,l}(x_i;\Delta)'\Psi_{i,1}\eta_{i,1}^2]\Big|.$$

Clearly, if $b_{p,0,k}(\cdot\,;\Delta)$ and $b_{p,0,l}(\cdot\,;\Delta)$ are basis functions with different supports, a_{kl} is zero. Now define the following function class

$$\mathcal{G}=\Big\{(x_1,\mathbf{w}_1)\mapsto b_{p,0,k}(x_1;\Delta)b_{p,0,l}(x_1;\Delta)\Psi_i\eta_{i,1}^2:1\leq k,l\leq J(p+1),\Delta\in\Pi\Big\}.$$

We have $\sup_{g\in\mathcal{G}}|g|_\infty\lesssim J$ and $\sup_{g\in\mathcal{G}}\mathbb{V}[g]\leq\sup_{g\in\mathcal{G}}\mathbb{E}[g^2]\lesssim J$, by Assumption SA-SM. Also, by Proposition 3.6.12 of Giné and Nickl (2016), the collection $\mathcal{G}$ is of VC type with a bounded index. Then, by Lemma SA-5.6,

$$\sup_{g\in\mathcal{G}}\Big|\frac{1}{n}\sum_{i=1}^n g(x_i)-\mathbb{E}[g(x_i)]\Big|\lesssim_{\mathbb{P}}\sqrt{J\log J/n},$$

which implies $\|\mathbb{E}_n[\widehat{\mathbf{b}}_{p,0}(x_i)\widehat{\mathbf{b}}_{p,0}(x_i)'\Psi_{i,1}\eta_{i,1}^2]-\mathbb{E}_{\widehat{\Delta}}[\widehat{\mathbf{b}}_{p,0}(x_i)\widehat{\mathbf{b}}_{p,0}(x_i)'\Psi_{i,1}\eta_{i,1}^2]\|\lesssim_{\mathbb{P}}\sqrt{J\log J/n}$.

Then, the lower bound on the minimum eigenvalue of $\bar{\mathbf{Q}}$ follows by Theorem 4.42 of Schumaker (2007) and Assumption SA-RP(i). The upper bound immediately follows by Assumption SA-RP(i) and Lemmas SA-5.3 and SA-5.4.

Given the above fact, it follows that $\|\bar{\mathbf{Q}}^{-1}\| \lesssim_{\mathbb{P}} 1$. Notice that $\bar{\mathbf{Q}}$ is a banded matrix with a finite band width. Then, the bounds on the elements of $\bar{\mathbf{Q}}^{-1}$ and $\|\bar{\mathbf{Q}}^{-1}\|_\infty$ hold by Theorem 2.2 of Demko (1977).

(ii) By Assumption SA-DGP and SA-SM, $\Psi_{i,1}\eta_{i,1}^2$ is bounded and bounded away from zero uniformly over $1 \leq i \leq n$. Then, $\mathbb{E}[\mathbf{b}_{p,s}(x_i)\mathbf{b}_{p,s}(x_i)'] \lesssim \mathbf{Q}_0 \lesssim \mathbb{E}[\mathbf{b}_{p,s}(x_i)\mathbf{b}_{p,s}(x_i)']$. The desired bounds on the minimum and maximum eigenvalues of $\mathbf{Q}_0$ follow from Lemma SA-3.5 of Cattaneo et al. (2024b).

Next, we show the convergence of $\bar{\mathbf{Q}}$ to $\mathbf{Q}_0$. Let α_{kl} be a generic (k,l)th entry of

$$\mathbb{E}_{\widehat{\Delta}}[\widehat{\mathbf{b}}_{p,0}(x_i)\widehat{\mathbf{b}}_{p,0}(x_i)'\Psi_{i,1}\eta_{i,1}^2]/J - \mathbb{E}[\mathbf{b}_{p,0}(x_i)\mathbf{b}_{p,0}(x_i)'\Psi_{i,1}\eta_{i,1}^2]/J.$$

By definition, it is either equal to zero or

$$\begin{aligned}
\alpha_{kl} =& \int_{\widehat{\mathcal{B}}_j}\Big(\frac{x-\hat{\tau}_j}{\hat{h}_j}\Big)^\ell \varphi(x_i)f_X(x)dx - \int_{\mathcal{B}_j}\Big(\frac{x-\tau_j}{h_j}\Big)^\ell \varphi(x_i)f_X(x)dx\\
=&\hat{h}_j\int_0^1 z^\ell\varphi(z\hat{h}_j+\hat{\tau}_j)f_X(z\hat{h}_j+\hat{\tau}_j)dz - h_j\int_0^1 z^\ell\varphi(zh_j+\tau_j)f_X(zh_j+\tau_j)dz\\
=&(\hat{h}_j-h_j)\int_0^1 z^\ell\varphi(z\hat{h}_j+\hat{\tau}_j)f_X(z\hat{h}_j+\hat{\tau}_j)dz\\
&+h_j\int_0^1 z^\ell\Big(\varphi(z\hat{h}_j+\hat{\tau}_j)f_X(z\hat{h}_j+\hat{\tau}_j)-\varphi(zh_j+\tau_j)f_X(zh_j+\tau_j)\Big)dz
\end{aligned}$$

for some $1 \leq j \leq J$ and $0 \leq \ell \leq 2p$ and $\varphi(x_i) = \mathbb{E}[\varkappa(x_i,\mathbf{w}_i)|x_i]$. By Assumptions SA-DGP and SA-SM and the argument in the proof of Lemma SA-3.5 of Cattaneo et al. (2024b),

$$\|\mathbb{E}_{\widehat{\Delta}}[\widehat{\mathbf{b}}_{p,0}(x_i)\widehat{\mathbf{b}}_{p,0}(x_i)'\Psi_{i,1}\eta_{i,1}^2] - \mathbf{Q}_0\| \lesssim_{\mathbb{P}} \mathfrak{r}_{\mathtt{RP}}.$$

Since $\bar{\mathbf{Q}}$ and $\mathbf{Q}_0$ are banded matrices with finite band widths. Then, the bound $\|\bar{\mathbf{Q}}^{-1}-\mathbf{Q}_0^{-1}\|_\infty$ hold by Theorem 2.2 of Demko (1977). This completes the proof. □

SA-5.3 Proof of Lemma SA-3.2

Proof. Since $\mathbb{E}[\psi(y_i,\eta_i)^2|x_i=x,\mathbf{w}_i=\mathbf{w}]$ and $(\eta^{(1)}(\mu_0(x)+\mathbf{w}'\boldsymbol{\gamma}_0))^2$ is bounded and bounded away from zero uniformly over $x\in\mathcal{X}$ and $\mathbf{w}\in\mathcal{W}$, $\mathbb{E}_n[\widehat{\mathbf{b}}_{p,s}(x_i)\widehat{\mathbf{b}}_{p,s}(x_i)'] \lesssim \bar{\boldsymbol{\Sigma}} \lesssim \mathbb{E}_n[\widehat{\mathbf{b}}_{p,s}(x_i)\widehat{\mathbf{b}}_{p,s}(x_i)']$. By the same argument in the proof of Lemma SA-3.1 (we can simply drop the additional term $\Psi_{i,1}\eta_{i,1}^2$ in $\bar{\mathbf{Q}}$), the eigenvalues of $\mathbb{E}_n[\widehat{\mathbf{b}}_{p,s}(x_i)\widehat{\mathbf{b}}_{p,s}(x_i)']$ and thus $\bar{\mathbf{\Sigma}}$ are bounded and bounded away from zero. Then, the desired results follow from Lemma SA-3.1 and the fact that $\inf_{x\in\mathcal{X}}\|\widehat{\mathbf{b}}^{(v)}_{p,s}(x)\| \gtrsim J^{1/2+v}$ w.p.a. 1 (it was shown in the proof of Lemma SA-3.6 of Cattaneo et al. (2024b)). $\square$

SA-5.4 Proof of Lemma SA-3.3

Proof. By Lemmas SA-5.3, SA-5.4 and SA-3.1, $\sup_{x\in\mathcal{X}}\|\widehat{\mathbf{b}}^{(v)}_{p,s}(x)\|_1 \lesssim_{\mathbb{P}} J^{1/2+v}$, $\|\bar{\mathbf{Q}}^{-1}\|_\infty \lesssim_{\mathbb{P}} 1$ and $\|\widehat{\mathbf{T}}_s\|_\infty \lesssim_{\mathbb{P}} 1$. Recall that by Assumption SA-SM, $\psi(y_i,\eta_i) = \psi^\dagger(y_i-\eta_i)\psi^\ddagger(\eta_i) = \psi^\dagger(\epsilon_i)\psi^\ddagger(\eta_i)$. Define the following function class

$$\mathcal{G} = \Big\{(x_1,\mathbf{w}_1,\epsilon_1)\mapsto b_{p,0,l}(x_1;\Delta)\eta^{(1)}(\mu_0(x_1)+\mathbf{w}_1'\boldsymbol{\gamma}_0)\psi^\dagger(\epsilon_1)\psi^\ddagger(\eta_1) : 1\le l\le J(p+1), \Delta\in\Pi\Big\}.$$

Then, $\sup_{g\in\mathcal{G}}|g| \lesssim \sqrt{J}|\psi^\dagger(\epsilon_1)|$, and hence take an envelop $\bar{G} = C\sqrt{J}|\psi^\dagger(\epsilon_1)|$ for some C large enough. Moreover, $\sup_{g\in\mathcal{G}}\mathbb{V}[g]\lesssim 1$ and $\mathcal{G}$ is of VC type with a bounded index. By Proposition 6.1 of Belloni et al. (2015),

$$\sup_{g\in\mathcal{G}}\Big|\frac{1}{n}\sum_{i=1}^n g(x_i,\epsilon_i)\Big| \lesssim_{\mathbb{P}} \sqrt{\frac{\log J}{n}} + \frac{J^{\frac{\nu}{2(\nu-2)}}\log J}{n} \lesssim \sqrt{\frac{\log J}{n}},$$

and the desired result follows. $\square$

SA-5.5 Proof of Lemma SA-3.4

Proof. Let $\hat{z}_i = \widehat{\mathbf{b}}_{p,s}(x_i)'\widehat{\boldsymbol{\beta}}_0+\mathbf{w}_i'\boldsymbol{\gamma}_0$ and $\mathfrak{r}(x_i,\mathbf{w}_i,y_i) := \mathfrak{r}(x_i,\mathbf{w}_i,y_i;\widehat{\Delta}) := \eta_{i,1}\psi(y_i,\eta_i)-\eta^{(1)}(\hat{z}_i)\psi(y_i,\eta(\hat{z}_i))$ $= A_1(x_i,\mathbf{w}_i,y_i)+A_2(x_i,\mathbf{w}_i,y_i)$ where

$$\begin{aligned}
A_1(x_i,\mathbf{w}_i,y_i) &:= A_1(x_i,\mathbf{w}_i,y_i;\widehat{\Delta}) := [\eta_{i,1}\psi^\ddagger(\eta_i)-\eta^{(1)}(\hat{z}_i)\psi^\ddagger(\eta(\hat{z}_i))]\psi^\dagger(y_i,\eta_i) \ \text{ and}\\
A_2(x_i,\mathbf{w}_i,y_i) &:= A_2(x_i,\mathbf{w}_i,y_i;\widehat{\Delta}) := \eta^{(1)}(\hat{z}_i)\psi^\ddagger(\eta(\hat{z}_i))[\psi^\dagger(y_i,\eta_i)-\psi^\dagger(y_i,\eta(\hat{z}_i))].
\end{aligned}$$

First, by Assumption SA-SM and Lemma SA-5.5, $\sup_{x\in\mathcal{X},\mathbf{w}\in\mathcal{W}}|\eta_{i,1}\psi^\ddagger(\eta_i)-\eta^{(1)}(\hat{z}_i)\psi^\ddagger(\eta(\hat{z}_i))| \lesssim$

J^{-p-1} w.p.a. 1. Also, for every $1 \le l \le K_{p,s}$ and $\Delta \in \Pi$,

$$\begin{aligned}
&b_{p,s,l}(x;\Delta)\Big(\eta_{i,1}\psi^{\ddagger}(\eta_i) - \eta^{(1)}(\mathbf{b}_{p,s}(x;\Delta)'\boldsymbol{\beta}_0(\Delta) + \mathbf{w}'\boldsymbol{\gamma}_0)\psi^{\ddagger}(\mathbf{b}_{p,s}(x;\Delta)'\boldsymbol{\beta}_0(\Delta) + \mathbf{w}'\boldsymbol{\gamma}_0)\Big) \\
&= b_{p,s,l}(x;\Delta)\eta_{i,1}\psi^{\ddagger}(\eta_i) - \\
&\quad b_{p,s,l}(x;\Delta)\eta^{(1)}\Big(\sum_{k=\underline{k}_l}^{\underline{k}_l+p} b_{p,s,k}(x;\Delta)\beta_{0,k}(\Delta) + \mathbf{w}'\boldsymbol{\gamma}_0\Big)\psi^{\ddagger}\Big(\sum_{k=\underline{k}_l}^{\underline{k}_l+p} b_{p,s,k}(x;\Delta)\beta_{0,k}(\Delta) + \mathbf{w}'\boldsymbol{\gamma}_0\Big)
\end{aligned}$$

for some integer $\underline{k}_l \in [1, K_{p,s}]$ where $\beta_{0,k}(\Delta)$ denotes the kth element in $\boldsymbol{\beta}_0(\Delta)$. Then, the function class $\mathcal{G} = \{(x, \mathbf{w}, y) \mapsto b_{p,s,l}(x;\Delta)A_1(x, \mathbf{w}, y;\Delta) : 1 \le l \le K_{p,s}, \Delta \in \Pi\}$ is of VC type with a bounded index. By the same argument given in the proof of Lemma SA-3.3,

$$\|\mathbb{E}_n[\widehat{\mathbf{b}}_{p,s}(x_i)A_1(x_i, \mathbf{w}_i, y_i)]\|_\infty \lesssim_{\mathbb{P}} J^{-p-1}\Big(\frac{\log J}{n}\Big)^{1/2}.$$

Next, let $\mathscr{F}_{XW\Delta}$ be the σ-field generated by $\{(x_i, \mathbf{w}_i)\}_{i=1}^n$ and $\widehat{\Delta}$. Note that

$$\begin{aligned}
\mathbb{E}_n[\widehat{\mathbf{b}}_{p,s}(x_i)A_2(x_i, \mathbf{w}_i, y_i)] = \mathbb{E}_n[\mathbb{E}[\widehat{\mathbf{b}}_{p,s}(x_i)A_2(x_i, \mathbf{w}_i, y_i)|\mathscr{F}_{XW\Delta}]] + \\
\mathbb{E}_n\Big[\widehat{\mathbf{b}}_{p,s}(x_i)A_2(x_i, \mathbf{w}_i, y_i) - \mathbb{E}[\widehat{\mathbf{b}}_{p,s}(x_i)A_2(x_i, \mathbf{w}_i, y_i)|\mathscr{F}_{XW\Delta}]\Big].
\end{aligned}$$

By Assumption SA-SM(iii) and Lemma SA-5.5,

$$\begin{aligned}
&\max_{1\le i\le n} |\mathbb{E}[A_2(x_i, \mathbf{w}_i, y_i)|\mathscr{F}_{XW\Delta}]| \\
&= \max_{1\le i\le n} |\eta^{(1)}(\widehat{\mathbf{b}}_{p,s}(x_i)'\widehat{\boldsymbol{\beta}}_0 + \mathbf{w}_i'\gamma_0)\Psi(x_i, \mathbf{w}_i; \eta(\hat{z}_i))| \lesssim_{\mathbb{P}} J^{-p-1}.
\end{aligned}$$

Then, $\|\mathbb{E}_n[\mathbb{E}[\widehat{\mathbf{b}}_{p,s}(x_i)A_2(x_i, \mathbf{w}_i, y_i)|\mathscr{F}_{XW\Delta}]]\|_\infty \lesssim_{\mathbb{P}} J^{-p-1-1/2}$ by the same argument in the proof of Lemma SA-3.1. On the other hand, define the following function class

$$\mathcal{G} := \Big\{(x, \mathbf{w}, y) \mapsto b_{p,s,l}(x;\Delta)A_2(x, \mathbf{w}, y;\Delta) : 1 \le l \le K_{p,s}, \Delta \in \Pi\Big\}.$$

By Assumption SA-SM, $\sup_{g\in\mathcal{G}} \|g\|_\infty \lesssim J^{1/2}$, and $\sup_{g\in\mathcal{G}} \mathbb{V}[g(x_i, \mathbf{w}_i, y_i)] \lesssim J^{-p-1}$. By a similar argument given before, this function class is of VC type with a bounded index. Then, as in the proof of Lemma SA-3.3, by Proposition 6.1 of Belloni et al. (2019),

$$\sup_{g\in\mathcal{G}}\Big|\frac{1}{n}\sum_{i=1}^{n}(g(x_i,\mathbf{w}_i,y_i)-\mathbb{E}[g(x_i,\mathbf{w}_i,y_i)])\Big|\lesssim_{\mathbb{P}} J^{-\frac{p+1}{2}}\sqrt{\frac{\log J}{n}}+\frac{J^{1/2}\log J}{n}.$$

Collecting these results, we conclude that

$$\widehat{\mathbf{b}}_{p,s}^{(v)}(x)'\bar{\mathbf{Q}}^{-1}\mathbb{E}[\widehat{\mathbf{b}}_{p,s}(x_i)\mathfrak{r}(x_i,\mathbf{w}_i,y_i)]\lesssim_{\mathbb{P}} J^{-p-1+v}+J^{\frac{2v-p-1}{2}}\Big(\frac{J\log J}{n}\Big)^{1/2}+\frac{J^{1+v}\log J}{n}.$$

The proof is complete. □

SA-5.6 Proof of Lemma SA-3.5

Proof. By convexity of $\rho(y;\eta(\cdot))$, we only need to consider $\boldsymbol{\beta}=\widehat{\boldsymbol{\beta}}_0+\varepsilon\boldsymbol{\alpha}/\sqrt{J}$ for any sufficiently small fixed $\varepsilon>0$ and $\boldsymbol{\alpha}\in\mathbb{R}^{K_{p,s}}$ such that $\|\boldsymbol{\alpha}\|=1$. For notational simplicity, let $\widehat{\mathbf{b}}_i:=\widehat{\mathbf{b}}_{p,s}(x_i)$. For this choice of $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}\in\mathbb{R}^d$,

$$\begin{aligned}\delta_i(\boldsymbol{\beta},\boldsymbol{\gamma})&=\rho(y_i;\eta(\widehat{\mathbf{b}}_i'\boldsymbol{\beta}+\mathbf{w}_i'\boldsymbol{\gamma}))-\rho(y_i;\eta(\widehat{\mathbf{b}}_i'\widehat{\boldsymbol{\beta}}_0+\mathbf{w}_i'\boldsymbol{\gamma}))\\&=\int_0^{\varepsilon\widehat{\mathbf{b}}_i'\boldsymbol{\alpha}/\sqrt{J}}\psi\Big(y_i,\eta(\widehat{\mathbf{b}}_i'\widehat{\boldsymbol{\beta}}_0+\mathbf{w}_i'\boldsymbol{\gamma}+t)\Big)\eta^{(1)}(\widehat{\mathbf{b}}_i'\widehat{\boldsymbol{\beta}}_0+\mathbf{w}_i'\boldsymbol{\gamma}+t)dt.\end{aligned}$$

Let $\mathscr{F}_{XW\Delta}$ be the σ-field generated by $\{(x_i,\mathbf{w}_i)\}_{i=1}^n$ and $\widehat{\Delta}$. We have

$$\mathbb{E}_n[\delta_i(\boldsymbol{\beta},\widehat{\boldsymbol{\gamma}})]=\frac{1}{\sqrt{n}}\mathbb{G}_n[\delta_i(\boldsymbol{\beta},\widehat{\boldsymbol{\gamma}})]+\mathbb{E}_n\Big[\mathbb{E}[\delta_i(\boldsymbol{\beta},\widehat{\boldsymbol{\gamma}})|\mathscr{F}_{XW\Delta}]\Big],$$

where $\mathbb{G}_n[\cdot]$ denotes $\sqrt{n}(\mathbb{E}_n[\cdot]-\mathbb{E}[\cdot|\mathscr{F}_{XW\Delta}])$, and $\mathbb{E}[\delta_i(\boldsymbol{\beta},\widehat{\boldsymbol{\gamma}})|\mathscr{F}_{XW\Delta}]:=\mathbb{E}[\delta_i(\boldsymbol{\beta},\boldsymbol{\gamma})|\mathscr{F}_{XW\Delta}]|_{\boldsymbol{\gamma}=\widehat{\boldsymbol{\gamma}}}$, i.e., the conditional expectation with $\widehat{\boldsymbol{\gamma}}$ viewed as fixed. By Assumption SA-SM,

$$\begin{aligned}\mathbb{E}[\delta_i(\boldsymbol{\beta},\widehat{\boldsymbol{\gamma}})|\mathscr{F}_{XW\Delta}]&=\int_0^{\varepsilon\widehat{\mathbf{b}}_i'\boldsymbol{\alpha}/\sqrt{J}}\Psi\Big(x_i,\mathbf{w}_i;\eta(\widehat{\mathbf{b}}_i'\widehat{\boldsymbol{\beta}}_0+\mathbf{w}_i'\widehat{\boldsymbol{\gamma}}+t)\Big)\eta^{(1)}(\widehat{\mathbf{b}}_i'\widehat{\boldsymbol{\beta}}_0+\mathbf{w}_i'\widehat{\boldsymbol{\gamma}}+t)dt\\=\int_0^{\varepsilon\widehat{\mathbf{b}}_i'\boldsymbol{\alpha}/\sqrt{J}}&\Psi_1(x_i,\mathbf{w}_i;\xi_{i,t})(\eta(\widehat{\mathbf{b}}_i'\widehat{\boldsymbol{\beta}}_0+\mathbf{w}_i'\widehat{\boldsymbol{\gamma}}+t)-\eta_i)\eta^{(1)}(\widehat{\mathbf{b}}_i'\widehat{\boldsymbol{\beta}}_0+\mathbf{w}_i'\widehat{\boldsymbol{\gamma}}+t)dt,\end{aligned}$$

where $\xi_{i,t}$ is between $\eta(\widehat{\mathbf{b}}_i'\widehat{\boldsymbol{\beta}}_0+\mathbf{w}_i'\widehat{\boldsymbol{\gamma}}+t)$ and $\eta(\mu_0(x_i)+\mathbf{w}_i'\boldsymbol{\gamma}_0)$ and we use the fact that $\Psi(x,\mathbf{w}_i;\eta_i)=0$. By Lemma SA-5.5, the fact that $\eta(\cdot)$ is strictly monotonic and $\widehat{\boldsymbol{\gamma}}-\boldsymbol{\gamma}_0=o_{\mathbb{P}}(\sqrt{J/n}+J^{-p-1})$ and the rate condition imposed, we have $\mathbb{E}_n[\mathbb{E}[\delta_i(\boldsymbol{\beta},\widehat{\boldsymbol{\gamma}})|\mathscr{F}_{XW\Delta}]]\gtrsim_{\mathbb{P}}\varepsilon^2\boldsymbol{\alpha}'\mathbb{E}_n[\widehat{\mathbf{b}}_i\widehat{\mathbf{b}}_i']\boldsymbol{\alpha}/J\gtrsim_{\mathbb{P}} J^{-1}\varepsilon^2$.

On the other hand, let $\mathcal{H} := \{\boldsymbol{\gamma} : \|\boldsymbol{\gamma} - \boldsymbol{\gamma}_0\| \leq C\mathfrak{r}_\gamma\}$ and define the following function class

$$\mathcal{G} := \Big\{(x_i, \mathbf{w}_i, y_i) \mapsto \delta_i(\boldsymbol{\beta}, \boldsymbol{\gamma}) : \boldsymbol{\alpha} \in \mathcal{S}^{K_{p,s}}, \boldsymbol{\gamma} \in \mathcal{H}\Big\}.$$

Note that

$$\begin{aligned}\delta_i(\boldsymbol{\beta}, \boldsymbol{\gamma}) = &\int_0^{\varepsilon\widehat{\mathbf{b}}_i'\boldsymbol{\alpha}/\sqrt{J}} \Big(\psi(y_i, \eta(\widehat{\mathbf{b}}_i'\widehat{\boldsymbol{\beta}}_0 + \mathbf{w}_i'\boldsymbol{\gamma} + t)) - \psi(y_i, \eta_i)\Big)\eta^{(1)}(\widehat{\mathbf{b}}_i'\widehat{\boldsymbol{\beta}}_0 + \mathbf{w}_i'\boldsymbol{\gamma} + t)dt + \\ &\int_0^{\varepsilon\widehat{\mathbf{b}}_i'\boldsymbol{\alpha}/\sqrt{J}} \psi(y_i, \eta_i)\eta^{(1)}(\widehat{\mathbf{b}}_i'\widehat{\boldsymbol{\beta}}_0 + \mathbf{w}_i'\boldsymbol{\gamma} + t)dt.\end{aligned}$$

By Assumption SA-SM, we have $\sup_{g\in\mathcal{G}} |g| \lesssim \varepsilon(1 + |\psi(y_i, \eta_i)|)$, $\| \max_{1\leq i\leq n} |\psi(y_i, \eta_i)| \|_{L_2(\mathbb{P})} \lesssim n^{1/\nu}$, $\sup_{g\in\mathcal{G}} \mathbb{E}_n[\mathbb{E}[g^2|\mathscr{F}_{XW\Delta}]] \lesssim_{\mathbb{P}} J^{-1}\varepsilon^2$, and the VC-index of $\mathcal{G}$ is bounded by $C'K_{p,s}$ for an absolute constant $C' > 0$. Therefore, by Lemma SA-5.6 and the rate restriction,

$$\sup_{g\in\mathcal{G}} \Big|\frac{1}{\sqrt{n}}\mathbb{G}_n[\delta_i(\boldsymbol{\beta}, \boldsymbol{\gamma})]\Big| \lesssim_{\mathbb{P}} J^{-1}\Big(\frac{J^2 \log J}{n}\Big)^{1/2}\varepsilon + J^{-1}\frac{J^2 \log J}{n^{1-\frac{1}{\nu}}}\varepsilon = o(\varepsilon/J).$$

Thus, for any fixed (sufficiently small) $\varepsilon > 0$, $\mathbb{E}_n[\delta_i(\boldsymbol{\beta}, \widehat{\boldsymbol{\gamma}})] > 0$ when n is sufficiently large. Thus, $\|\widehat{\boldsymbol{\beta}} - \widehat{\boldsymbol{\beta}}_0\| = o_{\mathbb{P}}(J^{-1/2})$, implying $\|\widehat{\boldsymbol{\beta}} - \widehat{\boldsymbol{\beta}}_0\|_\infty = o_{\mathbb{P}}(J^{-1/2})$ immediately. □

SA-5.7 Proof of Theorem SA-3.1

Proof. The proof is long. We divide it into several steps.

Step 0: We first prepare some notation and useful facts. To simplify the presentation, in this proof we drop the scaling factor $\sqrt{J}$ in the basis by defining

$$\breve{\mathbf{b}}_i := \widehat{\mathbf{b}}_{p,s}(x_i)/\sqrt{J} = (\widehat{b}_{p,s,1}(x_i), \cdots, \widehat{b}_{p,s,K_{p,s}}(x_i))'/\sqrt{J} \quad \text{and} \quad \breve{\boldsymbol{\beta}}_0 = \sqrt{J}\widehat{\boldsymbol{\beta}}_0.$$

Throughout the proof, $C, c, C_1, c_1, C_2, c_2, \cdots$ denote (strictly positive) absolute constants, $\mathscr{F}_{XW\Delta}$ denotes the σ-field generated by $\{(x_i, \mathbf{w}_i)\}_{i=1}^n$ and $\widehat{\Delta}$, and $\text{supp}(g(\cdot))$ denotes the support of a generic function $g(\cdot)$. Moreover, define

$$\begin{aligned}&\mathcal{V} = \{(v_1, \cdots, v_{K_{p,s}})' : \exists k \in \{1, \cdots, K_{p,s}\}, |v_\ell| \leq \varrho^{|k-\ell|}\varepsilon_n \text{ for } |\ell - k| \leq M_n \text{ and } v_\ell = 0 \text{ otherwise}\},\\ &\mathcal{H}_l = \{\mathbf{v} \in \mathbb{R}^{K_{p,s}} : \|\mathbf{v}\|_\infty \leq r_{l,n}\} \text{ for } l = 1, 2, \quad \text{and} \quad \mathcal{H}_3 = \{\mathbf{v} \in \mathbb{R}^d : \|\mathbf{v}\| \leq r_{3,n}\},\end{aligned}$$

where $\varrho \in (0,1)$ is the constant given in Lemma SA-3.1, $r_{1,n} = C_1[(J\log n/n)^{1/2} + J^{-p-1}]$, $r_{2,n} = \mathfrak{z}\mathfrak{r}_{2,n}$ for $\mathfrak{z} > 0$, $\varepsilon_n = \mathfrak{z}'\mathfrak{r}_{2,n}$ for $\mathfrak{z}' > 0$, $\mathfrak{r}_{2,n} = [(\frac{J\log n}{n})^{3/4}\log n + J^{-\frac{p+1}{2}}\sqrt{\frac{J}{n}}\log n + J^{-2p-2} + \mathfrak{r}_\gamma]$, $r_{3,n} = C\mathfrak{r}_\gamma$, and $M_n = c_1 \log n$. In the last step of the proof, we will consider $\mathfrak{z} = 2^\ell$, $\ell = L, L+1, \cdots, \bar{L}$ where $\bar{L}$ is the smallest number such that $2^{\bar{L}} r_{2n} \geq c$ for some sufficiently small constant $c > 0$, and ε_n is a quantity that we can choose. By Assumption SA-HLE, $\widehat{\boldsymbol{\gamma}} - \boldsymbol{\gamma}_0 \in \mathcal{H}_3$ with probability approaching one for C large enough, and by Lemma SA-3.5, $\sqrt{J}\widehat{\boldsymbol{\beta}} - \breve{\boldsymbol{\beta}}_0 \leq c$ with probability approaching one.

For any $\boldsymbol{\beta}_1 \in \mathcal{H}_1, \boldsymbol{\beta}_2 \in \mathcal{H}_2$, $\boldsymbol{\upsilon} \in \mathcal{V}$ and $\boldsymbol{\gamma} := \boldsymbol{\gamma}_0 + \boldsymbol{\gamma}_1$ with $\boldsymbol{\gamma}_1 \in \mathcal{H}_3$, define

$$
\begin{aligned}
\delta_i(\boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \boldsymbol{\upsilon}, \boldsymbol{\gamma}) &= \rho\Big(y_i; \eta(\breve{\mathbf{b}}_i'(\breve{\boldsymbol{\beta}}_0 + \boldsymbol{\beta}_1 + \boldsymbol{\beta}_2) + \mathbf{w}_i'\boldsymbol{\gamma})\Big) - \rho\Big(y_i; \eta(\breve{\mathbf{b}}_i'(\breve{\boldsymbol{\beta}}_0 + \boldsymbol{\beta}_1 + \boldsymbol{\beta}_2 - \boldsymbol{\upsilon}) + \mathbf{w}_i'\boldsymbol{\gamma})\Big) \\
&\quad - \Big[\eta(\breve{\mathbf{b}}_i'(\breve{\boldsymbol{\beta}}_0 + \boldsymbol{\beta}_1 + \boldsymbol{\beta}_2) + \mathbf{w}_i'\boldsymbol{\gamma}) - \eta(\breve{\mathbf{b}}_i'(\breve{\boldsymbol{\beta}}_0 + \boldsymbol{\beta}_1 + \boldsymbol{\beta}_2 - \boldsymbol{\upsilon}) + \mathbf{w}_i'\boldsymbol{\gamma})\Big] \\
&\qquad \times \psi(y_i, \eta(\breve{\mathbf{b}}_i'\breve{\boldsymbol{\beta}}_0 + \mathbf{w}_i'\boldsymbol{\gamma}_0)) \\
&= \int_{-\breve{\mathbf{b}}_i'\boldsymbol{\upsilon}}^0 \Big[\psi\Big(y_i, \eta(\breve{\mathbf{b}}_i'(\breve{\boldsymbol{\beta}}_0 + \boldsymbol{\beta}_1 + \boldsymbol{\beta}_2) + \mathbf{w}_i'\boldsymbol{\gamma} + t)\Big) - \psi\Big(y_i, \eta(\breve{\mathbf{b}}_i'\breve{\boldsymbol{\beta}}_0 + \mathbf{w}_i'\boldsymbol{\gamma}_0)\Big)\Big] \\
&\qquad \times \eta^{(1)}\Big(\breve{\mathbf{b}}_i'(\breve{\boldsymbol{\beta}}_0 + \boldsymbol{\beta}_1 + \boldsymbol{\beta}_2) + \mathbf{w}_i'\boldsymbol{\gamma} + t\Big) dt.
\end{aligned}
$$

Note that $\delta_i(\boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \boldsymbol{\upsilon}, \boldsymbol{\gamma}) \neq 0$ only if $\breve{\mathbf{b}}_i'\boldsymbol{\upsilon} \neq 0$. For each $\boldsymbol{\upsilon} \in \mathcal{V}$, let $\mathcal{J}_{\boldsymbol{\upsilon}} = \{j : \upsilon_j \neq 0\}$. By construction, the cardinality of $\mathcal{J}_{\boldsymbol{\upsilon}}$ is bounded by $2M_n + 1$. We have $\delta_i(\boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \boldsymbol{\upsilon}, \boldsymbol{\gamma}) \neq 0$ only if $\breve{b}_j(x_i) \neq 0$ for some $j \in \mathcal{J}_\upsilon$, which happens only when $x_i \in \text{supp}(\breve{b}_j(\cdot))$ for some $j \in \mathcal{J}_{\boldsymbol{\upsilon}}$. Let $\mathcal{I}_{\boldsymbol{\upsilon}} = \cup_{j \in \mathcal{J}_{\boldsymbol{\upsilon}}} \text{supp}(\breve{b}_j(\cdot))$. Since the basis functions are locally supported, $\mathcal{I}_{\boldsymbol{\upsilon}}$ includes at most $c_2 M_n$ (connected) intervals for all $\boldsymbol{\upsilon} \in \mathcal{V}$. Moreover, at most $c_3 M_n$ basis functions in $\breve{\mathbf{b}}(\cdot)$ have supports overlapping with $\mathcal{I}_{\boldsymbol{\upsilon}}$. Denote the set of indices for such basis functions by $\bar{\mathcal{J}}_{\boldsymbol{\upsilon}}$. Let $\breve{\beta}_{0,j}$, $\beta_{1,j}$ and $\beta_{2,j}$ be the jth entries of $\breve{\boldsymbol{\beta}}_0$, $\boldsymbol{\beta}_1$, and $\boldsymbol{\beta}_2$ respectively, and υ_j be the jth entry of $\boldsymbol{\upsilon}$. Based on the above observations, we have $\delta_i(\boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \boldsymbol{\upsilon}, \boldsymbol{\gamma}) \equiv \delta_i(\boldsymbol{\beta}_{1,\bar{\mathcal{J}}_{\boldsymbol{\upsilon}}}, \boldsymbol{\beta}_{2,\bar{\mathcal{J}}_{\boldsymbol{\upsilon}}}, \boldsymbol{\upsilon}, \boldsymbol{\gamma})$ where

$$
\begin{aligned}
\delta_i(\boldsymbol{\beta}_{1,\bar{\mathcal{J}}_{\boldsymbol{\upsilon}}}, \boldsymbol{\beta}_{2,\bar{\mathcal{J}}_{\boldsymbol{\upsilon}}}, \boldsymbol{\upsilon}, \boldsymbol{\gamma}) &:= \int_{-\sum\limits_{j \in \mathcal{J}_{\boldsymbol{\upsilon}}} \breve{b}_{i,j}\upsilon_j}^0 \Big[\psi\Big(y_i, \eta\Big(\sum_{l \in \bar{\mathcal{J}}_{\boldsymbol{\upsilon}}} \breve{b}_{i,l}(\breve{\beta}_{0,l} + \beta_{1,l} + \beta_{2,l}) + \mathbf{w}_i'\boldsymbol{\gamma} + t\Big)\Big) \\
&\quad -\psi\Big(y_i, \eta\Big(\sum_{l \in \bar{\mathcal{J}}_{\boldsymbol{\upsilon}}} \breve{b}_{i,l}\breve{\beta}_{0,l} + \mathbf{w}_i'\boldsymbol{\gamma}_0\Big)\Big)\Big] \times \eta^{(1)}\Big(\sum_{l \in \bar{\mathcal{J}}_{\boldsymbol{\upsilon}}} \breve{b}_{i,l}(\breve{\beta}_{0,l} + \beta_{1,l} + \beta_{2,l}) + \mathbf{w}_i'\boldsymbol{\gamma} + t\Big) dt \mathbb{1}_{i,\boldsymbol{\upsilon}},
\end{aligned}
$$

$\mathbb{1}_{i,\boldsymbol{\upsilon}} = \mathbb{1}(x_i \in \mathcal{I}_{\boldsymbol{\upsilon}})$, and $\boldsymbol{\beta}_{1,\bar{\mathcal{J}}_{\boldsymbol{\upsilon}}}$ and $\boldsymbol{\beta}_{2,\bar{\mathcal{J}}_{\boldsymbol{\upsilon}}}$ respectively denote the subvectors of $\boldsymbol{\beta}_1$ and $\boldsymbol{\beta}_2$ whose indices belong to $\bar{\mathcal{J}}_{\boldsymbol{\upsilon}}$. Accordingly, define the following function class

$$\mathcal{G} = \Big\{(x_i, \mathbf{w}_i, y_i) \mapsto \delta_i(\widetilde{\boldsymbol{\beta}}_1, \widetilde{\boldsymbol{\beta}}_2, \boldsymbol{\upsilon}, \boldsymbol{\gamma}) : \boldsymbol{\upsilon} \in \mathcal{V}, \widetilde{\boldsymbol{\beta}}_1 \in \mathbb{R}^{c_3 M_n}, \widetilde{\boldsymbol{\beta}}_2 \in \mathbb{R}^{c_3 M_n}, \\ \|\widetilde{\boldsymbol{\beta}}_1\|_\infty \leq r_{1,n}, \|\widetilde{\boldsymbol{\beta}}_2\|_\infty \leq r_{2,n}, \boldsymbol{\gamma} - \boldsymbol{\gamma}_0 \in \mathcal{H}_3\Big\}.$$

Step 1: We bound $\sup_{g\in\mathcal{G}} |\mathbb{E}_n[g(x_i, \mathbf{w}_i, y_i)] - \mathbb{E}[g(x_i, \mathbf{w}_i, y_i)|\mathscr{F}_{XW\Delta}]|$ in this step. Let $a_i(t) := \eta(\sum_{l\in\bar{\mathcal{J}}_{\boldsymbol{\upsilon}}} \breve{b}_{i,l}'\breve{\beta}_{0,l} + \mathbf{w}_i'\boldsymbol{\gamma}_0 + t)$. Define

$$\underline{a}_i = \min\Big\{a_i(0), a_i\Big(\sum_{l\in\bar{\mathcal{J}}_{\boldsymbol{\upsilon}}} \breve{b}_{i,l}(\beta_{1,l} + \beta_{2,l}) + \mathbf{w}_i'\boldsymbol{\gamma}_1\Big), a_i\Big(\sum_{l\in\bar{\mathcal{J}}_{\boldsymbol{\upsilon}}} \breve{b}_{i,l}(\beta_{1,l} + \beta_{2,l}) + \mathbf{w}_i'\boldsymbol{\gamma}_1 + \sum_{j\in\mathcal{J}_{\boldsymbol{\upsilon}}} \breve{b}_{i,j}\upsilon_j\Big)\Big\} \text{ and}$$
$$\bar{a}_i = \max\Big\{a_i(0), a_i\Big(\sum_{l\in\bar{\mathcal{J}}_{\boldsymbol{\upsilon}}} \breve{b}_{i,l}(\beta_{1,l} + \beta_{2,l}) + \mathbf{w}_i'\boldsymbol{\gamma}_1\Big), a_i\Big(\sum_{l\in\bar{\mathcal{J}}_{\boldsymbol{\upsilon}}} \breve{b}_{i,l}(\beta_{1,l} + \beta_{2,l}) + \mathbf{w}_i'\boldsymbol{\gamma}_1 + \sum_{j\in\mathcal{J}_{\boldsymbol{\upsilon}}} \breve{b}_{i,j}\upsilon_j\Big)\Big\}.$$

Consider the following two cases.

First, suppose that $(y_i - \bar{a}_i, y_i - \underline{a}_i)$ does not contain any discontinuity points. By Assumption SA-SM, for all t in the interval of integration $[-\sum_{j\in\mathcal{J}_{\boldsymbol{\upsilon}}} \breve{b}_{i,j}\upsilon_j, 0]$ (or $[0, -\sum_{j\in\mathcal{J}_{\boldsymbol{\upsilon}}} \breve{b}_{i,j}\upsilon_j]$),

$$\Big|\psi\Big(y_i, a_i\Big(\sum_{l\in\bar{\mathcal{J}}_{\boldsymbol{\upsilon}}} \breve{b}_{i,l}(\beta_{1,l} + \beta_{2,l}) + \mathbf{w}_i'\boldsymbol{\gamma} + t\Big)\Big) - \psi(y_i, a_i(0))\Big| \lesssim r_{1,n} + r_{2,n} + \varepsilon_n + r_{3,n}.$$

Second, if $(y_i - \bar{a}_i, y_i - \underline{a}_i)$ contains at least one discontinuity point, say $\jmath$. For any t in the interval of integration, by Assumption SA-SM,

$$\Big|\psi\Big(y_i, a_i\Big(\sum_{l\in\bar{\mathcal{J}}_{\boldsymbol{\upsilon}}} \breve{b}_{i,l}(\beta_{1,l}+\beta_{2,l})+\mathbf{w}_i'\boldsymbol{\gamma}+t\Big)\Big)-\psi(y_i, a_i(0))\Big| \lesssim 1+r_{3,n}+(1+|\psi(y_i,\eta_i)|)(r_{1,n}+r_{2,n}+\varepsilon_n+r_{3,n})$$

for any $(x_i, \mathbf{w}_i, y_i)$, and in this case $y_i \in (\jmath + \underline{a}_i, \jmath + \bar{a}_i)$. By Assumption SA-SM,

$$|\bar{a}_i - \underline{a}_i| \lesssim (r_{1,n} + r_{2,n} + r_{3,n} + \varepsilon_n)(|\eta_{i,1}| + r_{1,n} + r_{2,n} + r_{3,n} + \varepsilon_n).$$

By construction, for each $\boldsymbol{\upsilon} \in \mathcal{V}$, there exists some $k_{\boldsymbol{\upsilon}}$ such that $|\upsilon_\ell| \leq \varrho^{|\ell - k_{\boldsymbol{\upsilon}}|}\varepsilon_n$ for $|\ell - k_{\boldsymbol{\upsilon}}| \leq M_n$. Therefore, we can further write $\mathbb{1}_{i,\boldsymbol{\upsilon}} = \sum_{j:\widehat{\mathcal{B}}_j\subset\mathcal{I}_{\boldsymbol{\upsilon}}} \mathbb{1}_{i,\boldsymbol{\upsilon},j}$ where each $\mathbb{1}_{i,\boldsymbol{\upsilon},j}$ is an indicator of the subinterval involved in $\mathcal{I}_{\boldsymbol{\upsilon}}$, and the above facts imply that for any $x_i \in \widehat{\mathcal{B}}_l$ for some $\widehat{\mathcal{B}}_l \subset \mathcal{I}_{\boldsymbol{\upsilon}}$,

$$\mathbb{V}[\delta_i(\boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \boldsymbol{\upsilon}, \boldsymbol{\gamma})|\mathscr{F}_{XW\Delta}] \lesssim \varrho^{2|(p-s+1)l-k_{\boldsymbol{\upsilon}}|}\varepsilon_n^2(r_{1,n}+r_{2,n}+\varepsilon_n+r_{3,n})(|\eta_{i,1}|+r_{1,n}+r_{2,n}+\varepsilon_n+r_{3,n}).$$

In addition, since $\delta_i(\boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \boldsymbol{\upsilon}, \boldsymbol{\gamma}) \neq 0$ only if $x_i \in \mathcal{I}_{\boldsymbol{\upsilon}}$, for all $g \in \mathcal{G}$ (each corresponds to a particular $\boldsymbol{\upsilon}$),

$$\mathbb{E}_n[\mathbb{V}[g(x_i, \mathbf{w}_i, y_i)|\mathscr{F}_{XW\Delta}]] \lesssim \varepsilon_n^2(r_{1,n}+r_{2,n}+\varepsilon_n+r_{3,n}) \sum_{l:\widehat{\mathcal{B}}_l \subset \mathcal{I}_{\boldsymbol{\upsilon}}} \mathbb{E}_n[\mathbb{1}_{i,\boldsymbol{\upsilon},l}]\varrho^{2|(p-s+1)l-k_{\boldsymbol{\upsilon}}|}.$$

This inequality holds for any event in $\mathscr{F}_{XW\Delta}$. Define an event $\mathcal{A}_1$ on which $\sup_{1\leq j\leq J}\mathbb{E}_n[\mathbb{1}_{i,j}] \leq C_2J^{-1}$ for some large enough $C_2 > 0$ where $\mathbb{1}_{i,j} = \mathbb{1}(x_i \in \widehat{\mathcal{B}}_j)$. By the argument in Lemma SA-3.1, $\mathbb{P}(\mathcal{A}_1^c) \to 0$. On $\mathcal{A}_1$,

$$\bar{\sigma}^2 := \sup_{g\in\mathcal{G}} \mathbb{E}_n[\mathbb{V}[g(x_i, \mathbf{w}_i, y_i)|\mathscr{F}_{XW\Delta}]] \lesssim \varepsilon_n^2 J^{-1}(r_{1,n}+r_{2,n}+\varepsilon_n+r_{3,n}).$$

On the other hand,

$$\bar{G} := \sup_{g\in\mathcal{G}} |g(x_i, \mathbf{w}_i, y_i)| \lesssim \varepsilon_n(1+r_{3,n}+|\psi(y_i,\eta_i)|(r_{1,n}+r_{2,n}+\varepsilon_n+r_{3,n}))(|\eta_{i,1}|+r_{1,n}+r_{2,n}+\varepsilon_n+r_{3,n}).$$

Also, for any $g, \tilde{g} \in \mathcal{G}$, denote the corresponding parameters defining g and $\tilde{g}$ by $(\boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \boldsymbol{\upsilon}, \boldsymbol{\gamma})$ and $(\tilde{\boldsymbol{\beta}}_1, \tilde{\boldsymbol{\beta}}_2, \tilde{\boldsymbol{\upsilon}}, \tilde{\boldsymbol{\gamma}})$. We have

$$\begin{aligned}
\tilde{g}(x_i, \mathbf{w}_i, y_i) - g(x_i, \mathbf{w}_i, y_i) = &\int_0^{\mathit{\Lambda}_1} \Big[\psi(y_i, \eta(\breve{\mathbf{b}}_i'(\breve{\boldsymbol{\beta}}_0+\boldsymbol{\beta}_1+\boldsymbol{\beta}_2)+\mathbf{w}_i'\boldsymbol{\gamma}+t)) \\
&\quad - \psi(y_i, \eta(\breve{\mathbf{b}}_i'\breve{\boldsymbol{\beta}}_0+\mathbf{w}_i'\boldsymbol{\gamma}_0))\Big] \times \eta^{(1)}(\breve{\mathbf{b}}_i'(\breve{\boldsymbol{\beta}}_0+\boldsymbol{\beta}_1+\boldsymbol{\beta}_2)+\mathbf{w}_i'\boldsymbol{\gamma}+t)dt \\
&- \int_0^{\mathit{\Lambda}_2} \Big[\psi(y_i, \eta(\breve{\mathbf{b}}_i'(\breve{\boldsymbol{\beta}}_0+\boldsymbol{\beta}_1+\boldsymbol{\beta}_2-\boldsymbol{\upsilon})+\mathbf{w}_i'\boldsymbol{\gamma}+t)) \\
&\quad - \psi(y_i, \eta(\breve{\mathbf{b}}_i'\breve{\boldsymbol{\beta}}_0+\mathbf{w}_i'\boldsymbol{\gamma}_0))\Big] \times \eta^{(1)}(\breve{\mathbf{b}}_i'(\breve{\boldsymbol{\beta}}_0+\boldsymbol{\beta}_1+\boldsymbol{\beta}_2-\boldsymbol{\upsilon})+\mathbf{w}_i'\boldsymbol{\gamma}+t)dt \\
\lesssim &(1+\Lambda_1+\Lambda_2)(|\eta_{i,1}|+r_{1,n}+r_{2,n}+\mathit{\Lambda}_1+\mathit{\Lambda}_2+r_{3,n}) \\
&\times (\|(\tilde{\boldsymbol{\beta}}_1-\boldsymbol{\beta}_1\|_\infty+\|\tilde{\boldsymbol{\beta}}_2-\boldsymbol{\beta}_2)\|_\infty+\|\tilde{\boldsymbol{\upsilon}}-\boldsymbol{\upsilon}\|_\infty+\|\tilde{\boldsymbol{\gamma}}-\boldsymbol{\gamma}\|),
\end{aligned}$$

where $\varLambda_1 = \breve{\mathbf{b}}_i'(\tilde{\boldsymbol{\beta}}_1 + \tilde{\boldsymbol{\beta}}_2 - \boldsymbol{\beta}_1 - \boldsymbol{\beta}_2) + \mathbf{w}_i'(\tilde{\boldsymbol{\gamma}} - \boldsymbol{\gamma})$ and $\varLambda_2 = \varLambda_1 - \breve{\mathbf{b}}_i'(\tilde{\boldsymbol{\upsilon}} - \boldsymbol{\upsilon})$. Based on these observations,

$$\|\bar{G}\|_{\mathbb{P},2}\int_0^{\frac{\bar{\sigma}}{\|\bar{G}\|_{\mathbb{P},2}}}\sqrt{1+\sup_{\mathbb{Q}}\log N(\mathcal{G},L_2(\mathbb{Q}),t\|\bar{G}\|_{\mathbb{Q},2})}dt \lesssim \bar{\sigma}\Big(\sqrt{\log J}+\sqrt{\log n\log\frac{1}{\bar{\sigma}}}\Big)\lesssim \bar{\sigma}\log n,$$

where the supremum is taken over all finite discrete probability measures $\mathbb{Q}$. Then, by Lemma SA-5.6,

$$\mathbb{E}\bigg[\sup_{g\in\mathcal{G}}\Big|\mathbb{G}_n[g(x_i,\mathbf{w}_i,y_i)]\Big|\Big|\mathscr{F}_{XW\Delta}\bigg] \lesssim \bar{\sigma}\log n + \frac{\sqrt{\mathbb{E}[\bar{\bar{G}}^2]}\log^2 n}{\sqrt{n}},$$

where $\bar{\bar{G}} = \max_{1\leq i\leq n}\bar{G}(x_i,\mathbf{w}_i,y_i)$. Note that $(\mathbb{E}[\bar{\bar{G}}^2])^{1/2} \lesssim \varepsilon_n(1+n^{1/\nu}(r_{1,n}+r_{2,n}+r_{3,n}+\varepsilon_n))$.

Therefore, on $\mathcal{A}_1$ (whose probability approaches one),

$$\begin{aligned}&\sup_{\boldsymbol{\beta}_1\in\mathcal{H}_1,\boldsymbol{\beta}_2\in\mathcal{H}_2,\boldsymbol{\upsilon}\in\mathcal{V},\boldsymbol{\gamma}_1\in\mathcal{H}_3}\Big|\mathbb{E}_n\Big[\delta_i(\boldsymbol{\beta}_1,\boldsymbol{\beta}_2,\boldsymbol{\upsilon},\boldsymbol{\gamma})\Big]-\mathbb{E}_n\Big[\mathbb{E}[\delta_i(\boldsymbol{\beta}_1,\boldsymbol{\beta}_2,\boldsymbol{\upsilon},\boldsymbol{\gamma})|\mathscr{F}_{XW\Delta}]\Big]\Big|\\ \lesssim&\Bigg(J^{-1}\varepsilon_n\sqrt{\mathfrak{L}_n}\sqrt{\frac{J}{n}}\log n+\frac{\varepsilon_n(1+n^{1/\nu}\mathfrak{L}_n)(\log n)^2}{n}\Bigg)\end{aligned}$$

for $\mathfrak{L}_n = r_{1,n}+r_{2,n}+r_{3,n}+\varepsilon_n$.

Step 2: For $\widetilde{\mathbf{Q}} := \mathbb{E}_n[\breve{\mathbf{b}}_i\breve{\mathbf{b}}_i'\Psi_1(x_i,\mathbf{w}_i;\eta(\breve{\mathbf{b}}_i'\breve{\beta}_0+\mathbf{w}_i'\gamma_0))(\eta^{(1)}(\breve{\mathbf{b}}_i\breve{\beta}_0+\mathbf{w}_i'\gamma_0))^2]$, by Assumption SA-SM and the same argument in the proof of Lemma SA-3.1, $\|\bar{\mathbf{Q}}-\widetilde{\mathbf{Q}}\|_\infty\vee\|\bar{\mathbf{Q}}-\widetilde{\mathbf{Q}}\|\lesssim J^{-p-1}J^{-1}$. Therefore,

$$\sup_{\boldsymbol{\beta}_1\in\mathcal{H}_1,\boldsymbol{\beta}_2\in\mathcal{H}_2,\boldsymbol{\upsilon}\in\mathcal{V}}|\boldsymbol{\upsilon}'(\widetilde{\mathbf{Q}}-\bar{\mathbf{Q}})(\boldsymbol{\beta}_1+\boldsymbol{\beta}_2)|\lesssim J^{-p-2}\varepsilon_n(r_{1,n}+r_{2,n}).$$

In addition, by Lemmas SA-3.3 and SA-3.4, $\|\bar{\boldsymbol{\beta}}\|_\infty\leq r_{1,n}$ with probability approaching one for C_1 large enough, where

$$\bar{\boldsymbol{\beta}} := -\bar{\mathbf{Q}}^{-1}\mathbb{E}_n\Big[\breve{\mathbf{b}}_i\eta^{(1)}(\breve{\mathbf{b}}_i'\breve{\beta}_0+\mathbf{w}_i'\gamma_0)\psi\Big(y_i,\eta(\breve{\mathbf{b}}_i'\breve{\beta}_0+\mathbf{w}_i'\gamma_0)\Big)\Big].$$

Step 3: By Taylor expansion, we have

$$\begin{aligned}&\mathbb{E}_n\Big[\mathbb{E}[\delta_i(\boldsymbol{\beta}_1,\boldsymbol{\beta}_2,\boldsymbol{\upsilon},\boldsymbol{\gamma})|\mathscr{F}_{XW\Delta}]\Big]\\ =&\,\mathbb{E}_n\bigg[\int_{-\breve{\mathbf{b}}_i'\boldsymbol{\upsilon}}^0\Big\{\Psi(x_i,\mathbf{w}_i;\eta(\breve{\mathbf{b}}_i'(\breve{\boldsymbol{\beta}}_0+\boldsymbol{\beta}_1+\boldsymbol{\beta}_2)+\mathbf{w}_i'\boldsymbol{\gamma}+t))\\ &\qquad-\Psi(x_i,\mathbf{w}_i;\eta(\breve{\mathbf{b}}_i'\breve{\boldsymbol{\beta}}_0+\mathbf{w}_i'\gamma_0))\Big\}\times\eta^{(1)}\Big(\breve{\mathbf{b}}_i'(\breve{\boldsymbol{\beta}}_0+\boldsymbol{\beta}_1+\boldsymbol{\beta}_2)+\mathbf{w}_i'\boldsymbol{\gamma}+t\Big)dt\bigg]\end{aligned}$$

$$
\begin{aligned}
&= \mathbb{E}_n\bigg[\int_{-\breve{\mathbf{b}}_i'\boldsymbol{\upsilon}}^{0}\Big\{\Psi_1(x_i,\mathbf{w}_i;\eta(\breve{\mathbf{b}}_i'\breve{\boldsymbol{\beta}}_0+\mathbf{w}_i'\boldsymbol{\gamma}_0))\Big(\eta^{(1)}(\breve{\mathbf{b}}_i'\breve{\boldsymbol{\beta}}_0+\mathbf{w}_i'\boldsymbol{\gamma}_0)(\breve{\mathbf{b}}_i'(\boldsymbol{\beta}_1+\boldsymbol{\beta}_2)+\mathbf{w}_i'\boldsymbol{\gamma}_1+t)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad+\frac{1}{2}\eta^{(2)}(\xi_{i,t})(\breve{\mathbf{b}}_i'(\boldsymbol{\beta}_1+\boldsymbol{\beta}_2)+\mathbf{w}_i'\boldsymbol{\gamma}_1+t)^2\Big)\\
&\qquad\qquad+\frac{1}{2}\Psi_2(x_i,\mathbf{w}_i;\tilde{\xi}_{i,t})\Big(\eta(\breve{\mathbf{b}}_i'(\breve{\boldsymbol{\beta}}_0+\boldsymbol{\beta}_1+\boldsymbol{\beta}_2)+\mathbf{w}_i'\boldsymbol{\gamma}+t)-\eta(\breve{\mathbf{b}}_i'\breve{\boldsymbol{\beta}}_0+\mathbf{w}_i'\boldsymbol{\gamma}_0)\Big)^2\Big\}\\
&\qquad\qquad\qquad\times\Big(\eta^{(1)}(\breve{\mathbf{b}}_i'\breve{\boldsymbol{\beta}}_0+\mathbf{w}_i'\boldsymbol{\gamma}_0)+\eta^{(2)}(\check{\xi}_{i,t})(\breve{\mathbf{b}}_i'(\boldsymbol{\beta}_1+\boldsymbol{\beta}_2)+\mathbf{w}_i'\boldsymbol{\gamma}_1+t)\Big)dt\bigg]\\
&= \boldsymbol{\upsilon}'\widetilde{\mathbf{Q}}(\boldsymbol{\beta}_1+\boldsymbol{\beta}_2)+\boldsymbol{\upsilon}'\mathbb{E}_n[\mathbf{b}_i\widetilde{\varkappa}_i\mathbf{w}_i']\boldsymbol{\gamma}_1-\frac{1}{2}\boldsymbol{\upsilon}\widetilde{\mathbf{Q}}\boldsymbol{\upsilon}+\mathrm{I}+\mathrm{II}+\mathrm{III},
\end{aligned}
$$

where $\xi_{i,t}$ and $\check{\xi}_{i,t}$ are between $\breve{\mathbf{b}}_i'\breve{\boldsymbol{\beta}}_0+\mathbf{w}_i'\boldsymbol{\gamma}_0$ and $\breve{\mathbf{b}}_i'(\breve{\boldsymbol{\beta}}_0+\boldsymbol{\beta}_1+\boldsymbol{\beta}_2)+\mathbf{w}_i'\boldsymbol{\gamma}+t$, $\tilde{\xi}_{i,t}$ is between $\eta(\breve{\mathbf{b}}_i'\breve{\boldsymbol{\beta}}_0+\mathbf{w}_i'\boldsymbol{\gamma}_0)$ and $\eta(\breve{\mathbf{b}}_i'(\breve{\boldsymbol{\beta}}_0+\boldsymbol{\beta}_1+\boldsymbol{\beta}_2)+\mathbf{w}_i'\boldsymbol{\gamma}+t)$, $\Psi_2(x,\mathbf{w};\tau)=\frac{\partial^2}{\partial\tau^2}\Psi(x,\mathbf{w};\tau)$, $\widetilde{\varkappa}_i=\Psi_1(x_i,\mathbf{w}_i;\eta(\breve{\mathbf{b}}_i'\breve{\boldsymbol{\beta}}_0+\mathbf{w}_i'\boldsymbol{\gamma}_0))(\eta^{(1)}(\breve{\mathbf{b}}_i'\breve{\boldsymbol{\beta}}_0+\mathbf{w}_i'\boldsymbol{\gamma}_0))^2$, $\boldsymbol{\upsilon}'\mathbb{E}_n[\mathbf{b}_i\widetilde{\varkappa}_i\mathbf{w}_i']\boldsymbol{\gamma}_1\lesssim\varepsilon_n r_{3,n}/J$, $-\frac{1}{2}\boldsymbol{\upsilon}\widetilde{\mathbf{Q}}\boldsymbol{\upsilon}\lesssim\varepsilon_n^2/J$, and I,II, and III are defined and bounded as follows:

$$
\begin{aligned}
\mathrm{I}&=\mathbb{E}_n\bigg[\int_{-\breve{\mathbf{b}}_i'\boldsymbol{\upsilon}}^{0}\Psi_1(x_i;\eta(\breve{\mathbf{b}}_i'\breve{\boldsymbol{\beta}}_0+\mathbf{w}_i'\boldsymbol{\gamma}_0))\eta^{(1)}(\breve{\mathbf{b}}_i'\breve{\boldsymbol{\beta}}_0+\mathbf{w}_i'\boldsymbol{\gamma}_0)\\
&\qquad\qquad\times\eta^{(2)}(\check{\xi}_{i,t})(\breve{\mathbf{b}}_i'(\boldsymbol{\beta}_1+\boldsymbol{\beta}_2)+\mathbf{w}_i'\boldsymbol{\gamma}_1+t)^2dt\mathbb{1}_{i,\boldsymbol{\upsilon}}\bigg]\lesssim\varepsilon_nJ^{-1}(r_{1,n}+r_{2,n}+\varepsilon_n+r_{3,n})^2,\\
\mathrm{II}&=\mathbb{E}_n\bigg[\int_{-\breve{\mathbf{b}}_i'\boldsymbol{\upsilon}}^{0}\Psi_1(x_i;\eta(\breve{\mathbf{b}}_i'\breve{\boldsymbol{\beta}}_0+\mathbf{w}_i'\boldsymbol{\gamma}_0))\times\frac{1}{2}\eta^{(2)}(\xi_{i,t})(\breve{\mathbf{b}}_i'(\boldsymbol{\beta}_1+\boldsymbol{\beta}_2)+\mathbf{w}_i'\boldsymbol{\gamma}_1+t)^2\\
&\qquad\qquad\times\eta^{(1)}\Big(\breve{\mathbf{b}}_i'(\breve{\boldsymbol{\beta}}_0+\boldsymbol{\beta}_1+\boldsymbol{\beta}_2)+\mathbf{w}_i'\boldsymbol{\gamma}+t\Big)dt\mathbb{1}_{i,\boldsymbol{\upsilon}}\bigg]\lesssim\varepsilon_nJ^{-1}(r_{1,n}+r_{2,n}+\varepsilon_n+r_{3,n})^2,\\
\mathrm{III}&=\mathbb{E}_n\bigg[\int_{-\breve{\mathbf{b}}_i'\boldsymbol{\upsilon}}^{0}\frac{1}{2}\Psi_2(\tilde{\xi}_{i,t})\Big(\eta(\breve{\mathbf{b}}_i'(\breve{\boldsymbol{\beta}}_0+\boldsymbol{\beta}_1+\boldsymbol{\beta}_2)+\mathbf{w}_i'\boldsymbol{\gamma}+t)-\eta(\breve{\mathbf{b}}_i'\breve{\boldsymbol{\beta}}_0+\mathbf{w}_i'\boldsymbol{\gamma}_0)\Big)^2\\
&\qquad\qquad\times\eta^{(1)}\Big(\breve{\mathbf{b}}_i'(\breve{\boldsymbol{\beta}}_0+\boldsymbol{\beta}_1+\boldsymbol{\beta}_2)+\mathbf{w}_i'\boldsymbol{\gamma}+t\Big)dt\mathbb{1}_{i,\boldsymbol{\upsilon}}\bigg]\lesssim\varepsilon_nJ^{-1}(r_{1,n}+r_{2,n}+\varepsilon_n+r_{3,n})^2.
\end{aligned}
$$

These bounds hold uniformly for $\boldsymbol{\upsilon}\in\mathcal{V}$, $\boldsymbol{\beta}_1\in\mathcal{H}_1$, $\boldsymbol{\beta}_2\in\mathcal{H}_2$ and $\boldsymbol{\gamma}_1\in\mathcal{H}_3$ (that is, uniformly over the function class $\mathcal{G}$), and on an event $\mathcal{A}_1\cap\mathcal{A}_2$ where $\mathcal{A}_2=\{\lambda_{\max}(\widetilde{\mathbf{Q}})\leq c_4J^{-1}\}$ for some large enough $c_4>0$. Note that $\mathbb{P}(\mathcal{A}_1\cap\mathcal{A}_2)\to 1$ by Lemma SA-3.1.

Step 4: By Assumption SA-SM and Taylor's expansion,

$$
\begin{aligned}
\mathrm{IV}&=\mathbb{E}_n\bigg[\Big(\eta(\breve{\mathbf{b}}_i'(\breve{\boldsymbol{\beta}}_0+\boldsymbol{\beta}_1+\boldsymbol{\beta}_2)+\mathbf{w}_i'\boldsymbol{\gamma})-\eta(\breve{\mathbf{b}}_i'(\breve{\boldsymbol{\beta}}_0+\boldsymbol{\beta}_1+\boldsymbol{\beta}_2-\boldsymbol{\upsilon})+\mathbf{w}_i'\boldsymbol{\gamma})\Big)\psi(y_i,\eta(\breve{\mathbf{b}}_i'\breve{\boldsymbol{\beta}}_0+\mathbf{w}_i'\boldsymbol{\gamma}_0))\bigg]\\
&\quad-\mathbb{E}_n\Big[\boldsymbol{\upsilon}'\breve{\mathbf{b}}_i\psi(y_i,\eta(\breve{\mathbf{b}}_i'\breve{\boldsymbol{\beta}}_0+\mathbf{w}_i'\boldsymbol{\gamma}_0))\eta^{(1)}(\breve{\mathbf{b}}_i'\breve{\boldsymbol{\beta}}_0+\mathbf{w}_i'\boldsymbol{\gamma}_0)\Big]\\
&=\mathbb{E}_n\Big[\boldsymbol{\upsilon}'\breve{\mathbf{b}}_i\psi(y_i,\eta(\breve{\mathbf{b}}_i'\breve{\boldsymbol{\beta}}_0+\mathbf{w}_i'\boldsymbol{\gamma}_0))\Big(\eta^{(2)}(\xi_i)(\breve{\mathbf{b}}_i'(\boldsymbol{\beta}_1+\boldsymbol{\beta}_2-\boldsymbol{\upsilon})+\mathbf{w}_i'\boldsymbol{\gamma}_1)+\frac{1}{2}\eta^{(2)}(\tilde{\xi}_i)\boldsymbol{\upsilon}'\breve{\mathbf{b}}_i\Big)\Big]
\end{aligned}
$$

$$\lesssim J^{-1}((J\log n/n)^{1/2}+J^{-p-1})(\varepsilon_n+r_{1,n}+r_{2,n}+r_{3,n})\varepsilon_n,$$

where ξ_i is between $\breve{\mathbf{b}}_i'\breve{\boldsymbol{\beta}}_0+\mathbf{w}_i'\boldsymbol{\gamma}_0$ and $\breve{\mathbf{b}}_i'(\breve{\boldsymbol{\beta}}_0+\boldsymbol{\beta}_1+\boldsymbol{\beta}_2-\boldsymbol{\upsilon})+\mathbf{w}_i'\boldsymbol{\gamma}$ and $\tilde{\xi}_i$ is between $\breve{\mathbf{b}}_i'(\breve{\boldsymbol{\beta}}_0+\boldsymbol{\beta}_1+\boldsymbol{\beta}_2)+\mathbf{w}_i'\boldsymbol{\gamma}$ and $\breve{\mathbf{b}}_i'(\breve{\boldsymbol{\beta}}_0+\boldsymbol{\beta}_1+\boldsymbol{\beta}_2-\boldsymbol{\upsilon})+\mathbf{w}_i'\boldsymbol{\gamma}$. The last line holds on the event

$$\mathcal{A}_3=\Big\{\sup\Big(\Big\|\mathbb{E}_n\Big[\breve{\mathbf{b}}_i\breve{\mathbf{b}}_i'\psi(y_i,\eta(\breve{\mathbf{b}}_i'\breve{\boldsymbol{\beta}}_0+\mathbf{w}_i'\boldsymbol{\gamma}_0))\eta^{(2)}(\varpi_i)\Big]\Big\|_\infty+ \Big\|\mathbb{E}_n\Big[\breve{\mathbf{b}}_i\psi(y_i,\eta(\breve{\mathbf{b}}_i'\breve{\boldsymbol{\beta}}_0+\mathbf{w}_i'\boldsymbol{\gamma}_0))\eta^{(2)}(\varpi_i)\mathbf{w}_i\Big]\Big\|_\infty\Big)\lesssim J^{-1}\Big(\Big(\frac{J\log n}{n}\Big)^{1/2}+J^{-p-1}\Big)\Big\},$$

where the supremum is taken over $\boldsymbol{\beta}_1\in\mathcal{H}_1,\boldsymbol{\beta}_2\in\mathcal{H}_2,\boldsymbol{\upsilon}\in\mathcal{V},\boldsymbol{\gamma}_1\in\mathcal{H}_3$ and ϖ_i within the range of ξ_i or $\tilde{\xi}_i$. Note that $\mathbb{E}[\psi(y_i,\eta_i)|\mathscr{F}_{XW\Delta}]=0$ and $\breve{\mathbf{b}}_i'\breve{\boldsymbol{\beta}}_0-\mu_0(x_i)\lesssim J^{-p-1}$. Then, we can use the argument in the proof of Lemmas SA-3.3 and SA-3.4 to obtain $\mathbb{P}(\mathcal{A}_3)\to 1$ by choosing $C_3>0$ sufficiently large.

Step 5: Let $\bar{\boldsymbol{\upsilon}}=c_5\varepsilon_nJ^{-1}[\bar{\mathbf{Q}}^{-1}]_{k\cdot}$ for some k such that $|\beta_{2,k}|=\|\boldsymbol{\beta}_2\|_\infty$ for some $c_5>0$ where $[\bar{\mathbf{Q}}^{-1}]_{k\cdot}$ denotes the kth row of $\bar{\mathbf{Q}}^{-1}$. Note that $\boldsymbol{\upsilon}'\bar{\mathbf{Q}}\boldsymbol{\beta}_2=\beta_{2,k}$. Take $\boldsymbol{\upsilon}=(\upsilon_1,\cdots,\upsilon_{K_{p,s}})$ where $\upsilon_j=\bar{\upsilon}_j$ for $|j-k|\leq M_n$ and zero otherwise. Clearly, $\boldsymbol{\upsilon}\in\mathcal{V}$ on an event $\mathcal{A}_4$ with $\mathbb{P}(\mathcal{A}_4)\to 1$. On $\mathcal{A}_2\cap\mathcal{A}_4$,

$$|(\boldsymbol{\upsilon}-\bar{\boldsymbol{\upsilon}})'\bar{\mathbf{Q}}\boldsymbol{\beta}_2|\lesssim\varepsilon_nJ^{-1}r_{2,n}n^{-c_6}$$

for some large $c_6>0$ if we let c_1 be sufficiently large.

Step 6: Finally, partition the whole parameter space into shells: $\mathcal{O}=\cup_{\ell=-\infty}^{\bar{L}}\mathcal{O}_\ell$ where $\mathcal{O}_\ell=\{\boldsymbol{\beta}\in\mathbb{R}^{K_{p,s}}:2^{\ell-1}\mathfrak{r}_{2,n}\leq\|\boldsymbol{\beta}-\breve{\boldsymbol{\beta}}_0-\bar{\boldsymbol{\beta}}\|_\infty\leq 2^\ell\mathfrak{r}_{2,n}\}$ for the smallest $\bar{L}$ such that $2^{\bar{L}}r_{2,n}\geq c$, and $\bar{\mathbf{Q}}\bar{\boldsymbol{\beta}}=-\mathbb{E}_n[\breve{\mathbf{b}}_i\eta^{(1)}(\breve{\mathbf{b}}_i'\breve{\boldsymbol{\beta}}_0+\mathbf{w}_i'\boldsymbol{\gamma}_0)\psi(y_i,\eta(\breve{\mathbf{b}}_i'\breve{\boldsymbol{\beta}}_0+\mathbf{w}_i'\boldsymbol{\gamma}_0))]$. Define $\mathcal{A}=\cap_{j=1}^4\mathcal{A}_j$. Then, for some constant $L\leq\bar{L}$, we have by Lemma SA-3.5 and the results given in the previous steps,

$$\begin{aligned}
&\mathbb{P}(\|\breve{\boldsymbol{\beta}}-\breve{\boldsymbol{\beta}}_0-\bar{\boldsymbol{\beta}}\|_\infty\geq 2^Lr_{2,n}|\mathscr{F}_{XW\Delta})\\
\leq{}&\mathbb{P}\Big(\bigcup_{\ell=L}^{\bar{L}}\Big\{\inf_{\boldsymbol{\beta}\in\mathcal{O}_\ell}\sup_{\boldsymbol{\upsilon}\in\mathcal{V}}\mathbb{E}_n[\rho(y_i;\eta(\breve{\mathbf{b}}_i'\boldsymbol{\beta}+\mathbf{w}_i'\widehat{\boldsymbol{\gamma}}))-\rho(y_i;\eta(\breve{\mathbf{b}}_i'(\boldsymbol{\beta}-\boldsymbol{\upsilon})+\mathbf{w}_i'\widehat{\boldsymbol{\gamma}}))]<0\Big\}\Big|\mathscr{F}_{XW\Delta}\Big)+o_{\mathbb{P}}(1)\\
={}&\mathbb{P}\Big(\bigcup_{\ell=L}^{\bar{L}}\Big\{\inf_{\boldsymbol{\beta}\in\mathcal{O}_\ell}\sup_{\boldsymbol{\upsilon}\in\mathcal{V}}\Big\{\mathbb{E}\Big[\rho(y_i;\eta(\breve{\mathbf{b}}_i'\boldsymbol{\beta}+\mathbf{w}_i'\widehat{\boldsymbol{\gamma}}))-\rho(y_i;\eta(\breve{\mathbf{b}}_i'(\boldsymbol{\beta}-\boldsymbol{\upsilon})+\mathbf{w}_i'\widehat{\boldsymbol{\gamma}}))\\
&\qquad-[\eta(\breve{\mathbf{b}}_i'\boldsymbol{\beta}+\mathbf{w}_i'\widehat{\boldsymbol{\gamma}})-\eta(\breve{\mathbf{b}}_i'(\boldsymbol{\beta}-\boldsymbol{\upsilon})+\mathbf{w}_i'\widehat{\boldsymbol{\gamma}})]\psi(y_i,\eta(\breve{\mathbf{b}}_i'\breve{\boldsymbol{\beta}}_0+\mathbf{w}_i'\widehat{\boldsymbol{\gamma}}))|\mathscr{F}_{XW\Delta}\Big]+
\end{aligned}$$

$$
\begin{aligned}
&\mathbb{E}_n\Big[(\eta(\breve{\mathbf{b}}_i'\boldsymbol{\beta}+\mathbf{w}_i'\widehat{\boldsymbol{\gamma}})-\eta(\breve{\mathbf{b}}_i'(\boldsymbol{\beta}-\boldsymbol{\upsilon})+\mathbf{w}_i'\widehat{\boldsymbol{\gamma}}))\psi(y_i,\eta(\breve{\mathbf{b}}_i'\breve{\boldsymbol{\beta}}_0+\mathbf{w}_i'\widehat{\boldsymbol{\gamma}}))\Big]+\\
&\frac{1}{\sqrt{n}}\mathbb{G}_n\Big[\rho(y_i;\eta(\breve{\mathbf{b}}_i'\boldsymbol{\beta}+\mathbf{w}_i'\widehat{\boldsymbol{\gamma}}))-\rho(y_i;\eta(\breve{\mathbf{b}}_i'(\boldsymbol{\beta}-\boldsymbol{\upsilon})+\mathbf{w}_i'\widehat{\boldsymbol{\gamma}}))-\\
&[\eta(\breve{\mathbf{b}}_i'\boldsymbol{\beta}+\mathbf{w}_i'\widehat{\boldsymbol{\gamma}})-\eta(\breve{\mathbf{b}}_i'(\boldsymbol{\beta}-\boldsymbol{\upsilon})+\mathbf{w}_i'\widehat{\boldsymbol{\gamma}})]\psi(y_i,\eta(\breve{\mathbf{b}}_i'\breve{\boldsymbol{\beta}}_0+\mathbf{w}_i'\widehat{\boldsymbol{\gamma}}))\Big]\Big\}<0\Big\}\Big|\mathscr{F}_{XW\Delta}\Big)+o_{\mathbb{P}}(1)\\
\leq\ &\mathbb{P}\Big(\bigcup_{\ell=L}^{\bar{L}}\Big\{\sup_{\boldsymbol{\beta}_1\in\mathcal{H}_1}\sup_{\boldsymbol{\beta}_2\in\mathcal{H}_{2,\ell}}\sup_{\boldsymbol{\gamma}_1\in\mathcal{H}_3}\sup_{\boldsymbol{\upsilon}\in\mathcal{V}}\frac{1}{\sqrt{n}}\Big|(\mathbb{1}(\mathcal{A}_1)+\mathbb{1}(\mathcal{A}_1^c))\mathbb{G}_n[\delta_i(\boldsymbol{\beta}_1,\boldsymbol{\beta}_2,\boldsymbol{\upsilon},\boldsymbol{\gamma})]\Big|>\\
&\qquad C_4J^{-1}2^\ell r_{2,n}\varepsilon_n\Big\}\cap\mathcal{A}\Big|\mathscr{F}_{XW\Delta}\Big)+o_{\mathbb{P}}(1)\\
\leq\ &\sum_{\ell=L}^{\bar{L}}(C_6J^{-1}2^\ell\mathfrak{r}_{2,n}\varepsilon_n)^{-1}\mathbb{1}(\mathcal{A}_1)\mathbb{E}\Big[\sup_{\boldsymbol{\beta}_1\in\mathcal{H}_1}\sup_{\boldsymbol{\beta}_2\in\mathcal{H}_{2,\ell}}\sup_{\boldsymbol{\gamma}_1\in\mathcal{H}_3}\sup_{\boldsymbol{\upsilon}\in\mathcal{V}}\frac{1}{\sqrt{n}}\mathbb{G}_n[\delta_i(\boldsymbol{\beta}_1,\boldsymbol{\beta}_2,\boldsymbol{\upsilon},\boldsymbol{\gamma})]\Big|\mathscr{F}_{XW\Delta}\Big]+o_{\mathbb{P}}(1),
\end{aligned}
$$

where $\mathbb{G}_n[\cdot]$ is understood as $\sqrt{n}(\mathbb{E}_n[\cdot]-\mathbb{E}[\cdot|\mathscr{F}_{XW}])$ in the above, we let $\varepsilon_n=2^Lr_{2,n}$, and $\mathbb{1}(\mathcal{A}_1)$ is an indicator of the event $\mathcal{A}_1$. Using the result in Step 1 and the rate condition, the first term in the last line can be made arbitrarily small by choosing L large enough, when n is sufficiently large. Then, the proof for part (i) is complete.

Step 7: To show part (ii) and part (iii), by Taylor expansion and the result in part (i),

$$
\begin{aligned}
&\eta(\widehat{\mu}(x)+\widehat{\mathsf{w}}'\widehat{\boldsymbol{\gamma}})-\eta(\mu_0(x)+\mathsf{w}'\boldsymbol{\gamma}_0)\\
=\ &\eta^{(1)}(\mu_0(x)+\mathsf{w}'\boldsymbol{\gamma}_0)\Big(\widehat{\mathbf{b}}_{p,s}(x)'\widehat{\boldsymbol{\beta}}-\mu_0(x)\Big)\\
&\qquad+O_{\mathbb{P}}\Big(\|\widehat{\mathsf{w}}-\mathsf{w}\|+\|\widehat{\boldsymbol{\gamma}}-\boldsymbol{\gamma}_0\|+\frac{J\log n}{n}+J^{-2p-2}+\mathfrak{r}_{2,n}^2\Big)\\
=\ &-\eta^{(1)}(\mu_0(x)+\mathsf{w}'\boldsymbol{\gamma}_0)\widehat{\mathbf{b}}_{p,s}(x)'\bar{\mathbf{Q}}^{-1}\mathbb{E}_n[\widehat{\mathbf{b}}_{p,s}(x_i)\eta_{i,1}\psi(y_i,\eta_i)]\\
&\qquad+O_{\mathbb{P}}\Big(J^{-p-1}+\Big(\frac{J\log n}{n}\Big)^{3/4}\log n+J^{-\frac{p+1}{2}}\Big(\frac{J\log^2 n}{n}\Big)^{1/2}+\mathfrak{r}_\gamma+\|\widehat{\mathsf{w}}-\mathsf{w}\|\Big),
\end{aligned}
$$

and

$$
\begin{aligned}
&\eta^{(1)}(\widehat{\mu}(x)+\widehat{\mathsf{w}}'\widehat{\boldsymbol{\gamma}})\widehat{\mu}^{(1)}(x)-\eta^{(1)}(\mu_0(x)+\mathsf{w}'\boldsymbol{\gamma}_0)\mu_0^{(1)}(x)\\
=\ &\eta^{(1)}(\mu_0(x)+\mathsf{w}'\boldsymbol{\gamma}_0)\Big(\widehat{\mu}^{(1)}(x)-\mu_0^{(1)}(x)\Big)\\
&+O_{\mathbb{P}}\Big(\Big(\frac{J\log n}{n}\Big)^{1/2}+J^{-p-1}+\|\widehat{\mathsf{w}}-\mathsf{w}\|+\mathfrak{r}_{2,n}\Big)O_{\mathbb{P}}\Big(1+J\Big(\Big(\frac{J\log n}{n}\Big)^{1/2}+J^{-p-1}+\mathfrak{r}_{2,n}\Big)\Big)\\
=\ &-\eta^{(1)}(\mu_0(x)+\mathsf{w}'\boldsymbol{\gamma}_0)\widehat{\mathbf{b}}_{p,s}^{(1)}(x)'\bar{\mathbf{Q}}^{-1}\mathbb{E}_n[\widehat{\mathbf{b}}_{p,s}(x_i)\eta_{i,1}\psi(y_i,\eta_i)]+\\
&O_{\mathbb{P}}\Big(\Big(\frac{J\log n}{n}\Big)^{1/2}+J^{-p}+J\Big(\frac{J\log n}{n}\Big)^{3/4}\log n+J^{-\frac{p-1}{2}}\Big(\frac{J\log^2 n}{n}\Big)^{1/2}+J\mathfrak{r}_\gamma
\end{aligned}
$$

$$+\|\widehat{\mathsf{w}}-\mathsf{w}\|\Big(1+\Big(\frac{J^3\log n}{n}\Big)^{1/2}\Big)\Big).$$

In the above derivation the probability bound holds uniformly over $x\in\mathcal{X}$ as well. Then the proof is complete. $\square$

SA-5.8 Proof of Theorem SA-3.2

Proof. Since $\widehat{\epsilon}_i:=\epsilon_i+\eta_i-\widehat{\eta}_i=:\epsilon_i+u_i$, we can write

$$\begin{aligned}
&\mathbb{E}_n[\widehat{\mathbf{b}}_{p,s}(x_i)\widehat{\mathbf{b}}_{p,s}(x_i)'\widehat{\eta}_{i,1}^2\psi^\ddagger(\widehat{\eta}_i)^2\psi^\dagger(\widehat{\epsilon}_i)^2]-\mathbb{E}[\mathbf{b}_{p,s}(x_i)\mathbf{b}_{p,s}(x_i)'\eta_{i,1}^2\sigma^2(x_i,\mathbf{w}_i)]\\
=&\,\mathbb{E}_n\Big[\widehat{\mathbf{b}}_{p,s}(x_i)\widehat{\mathbf{b}}_{p,s}(x_i)'\widehat{\eta}_{i,1}^2\psi^\ddagger(\widehat{\eta}_i)^2\Big(\psi^\dagger(\epsilon_i+u_i)^2-\psi^\dagger(\epsilon_i)^2\Big)\Big]\\
&+\mathbb{E}_n\Big[\widehat{\mathbf{b}}_{p,s}(x_i)\widehat{\mathbf{b}}_{p,s}(x_i)'\Big(\widehat{\eta}_{i,1}^2\psi^\ddagger(\widehat{\eta}_i)^2-\eta_{i,1}^2\psi^\ddagger(\eta_i)^2\Big)\psi^\dagger(\epsilon_i)^2\Big]\\
&+\mathbb{E}_n[\widehat{\mathbf{b}}_{p,s}(x_i)\widehat{\mathbf{b}}_{p,s}(x_i)'\eta_{i,1}^2(\psi(y_i,\eta_i)^2-\sigma^2(x_i,\mathbf{w}_i))]\\
&+\Big(\mathbb{E}_n[\widehat{\mathbf{b}}_{p,s}(x_i)\widehat{\mathbf{b}}_{p,s}(x_i)'\eta_{i,1}^2\sigma^2(x_i,\mathbf{w}_i)]-\mathbb{E}[\mathbf{b}_{p,s}(x_i)\mathbf{b}_{p,s}(x_i)'\eta_{i,1}^2\sigma^2(x_i,\mathbf{w}_i)]\Big)\\
=:&\,\mathbf{V}_1+\mathbf{V}_2+\mathbf{V}_3+\mathbf{V}_4.
\end{aligned}$$

We bound each term in the following. The first part of the theorem only concerns $\mathbf{V}_1+\mathbf{V}_2+\mathbf{V}_3$, and the second part needs a bound on $\mathbf{V}_4$ as well where the additional Assumption SA-RP(ii) is used.

Step 1: For $\mathbf{V}_1$, we further write $\mathbf{V}_1=\mathbf{V}_{11}+\mathbf{V}_{12}$ where

$$\begin{aligned}
\mathbf{V}_{11}&:=\mathbb{E}_n\Big[\widehat{\mathbf{b}}_{p,s}(x_i)\widehat{\mathbf{b}}_{p,s}(x_i)'\eta_{i,1}^2\psi^\ddagger(\eta_i)^2\Big(\psi^\dagger(\epsilon_i+u_i)^2-\psi^\dagger(\epsilon_i)^2\Big)\Big],\\
\mathbf{V}_{12}&:=\mathbb{E}_n\Big[\widehat{\mathbf{b}}_{p,s}(x_i)\widehat{\mathbf{b}}_{p,s}(x_i)'\Big(\widehat{\eta}_{i,1}^2\psi^\ddagger(\widehat{\eta}_i)^2-\eta_{i,1}^2\psi^\ddagger(\eta_i)^2\Big)\Big(\psi^\dagger(\epsilon_i+u_i)^2-\psi^\dagger(\epsilon_i)^2\Big)\Big].
\end{aligned}$$

Let $r_{1,n}=C_1(J\log n/n)^{1/2}+J^{-p-1}$ for a constant $C_1>0$. By Assumption SA-SM and Corollary SA-3.1, $\max_{1\leq i\leq n}|u_i|\leq r_{1,n}$ with arbitrarily large probability for C_1 sufficiently large. For $\mathbf{V}_{11}$, let $\mathcal{J}$ be the set of all discontinuity points of $\psi(\cdot)$. Define $\mathbb{1}_{i,\mathcal{D}}:=\mathbb{1}(\epsilon_i\in\mathcal{D})$ and $\mathbb{1}_{i,\mathcal{D}^c}:=(1-\mathbb{1}_{i,\mathcal{D}})$ where $\mathcal{D}:=\{a:|a-\jmath|\leq r_{1,n}\text{ for some }\jmath\in\mathcal{J}\}$. Define

$$\mathbf{V}_{111}:=\mathbb{E}_n\Big[\widehat{\mathbf{b}}_{p,s}(x_i)\widehat{\mathbf{b}}_{p,s}(x_i)'\eta_{i,1}^2\psi^\ddagger(\eta_i)^2\Big(\psi^\dagger(\epsilon_i+u_i)^2-\psi^\dagger(\epsilon_i)^2\Big)\mathbb{1}_{i,\mathcal{D}}\Big],$$

$$\mathbf{V}_{112} := \mathbb{E}_n\Big[\widehat{\mathbf{b}}_{p,s}(x_i)\widehat{\mathbf{b}}_{p,s}(x_i)'\eta_{i,1}^2\psi^{\ddagger}(\eta_i)^2\Big(\psi^{\dagger}(\epsilon_i+u_i)^2-\psi^{\dagger}(\epsilon_i)^2\Big)\mathbb{1}_{i,\mathcal{D}^c}\Big].$$

By definition of $\mathcal{D}$ and Assumption SA-SM,

$$\|\mathbf{V}_{111}\| \lesssim \|\mathbb{E}_n[\widehat{\mathbf{b}}_{p,s}(x_i)\widehat{\mathbf{b}}_{p,s}(x_i)'\mathbb{E}[\mathbb{1}_{i,\mathcal{D}}|\mathscr{F}_{XW\Delta}]]\| + \|\mathbb{E}_n[\widehat{\mathbf{b}}_{p,s}(x_i)\widehat{\mathbf{b}}_{p,s}(x_i)'(\mathbb{1}_{i,\mathcal{D}}-\mathbb{E}[\mathbb{1}_{i,\mathcal{D}}|\mathscr{F}_{XW\Delta}])]\|.$$

By Assumption SA-SM and Lemma SA-3.5 of Cattaneo et al. (2024b), the first term on the right hand side is $O_{\mathbb{P}}(r_{1,n})$. For the second term, conditional on $\mathscr{F}_{XW\Delta}$, it is an independent sequence with mean zero. Thus, we can apply the argument given in Step 3 below and conclude that the second term is $O_{\mathbb{P}}(\sqrt{r_{1,n}J\log J/n}+J\log J/n)$. In this case, the indicator $\mathbb{1}_{i,\mathcal{D}}$ is trivially bounded uniformly.

On the other hand, by Assumption SA-SM,

$$\|\mathbf{V}_{112}\| \lesssim r_{1,n}\|\mathbb{E}_n[\widehat{\mathbf{b}}_{p,s}(x_i)\widehat{\mathbf{b}}_{p,s}(x_i)'\eta_{i,1}^2\psi^{\ddagger}(\eta_i)^2|\psi^{\dagger}(\epsilon_i+u_i)+\psi^{\dagger}(\epsilon_i)|]\|.$$

Since $|c|\leq\frac{1}{2}(1+c^2)$ for any scalar c, we have

$$\mathbb{E}_n\Big[\widehat{\mathbf{b}}_{p,s}(x_i)\widehat{\mathbf{b}}_{p,s}(x_i)'\eta_{i,1}^2\psi^{\ddagger}(\eta_i)^2|\psi^{\dagger}(\epsilon_i)|\Big] \leq \frac{1}{2}\mathbb{E}_n\Big[\widehat{\mathbf{b}}_{p,s}(x_i)\widehat{\mathbf{b}}_{p,s}(x_i)'\eta_{i,1}^2\psi^{\ddagger}(\eta_i)^2(1+\psi^{\dagger}(\epsilon_i)^2)\Big] \lesssim_{\mathbb{P}} 1,$$

by Lemma SA-3.1 and the result in Step 3. In addition, we further write

$$\begin{aligned}&\mathbb{E}_n\Big[\widehat{\mathbf{b}}_{p,s}(x_i)\widehat{\mathbf{b}}_{p,s}(x_i)'\eta_{i,1}^2\psi^{\ddagger}(\eta_i)^2|\psi^{\dagger}(\epsilon_i+u_i)|\Big]\\=&\mathbb{E}_n\Big[\widehat{\mathbf{b}}_{p,s}(x_i)\widehat{\mathbf{b}}_{p,s}(x_i)'\eta_{i,1}^2\psi^{\ddagger}(\eta_i)^2|\psi^{\dagger}(\epsilon_i)+(\psi^{\dagger}(\epsilon_i+u_i)-\psi^{\dagger}(\epsilon_i))|\Big].\end{aligned}$$

Repeat the previous argument to bound this term. We conclude that $\|\mathbf{V}_{11}\|\lesssim_{\mathbb{P}} r_{1,n}$.

$\mathbf{V}_{12}$ can be treated using the previous argument combined with the argument given in Step 2 and the result in Step 3. It leads to $\|\mathbf{V}_{12}\|\lesssim_{\mathbb{P}} r_{1,n}$.

Step 2: For $\mathbf{V}_2$, by Assumption SA-SM, Corollary SA-3.1 and the argument given later in Step 3, we have

$$\|\mathbf{V}_2\| \leq \max_{1\leq i\leq n}|\widehat{\eta}_{i,1}^2\psi^{\ddagger}(\widehat{\eta}_i)^2-\eta_{i,1}^2\psi^{\ddagger}(\eta_i)^2|\|\mathbb{E}_n[\widehat{\mathbf{b}}_{p,s}(x_i)\widehat{\mathbf{b}}_{p,s}(x_i)'\psi^{\dagger}(\epsilon_i)^2]\| \lesssim_{\mathbb{P}} (J\log n/n)^{1/2}+J^{-p-1}.$$

Step 3: For $\mathbf{V}_3$, in view of Lemmas SA-5.2 and SA-5.3, it suffices to show that

$$\sup_{\Delta\in\Pi}\left\|\mathbb{E}_n[\mathbf{b}_{p,0}(x_i;\Delta)\mathbf{b}_{p,0}(x_i;\Delta)'\eta_{i,1}^2(\psi(y_i,\eta_i)^2-\sigma^2(x_i,\mathbf{w}_i))]\right\|\lesssim_{\mathbb{P}}\left(\frac{J\log J}{n^{\frac{\nu-2}{\nu}}}\right)^{1/2}.$$

For notational simplicity, we write $\varphi_i=\psi(y_i,\eta_i)^2-\sigma^2(x_i,\mathbf{w}_i)$, $\varphi_i^-=\varphi_i\mathbb{1}(|\varphi_i|\leq M)-\mathbb{E}[\varphi_i\mathbb{1}(|\varphi_i|\leq M)|x_i,\mathbf{w}_i]$, $\varphi_i^+=\varphi_i\mathbb{1}(|\varphi_i|>M)-\mathbb{E}[\varphi_i\mathbb{1}(|\varphi_i|>M)|x_i,\mathbf{w}_i]$ for some $M>0$ to be specified later. Since $\mathbb{E}[\varphi_i|x_i,\mathbf{w}_i]=0$, $\varphi_i=\varphi_i^-+\varphi_i^+$. Then, define a function class

$$\mathcal{G}=\Big\{(x_1,\mathbf{w}_1,\varphi_1)\mapsto b_{p,0,l}(x_1;\Delta)b_{p,0,k}(x_1;\Delta)\eta_{i,1}^2\varphi_1:1\leq l\leq J(p+1),1\leq k\leq J(p+1),\Delta\in\Pi\Big\}.$$

For $g\in\mathcal{G}$, $\sum_{i=1}^n g(x_i,\mathbf{w}_i,\varphi_i)=\sum_{i=1}^n g(x_i,\mathbf{w}_i,\varphi_i^+)+\sum_{i=1}^n g(x_i,\mathbf{w}_i,\varphi_i^-)$.

For the truncated piece, we have $\sup_{g\in\mathcal{G}}|g(x_i,\mathbf{w}_i,\varphi_i^-)|\lesssim JM$, and

$$\begin{aligned}\sup_{g\in\mathcal{G}}\mathbb{V}[g(x_1,\mathbf{w}_1,\varphi_1^-)]&\lesssim\sup_{x\in\mathcal{X},\mathbf{w}\in\mathcal{W}}\mathbb{E}[(\varphi_i^-)^2|x_i=x,\mathbf{w}_i=\mathbf{w}]\sup_{\Delta\in\Pi}\sup_{1\leq l,k\leq J(p+1)}\mathbb{E}[b_{p,0,l}^2(x_i;\Delta)b_{p,0,k}^2(x_i;\Delta)\eta_{i,1}^4]\\&\lesssim JM\sup_{x\in\mathcal{X},\mathbf{w}\in\mathcal{W}}\mathbb{E}\Big[|\varphi_1|\Big|x_i=x\Big]\lesssim JM.\end{aligned}$$

The VC condition holds by the same argument given in the proof of Lemma SA-3.1. Then, by Lemma SA-5.6,

$$\mathbb{E}\Big[\sup_{g\in\mathcal{G}}\Big|\mathbb{E}_n[g(x_i,\mathbf{w}_i,\varphi_i^-)]\Big|\Big]\lesssim\sqrt{\frac{JM\log(JM)}{n}}+\frac{JM\log(JM)}{n}.$$

Regarding the tail, we apply Theorem 2.14.1 of van der Vaart and Wellner (1996) and obtain

$$\begin{aligned}\mathbb{E}\Big[\sup_{g\in\mathcal{G}}\Big|\mathbb{E}_n[g(x_i,\mathbf{w}_i,\varphi_i^+)]\Big|\Big]&\lesssim\frac{1}{\sqrt{n}}J\mathbb{E}\Big[\sqrt{\mathbb{E}_n[|\varphi_i^+|^2]}\Big]\\&\leq\frac{1}{\sqrt{n}}J(\mathbb{E}[\max_{1\leq i\leq n}|\varphi_i^+|])^{1/2}(\mathbb{E}[\mathbb{E}_n[|\varphi_i^+|])^{1/2}\\&\lesssim\frac{J}{\sqrt{n}}\cdot\frac{n^{\frac{1}{\nu}}}{M^{(\nu-2)/4}},\end{aligned}$$

where the second line follows from Cauchy-Schwarz inequality and the third line uses the fact that

$$\mathbb{E}[\max_{1\leq i\leq n}|\varphi_i^+|]\lesssim\mathbb{E}[\max_{1\leq i\leq n}\psi(y_i,\eta_i)^2]\lesssim n^{2/\nu}\quad\text{and}\quad\mathbb{E}[\mathbb{E}_n[|\varphi_i^+|]]\leq\mathbb{E}[|\varphi_1|^+|]\lesssim\frac{\mathbb{E}[|\psi(y_1,\eta_1)|^\nu]}{M^{(\nu-2)/2}}.$$

Then the desired result follows simply by setting $M = J^{\frac{2}{\nu-2}}$ and the sparsity of the basis.

Step 4: For $\mathbf{V}_4$, since by Assumption SA-SM, $\sup_{x\in\mathcal{X},\mathbf{w}\in\mathcal{W}} \mathbb{E}[\psi(y_i,\eta_i)^2|x_i = x] \lesssim 1$. Then, by the same argument given in the proof of Lemma SA-3.1,

$$\sup_{\Delta\in\Pi}\Big\|\frac{1}{\sqrt{n}}\mathbb{G}_n[\mathbf{b}_{p,s}(x_i;\Delta)\mathbf{b}_{p,s}(x_i;\Delta)'\eta_{i,1}^2\sigma^2(x_i,\mathbf{w}_i)]\Big\| \lesssim_{\mathbb{P}} \sqrt{J\log J/n} \quad \text{and}$$
$$\Big\|\mathbb{E}_{\widehat{\Delta}}\Big[\widehat{\mathbf{b}}_{p,s}(x_i)\widehat{\mathbf{b}}_{p,s}(x_i)'\eta_{i,1}^2\psi(y_i,\eta_i)^2\Big] - \mathbb{E}\Big[\mathbf{b}_{p,s}(x_i)\mathbf{b}_{p,s}(x_i)'\eta_{i,1}^2\psi(y_i,\eta_i)^2\Big]\Big\| \lesssim_{\mathbb{P}} \sqrt{J\log J/n} + \mathfrak{r}_{\mathtt{RP}}.$$

The proof for the first conclusion is complete.

Step 5: The results about $\widehat{\Omega}_{\mu^{(v)}}(x)$, $\widehat{\Omega}_{\vartheta}(x)$ and $\widehat{\Omega}_{\zeta}(x)$ follow by Assumptions SA-SM and SA-HLE, Lemmas SA-5.4 and SA-3.1, and Corollary SA-3.1. The proof is complete.

□

SA-5.9 Proof of Theorem SA-3.3

Proof. We first show that for each fixed $x \in \mathcal{X}$,

$$\bar{\Omega}_{\mu^{(v)}}(x)^{-1/2}\widehat{\mathbf{b}}^{(v)}_{p,s}(x)'\bar{\mathbf{Q}}^{-1}\mathbb{G}_n[\widehat{\mathbf{b}}_{p,s}(x_i)\eta_{i,1}\psi(y_i,\eta_i)] =: \mathbb{G}_n[a_i\psi(y_i,\eta_i)]$$

is asymptotically normal. Conditional on $\mathscr{F}_{XW\Delta}$, the σ-field generated by $\{(x_i,\mathbf{w}_i)\}_{i=1}^n$ and $\widehat{\Delta}$, it is an independent mean-zero sequence over i with variance equal to 1. Then by Berry-Esseen inequality,

$$\sup_{u\in\mathbb{R}}\Big|\mathbb{P}(\mathbb{G}_n[a_i\psi(y_i,\eta_i)] \le u|) - \Phi(u)\Big| \le \min\left(1,\ \frac{\sum_{i=1}^n \mathbb{E}[|a_i\psi(y_i,\eta_i)|^3|\mathscr{F}_{XW\Delta}]}{n^{3/2}}\right).$$

By Lemmas SA-5.4, SA-3.1 and SA-3.2,

$$\begin{aligned}
&\frac{1}{n^{3/2}}\sum_{i=1}^n\mathbb{E}\Big[|a_i\psi(y_i,\eta_i)|^3\Big|\mathscr{F}_{XW\Delta}\Big]\\
&\lesssim \bar{\Omega}_{\mu^{(v)}}(x)^{-3/2}\frac{1}{n^{3/2}}\sum_{i=1}^n\mathbb{E}\Big[|\widehat{\mathbf{b}}^{(v)}_{p,s}(x)'\bar{\mathbf{Q}}^{-1}\widehat{\mathbf{b}}_{p,s}(x_i)\eta_{i,1}\psi(y_i,\eta_i)|^3\Big|\mathscr{F}_{XW\Delta}\Big]\\
&\lesssim \bar{\Omega}_{\mu^{(v)}}(x)^{-3/2}\frac{1}{n^{3/2}}\sum_{i=1}^n|\widehat{\mathbf{b}}^{(v)}_{p,s}(x)'\bar{\mathbf{Q}}^{-1}\widehat{\mathbf{b}}_{p,s}(x_i)|^3
\end{aligned}$$

$$
\begin{aligned}
&\leq \bar{\Omega}_{\mu^{(v)}}(x)^{-3/2}\frac{\sup_{x\in\mathcal{X}}\sup_{z\in\mathcal{X}}|\widehat{\mathbf{b}}^{(v)}_{p,s}(x)'\bar{\mathbf{Q}}^{-1}\widehat{\mathbf{b}}_{p,s}(z)|}{n^{3/2}}\sum_{i=1}^{n}|\widehat{\mathbf{b}}^{(v)}_{p,s}(x)'\bar{\mathbf{Q}}^{-1}\widehat{\mathbf{b}}_{p,s}(x_i)|^2\\
&\lesssim_{\mathbb{P}} \frac{1}{J^{3/2+3v}}\cdot\frac{J^{1+v}}{\sqrt{n}}\cdot J^{1+2v}\to 0
\end{aligned}
$$

since $J/n = o(1)$. By Theorem SA-3.2, the above weak convergence still holds if $\bar{\Omega}_{\mu^{(v)}}(x)$ is replaced by $\widehat{\Omega}_{\mu^{(v)}}(x)$. Then, the desired results follow by Theorem SA-3.1. □

SA-5.10 Proof of Theorem SA-3.4

Proof. We let $\widehat{\boldsymbol{\beta}}_0$ and $\widehat{r}_{0,v}$ be defined as in Lemma SA-5.5. By Lemmas SA-5.5 and SA-3.1, Theorem SA-3.1 and the results given in the proof of Lemma SA-3.4, we have

$$
\begin{aligned}
\widehat{\mu}^{(v)}(x)-\mu_0^{(v)}(x) &= \widehat{\mathbf{b}}_{p,s}(x_i)'(\widehat{\boldsymbol{\beta}}-\widehat{\boldsymbol{\beta}}_0)-\widehat{r}_{0,v}(x)\\
&= -\,\widehat{\mathbf{b}}^{(v)}_{p,s}(x)'\mathbf{Q}_0^{-1}\mathbb{E}_n[\widehat{\mathbf{b}}_{p,s}(x_i)\eta_{i,1}\psi(y_i,\eta_i)]-\widehat{\mathbf{b}}^{(v)}_{p,s}(x)'\mathbf{Q}_0^{-1}\mathbb{E}_n[\widehat{\mathbf{b}}_{p,s}(x_i)\eta_{i,1}\Psi(x_i,\mathbf{w}_i;\check{\eta}_i)]\\
&\qquad -\widehat{r}_{0,v}(x)+O_{\mathbb{P}}\Big(J^v\Big\{\Big(\frac{J\log n}{n}\Big)^{3/4}\sqrt{\log n}+J^{-\frac{p+1}{2}}\Big(\frac{J\log^2 n}{n}\Big)^{1/2}+\mathfrak{r}_\gamma\Big\}\Big),
\end{aligned}
$$

where $\check{\eta}_i=\eta(\widehat{\mathbf{b}}_{p,s}(x_i)'\widehat{\boldsymbol{\beta}}_0+\mathbf{w}_i'\boldsymbol{\gamma}_0)$. Recall that the $O_{\mathbb{P}}(\cdot)$ in the last line holds uniformly over $x\in\mathcal{X}$, and thus the integral of the squared remainder is $o_{\mathbb{P}}(J^{1+2v}/n+J^{-2(p+1-v)})$ by the rate condition imposed. Then,

$$
\begin{aligned}
\texttt{AISE}_{\mu^{(v)}} = \int_{\mathcal{X}}\Big(&\widehat{\mathbf{b}}^{(v)}_{p,s}(x)'\mathbf{Q}_0^{-1}\mathbb{E}_n[\widehat{\mathbf{b}}_{p,s}(x_i)\eta_{i,1}\psi(y_i,\eta_i)]\\
&+\widehat{\mathbf{b}}^{(v)}_{p,s}(x)'\mathbf{Q}_0^{-1}\mathbb{E}_n[\widehat{\mathbf{b}}_{p,s}(x_i)\eta_{i,1}\Psi(x_i,\mathbf{w}_i;\check{\eta}_i)]+\widehat{r}_{0,v}(x)\Big)^2\omega(x)dx.
\end{aligned}
$$

Next, taking conditional expectation given $\mathbf{X}$, $\mathbf{W}$ and $\widehat{\Delta}$ and using the argument in the proof of Lemma SA-3.1 again, we have

$$
\begin{aligned}
\mathbb{E}[\texttt{AISE}_{\mu^{(v)}}|\mathbf{X},\mathbf{W},\widehat{\Delta}] &= \frac{1}{n}\operatorname{trace}\Big(\mathbf{Q}_0^{-1}\boldsymbol{\Sigma}_0\mathbf{Q}_0^{-1}\int_{\mathcal{X}}\mathbf{b}^{(v)}_{p,s}(x)\mathbf{b}^{(v)}_{p,s}(x)'\omega(x)dx\Big)+o_{\mathbb{P}}(J^{2v+1}/n)\\
&\quad+\int_{\mathcal{X}}\Big(\widehat{\mathbf{b}}^{(v)}_{p,s}(x)'\widehat{\boldsymbol{\beta}}_0-\mu_0^{(v)}(x)\Big)^2\omega(x)dx\\
&\quad+\int_{\mathcal{X}}\Big(\widehat{\mathbf{b}}^{(v)}_{p,s}(x)'\mathbf{Q}_0^{-1}\mathbb{E}_n[\widehat{\mathbf{b}}_{p,s}(x_i)\eta_{i,1}\Psi(x_i,\mathbf{w}_i;\check{\eta}_i)]\Big)^2\omega(x)dx\\
&\quad+2\int_{\mathcal{X}}\widehat{\mathbf{b}}^{(v)}_{p,s}(x)'\mathbf{Q}_0^{-1}\mathbb{E}_n[\widehat{\mathbf{b}}_{p,s}(x_i)\eta_{i,1}\Psi(x_i,\mathbf{w}_i;\check{\eta}_i)]\widehat{r}_{0,v}(x)\omega(x)dx.
\end{aligned}
$$

By Assumption SA-SM, $\Psi(x_i, \mathbf{w}_i; \check{\eta}_i) = -\Psi_1(x_i, \mathbf{w}_i; \eta_{i,0})\eta_{i,1}\widehat{r}_0(x_i) + O_{\mathbb{P}}(J^{-2p-2})$ where $O_{\mathbb{P}}(\cdot)$ holds uniformly over i. The terms in the last three lines correspond to the integrated squared bias. Also, using the same argument in the proof of Lemma SA-3.1, $\mathbb{E}_n[\cdot]$ in the last two lines can be safely replaced by $\mathbb{E}_{\widehat{\Delta}}[\cdot]$, which only introduces some additional approximation error of order $o_{\mathbb{P}}(J^{-2p-2+2v})$.

The proof of Theorem SA-3.4 in Cattaneo et al. (2024b) shows that

$$
\begin{aligned}
\widehat{r}_{0,v}(x) &= \mu_0^{(v)}(x) - \widehat{\mathbf{b}}_{p,s}^{(v)}(x)'\widehat{\boldsymbol{\beta}}_0 \\
&= \frac{J^{-p-1+v}\mu_0^{(p+1)}(x)}{(p+1-v)!f_X(x)^{p+1-v}}\mathscr{E}_{p+1-v}\Big(\frac{x-\hat{\tau}_x^{\mathtt{L}}}{\hat{h}_x}\Big) \\
&\quad - J^{-p-1}\widehat{\mathbf{b}}_{p,s}^{(v)}(x)'\mathbf{Q}_0^{-1}\mathbf{T}_s\mathbb{E}_{\widehat{\Delta}}\Big[\widehat{\mathbf{b}}_{p,0}(x_i)\frac{\mu_0^{(p+1)}(x_i)}{(p+1)!f_X(x_i)^{p+1}}\mathscr{E}_{p+1}\Big(\frac{x_i-\hat{\tau}_{x_i}^{\mathtt{L}}}{\hat{h}_{x_i}}\Big)\Big] + o_{\mathbb{P}}(J^{-p-1+v}),
\end{aligned}
$$

where $\hat{\tau}_x^{\mathtt{L}}$ is the start of the (random) interval in $\widehat{\Delta}$ containing x and $\hat{h}_x$ denotes its length. Then, using the same argument as in the proof of Theorem SA-3.4 in Cattaneo et al. (2024b), we can approximate the integrated squared bias by the analogue based on the non-random partition Δ_0, i.e., $\int_{\mathcal{X}}(r_{0,v}^{\dagger}(x) - \mathbf{b}_{p,s}^{(v)}(x)'\mathbf{Q}_0^{-1}\mathbb{E}[\mathbf{b}_{p,s}(x_i)\varkappa(x_i,\mathbf{w}_i)r_{0,0}^{\dagger}(x_i)])^2\omega(x)dx$ where

$$
\begin{aligned}
r_{0,v}^{\dagger}(x) = &\frac{J^{-p-1+v}\mu_0^{(p+1)}(x)}{(p+1-v)!f_X(x)^{p+1-v}}\mathscr{E}_{p+1-v}\Big(\frac{x-\tau_x^{\mathtt{L}}}{h_x}\Big) \\
&- J^{-p-1}\widehat{\mathbf{b}}_{p,s}^{(v)}(x)'\mathbf{Q}_0^{-1}\mathbf{T}_s\mathbb{E}\Big[\mathbf{b}_{p,0}(x_i)\frac{\mu_0^{(p+1)}(x_i)}{(p+1)!f_X(x_i)^{p+1}}\mathscr{E}_{p+1}\Big(\frac{x_i-\tau_{x_i}^{\mathtt{L}}}{h_{x_i}}\Big)\Big].
\end{aligned}
$$

The expression of the bias term can be further simplified. For both $R_v(x) = r_{0,v}^{\dagger}(x)$ and $R_v(x) = r_{0,v}^{\star}(x)$, there exists some vector $\boldsymbol{\beta}$ such that $\sup_{x\in\mathcal{X}}|\mu_0(x) - \mathbf{b}_{p,s}(x_i)'\boldsymbol{\beta} - R_v(x)| = o(J^{-p-1+v})$ (see Lemma SA-5.5 and Lemma SA-6.1 of Cattaneo et al. (2020)). Define

$$
r_{0,v}^{\mathtt{P}}(x) = \mu_0^{(v)}(x) - \mathbf{b}_{p,s}^{(v)}(x)'\mathbf{Q}_0^{-1}\mathbb{E}[\mathbf{b}_{p,s}(x_i)\varkappa(x_i,\mathbf{w}_i)\mu_0(x_i)].
$$

Then, it follows that $r_{0,v}^{\mathtt{P}}(x) = R_v(x) - \mathbf{b}_{p,s}(x)'\mathbf{Q}_0^{-1}\mathbb{E}[\mathbf{b}_{p,s}(x_i)\varkappa(x_i,\mathbf{w}_i)R_0(x_i)] + o(J^{-p-1+v})$. Thus,

$$
\begin{aligned}
&\{r_{0,v}^{\dagger}(x) - \mathbf{b}_{p,s}^{(v)}(x)'\mathbf{Q}_0^{-1}\mathbb{E}[\mathbf{b}_{p,s}(x_i)\varkappa(x_i,\mathbf{w}_i)r_{0,0}^{\dagger}(x_i)]\} \\
&-\{[r_{0,v}^{\star}(x) - \mathbf{b}_{p,s}^{(v)}(x)'\mathbf{Q}_0^{-1}\mathbb{E}[\mathbf{b}_{p,s}(x_i)\varkappa(x_i,\mathbf{w}_i)r_{0,0}^{\star}(x_i)]]\} = o(J^{-p-1+v}).
\end{aligned}
$$

Therefore, the expression of $\mathscr{B}_n(p,s,v)$ given in the theorem holds.

Finally, the desired results in part (ii) and part (iii) follow by Theorem SA-3.1, the rate condition imposed and the same argument for part (i). $\square$

SA-5.11 Proof of Theorem SA-3.5

Proof. The proof is divided into several steps.

Step 1: Note that

$$\begin{aligned}
&\sup_{x\in\mathcal{X}}\left|\frac{\widehat{\mu}^{(v)}(x)-\mu_0^{(v)}(x)}{\sqrt{\widehat{\Omega}_{\mu^{(v)}}(x)/n}}-\frac{\widehat{\mu}^{(v)}(x)-\mu_0^{(v)}(x)}{\sqrt{\bar{\Omega}_{\mu^{(v)}}(x)/n}}\right| \\
\leq &\sup_{x\in\mathcal{X}}\left|\frac{\widehat{\mu}^{(v)}(x)-\mu_0^{(v)}(x)}{\sqrt{\bar{\Omega}_{\mu^{(v)}}(x)/n}}\right|\sup_{x\in\mathcal{X}}\left|\frac{\widehat{\Omega}_{\mu^{(v)}}(x)^{1/2}-\bar{\Omega}_{\mu^{(v)}}(x)^{1/2}}{\widehat{\Omega}_{\mu^{(v)}}(x)^{1/2}}\right| \\
\lesssim_{\mathbb{P}} &\Big(\sqrt{\log n}+\sqrt{n}J^{-p-1-1/2}\Big)\Big(J^{-p-1}+\sqrt{\frac{J\log n}{n^{1-\frac{2}{\nu}}}}\Big)
\end{aligned}$$

where the last step uses Lemma SA-3.2 and Corollary SA-3.1. Then, in view of Lemmas SA-5.5, SA-3.4, Theorems SA-3.1, SA-3.2 and the rate restriction given in the lemma, we have

$$\sup_{x\in\mathcal{X}}\left|\frac{\widehat{\mu}^{(v)}(x)-\mu_0^{(v)}(x)}{\sqrt{\widehat{\Omega}_{\mu^{(v)}}(x)/n}}+\frac{\widehat{\mathbf{b}}_{p,s}^{(v)}(x)'\bar{\mathbf{Q}}^{-1}}{\sqrt{\bar{\Omega}_{\mu^{(v)}}(x)}}\mathbb{G}_n[\widehat{\mathbf{b}}_{p,s}(x_i)\eta_{i,1}\psi(y_i,\eta_i)]\right|=o_{\mathbb{P}}(a_n^{-1}).$$

Step 2: Let us write $\mathscr{K}(x,x_i)=\Omega_{\mu^{(v)}}(x)^{-1/2}\widehat{\mathbf{b}}_{p,s}^{(v)}(x)'\bar{\mathbf{Q}}^{-1}\widehat{\mathbf{b}}_{p,s}(x_i)$ (the dependence of $\widehat{\mathbf{b}}_{p,s}^{(v)}(x)$, $\bar{\mathbf{Q}}$ and $\bar{\Omega}_{\mu^{(v)}}(x)$ on $\mathbf{X}$, $\mathbf{W}$ and $\widehat{\Delta}$ is omitted for simplicity), and $\tilde{\sigma}^2(x_i,\mathbf{w}_i)=\mathbb{E}[\psi^\dagger(\epsilon_i)^2|x_i,\mathbf{w}_i]$. Now we rearrange $\{x_i\}_{i=1}^n$ as a sequence of order statistics $\{x_{(i)}\}_{i=1}^n$, i.e., $x_{(1)}\leq\cdots\leq x_{(n)}$. Accordingly, $\{\epsilon_i\}_{i=1}^n$, $\{\mathbf{w}_i\}_{i=1}^n$ and $\{\tilde{\sigma}^2(x_i,\mathbf{w}_i)\}_{i=1}^n$ are ordered as concomitants $\{\epsilon_{[i]}\}_{i=1}^n$, $\{\mathbf{w}_{[i]}\}$ and $\{\tilde{\sigma}^2_{[i]}\}_{i=1}^n$ where $\tilde{\sigma}^2_{[i]}=\tilde{\sigma}^2(x_{(i)},\mathbf{w}_{[i]})$. Clearly, conditional on $\mathscr{F}_{XW\Delta}$ (the σ-field generated by $\{(x_i,\mathbf{w}_i)\}$ and $\widehat{\Delta}$), $\{\psi^\dagger(\epsilon_{[i]})\}_{i=1}^n$ is still an independent mean-zero sequence. Then by Assumptions SA-DGP, SA-SM and the result of Sakhanenko (1991), there exists a sequence of i.i.d. standard normal random variables $\{\zeta_{[i]}\}_{i=1}^n$ such that

$$\max_{1\leq\ell\leq n}|S_\ell|:=\max_{1\leq\ell\leq n}\left|\sum_{i=1}^{\ell}\eta^{(1)}(\mu_0(x_{(i)})+\mathbf{w}_{[i]}'\gamma_0)\psi^\ddagger(\eta(\mu_0(x_{(i)})+\mathbf{w}_{[i]}'\gamma_0))\psi^\dagger(\epsilon_{[i]})\right.$$

$$-\sum_{i=1}^{\ell}\eta^{(1)}(\mu_0(x_{(i)})+\mathbf{w}_{[i]}'\boldsymbol{\gamma}_0)\psi^{\ddagger}(\eta(\mu_0(x_{(i)})+\mathbf{w}_{[i]}'\boldsymbol{\gamma}_0))\tilde{\sigma}_{[i]}\zeta_{[i]}\Big| \lesssim_{\mathbb{P}} n^{\frac{1}{\nu}}.$$

Then, using summation by parts,

$$\begin{aligned}
&\sup_{x\in\mathcal{X}}\left|\sum_{i=1}^{n}\mathscr{K}(x,x_{(i)})\eta^{(1)}(\mu_0(x_{(i)})+\mathbf{w}_{[i]}'\boldsymbol{\gamma}_0)\psi^{\ddagger}(\eta(\mu_0(x_{(i)})+\mathbf{w}_{[i]}'\boldsymbol{\gamma}_0))(\psi^{\dagger}(\epsilon_{[i]})-\tilde{\sigma}_{[i]}\zeta_{[i]})\right|\\
&=\sup_{x\in\mathcal{X}}\left|\mathscr{K}(x,x_{(n)})S_n-\sum_{i=1}^{n-1}S_i\left(\mathscr{K}(x,x_{(i+1)})-\mathscr{K}(x,x_{(i)})\right)\right|\\
&\leq \sup_{x\in\mathcal{X}}\max_{1\leq i\leq n}|\mathscr{K}(x,x_i)||S_n|+\sup_{x\in\mathcal{X}}\left|\frac{\widehat{\mathbf{b}}_{p,s}^{(v)}(x)'\bar{\mathbf{Q}}^{-1}}{\sqrt{\bar{\Omega}_{\mu^{(v)}}(x)}}\sum_{i=1}^{n-1}S_i\Big(\widehat{\mathbf{b}}_{p,s}(x_{(i+1)})-\widehat{\mathbf{b}}_{p,s}(x_{(i)})\Big)\right|\\
&\leq \sup_{x\in\mathcal{X}}\max_{1\leq i\leq n}|\mathscr{K}(x,x_i)||S_n|+\sup_{x\in\mathcal{X}}\left\|\frac{\bar{\mathbf{Q}}^{-1}\widehat{\mathbf{b}}_{p,s}^{(v)}(x)}{\sqrt{\bar{\Omega}_{\mu^{(v)}}(x)}}\right\|_1\left\|\sum_{i=1}^{n-1}S_i\Big(\widehat{\mathbf{b}}_{p,s}(x_{(i+1)})-\widehat{\mathbf{b}}_{p,s}(x_{(i)})\Big)\right\|_\infty.
\end{aligned}$$

By Lemmas SA-5.4, SA-3.1 and SA-3.2, $\sup_{x\in\mathcal{X}}\sup_{x_i\in\mathcal{X}}|\mathscr{K}(x,x_i)|\lesssim_{\mathbb{P}}\sqrt{J}$, and

$$\sup_{x\in\mathcal{X}}\left\|\frac{\bar{\mathbf{Q}}^{-1}\widehat{\mathbf{b}}_{p,s}^{(v)}(x)}{\sqrt{\bar{\Omega}_{\mu^{(v)}}(x)}}\right\|_1\lesssim_{\mathbb{P}}1.$$

Then, notice that

$$\max_{1\leq l\leq K_{p,s}}\left|\sum_{i=1}^{n-1}\Big(\widehat{b}_{p,s,l}(x_{(i+1)})-\widehat{b}_{p,s,l}(x_{(i)})\Big)S_l\right|\leq\max_{1\leq l\leq K_{p,s}}\sum_{i=1}^{n-1}\left|\widehat{b}_{p,s,l}(x_{(i+1)})-\widehat{b}_{p,s,l}(x_{(i)})\right|\max_{1\leq\ell\leq n}\left|S_\ell\right|.$$

By construction of the ordering, $\max_{1\leq l\leq K_{p,s}}\sum_{i=1}^{n-1}\left|\widehat{b}_{p,s,l}(x_{(i+1)})-\widehat{b}_{p,s,l}(x_{(i)})\right|\lesssim\sqrt{J}$. Under the rate restriction in the theorem, this suffices to show that for any $\xi>0$,

$$\mathbb{P}\left(\sup_{x\in\mathcal{X}}\left|\mathbb{G}_n[\mathscr{K}(x,x_i)\eta^{(1)}(\mu_0(x_i)+\mathbf{w}_i'\boldsymbol{\gamma}_0)(\psi(y_i,\eta_i)-\sigma(x_i,\mathbf{w}_i)\zeta_i)]\right|>\xi a_n^{-1}\middle|\mathscr{F}_{XW\Delta}\right)=o_{\mathbb{P}}(1),$$

where we recover the original ordering. Since $\mathbb{G}_n[\widehat{\mathbf{b}}_{p,s}(x_i)\zeta_i\sigma(x_i,\mathbf{w}_i)\eta_{i,1}]=_{d|\mathscr{F}_{XW\Delta}}\mathbf{N}(0,\bar{\boldsymbol{\Sigma}})$ ($=_{d|\mathscr{F}_{XW}}$ denotes "equal in distribution conditional on $\mathscr{F}_{XW\Delta}$"), the above steps construct the following approximating process:

$$\bar{Z}_{\mu^{(v)},p}(x):=\frac{\widehat{\mathbf{b}}_{p,s}^{(v)}(x)'\bar{\mathbf{Q}}^{-1}}{\sqrt{\bar{\Omega}_{\mu^{(v)}}(x)}}\bar{\boldsymbol{\Sigma}}^{1/2}\mathbf{N}_{K_{p,s}}.$$

Step 3: Suppose that Assumption SA-RP(ii) also holds. Note that

$$
\begin{aligned}
&\sup_{x\in\mathcal{X}}|\bar{Z}_{\mu^{(v)},p}(x)-Z_{\mu^{(v)},p}(x)|\\
\leq&\sup_{x\in\mathcal{X}}\left|\frac{\widehat{\mathbf{b}}^{(v)}(x)'(\bar{\mathbf{Q}}^{-1}-\mathbf{Q}_0^{-1})}{\sqrt{\Omega_{\mu^{(v)}}(x)}}\bar{\mathbf{\Sigma}}^{1/2}\mathbf{N}_{K_{p,s}}\right|+\sup_{x\in\mathcal{X}}\left|\frac{\widehat{\mathbf{b}}^{(v)}(x)'\mathbf{Q}_0^{-1}}{\sqrt{\Omega_{\mu^{(v)}}(x)}}\Big(\bar{\mathbf{\Sigma}}^{1/2}-\mathbf{\Sigma}_0^{1/2}\Big)\mathbf{N}_{K_{p,s}}\right|+\\
&\sup_{x\in\mathcal{X}}\left|\frac{\widehat{\mathbf{b}}_{p,0}^{(v)}(x)'(\widehat{\mathbf{T}}_s-\mathbf{T}_s)\mathbf{Q}_0^{-1}}{\sqrt{\Omega_{\mu^{(v)}}(x)}}\mathbf{\Sigma}_0^{1/2}\mathbf{N}_{K_{p,s}}\right|+\sup_{x\in\mathcal{X}}\left|\Big(\frac{1}{\sqrt{\bar{\Omega}_{\mu^{(v)}}(x)}}-\frac{1}{\sqrt{\Omega_{\mu^{(v)}}(x)}}\Big)\widehat{\mathbf{b}}_{p,0}^{(v)}(x)'\widehat{\mathbf{T}}_s\bar{\mathbf{Q}}^{-1}\bar{\mathbf{\Sigma}}^{1/2}\mathbf{N}_{K_{p,s}}\right|\\
=&\,I+II+III+IV,
\end{aligned}
$$

where each term on the right-hand side is a mean-zero Gaussian process conditional on $\mathscr{F}_{XW\Delta}$. By Theorem SA-3.2 (see Step 4 of its proof), $\sup_{x\in\mathcal{X}}|\bar{\Omega}_{\mu^{(v)}}(x)-\Omega_{\mu^{(v)}}(x)|\lesssim_{\mathbb{P}} J^{1+2v}(\sqrt{J\log n/n}+\mathfrak{r}_{\mathtt{RP}})$. By a similar calculation given in Step 1 and the rate condition imposed, the last term is $o_{\mathbb{P}}(a_n^{-1})$. By Lemmas SA-5.3 and SA-3.1, $\|\bar{\mathbf{Q}}^{-1}-\mathbf{Q}_0^{-1}\|\lesssim_{\mathbb{P}}\sqrt{J\log J/n}$ and $\|\widehat{\mathbf{T}}_s-\mathbf{T}_s\|\lesssim_{\mathbb{P}}\sqrt{J\log J/n}$. Also, using the argument in the proof of Lemma SA-5.4 and Theorem X.3.8 of Bhatia (2013), $\|\bar{\mathbf{\Sigma}}^{1/2}-\mathbf{\Sigma}_0^{1/2}\|\lesssim_{\mathbb{P}}\sqrt{J\log J/n}$. By Gaussian Maximal Inequality (van der Vaart and Wellner, 1996, Corollary 2.2.8),

$$
\mathbb{E}\Big[I+II+III\Big|\mathscr{F}_{XW\Delta}\Big]\lesssim_{\mathbb{P}}\sqrt{\log J}\Big(\|\bar{\mathbf{\Sigma}}^{1/2}-\mathbf{\Sigma}_0^{1/2}\|+\|\bar{\mathbf{Q}}^{-1}-\mathbf{Q}_0^{-1}\|+\|\widehat{\mathbf{T}}_s-\mathbf{T}_s\|\Big)=o_{\mathbb{P}}(a_n^{-1})
$$

where the last line follows from the imposed rate restriction. Then the proof for part (i) is complete.

The results in parts (ii) and (iii) immediately follow by Theorem SA-3.1 and the fact that the leading variance term in the Bahadur representation for $\widehat{\vartheta}(x,\widehat{\mathsf{w}})$ or $\widehat{\zeta}(x,\widehat{\mathsf{w}})$ differs from that for $\widehat{\mu}(x)$ or $\widehat{\mu}^{(1)}(x)$ up to a sign only. □

SA-5.12 Proof of Theorem SA-3.6

Proof. This conclusion follows from Lemmas SA-5.4, SA-3.1, Theorem SA-3.2 and Gaussian Maximal Inequality as applied in Step 3 in the proof of Theorem SA-3.5. □

SA-5.13 Proof of Theorem SA-3.7

Proof. We first show that

$$\sup_{u\in\mathbb{R}}\Big|\mathbb{P}\Big(\sup_{x\in\mathcal{X}}|T_{\mu^{(v)},p}(x)|\leq u\Big)-\mathbb{P}\Big(\sup_{x\in\mathcal{X}}|Z_{\mu^{(v)},p}(x)|\leq u\Big)\Big|=o(1).$$

By Theorem SA-3.5, there exists a sequence of constants ξ_n such that $\xi_n=o(1)$ and

$$\mathbb{P}\Big(\Big|\sup_{x\in\mathcal{X}}|T_{\mu^{(v)},p}(x)|-\sup_{x\in\mathcal{X}}|Z_{\mu^{(v)},p}(x)|\Big|>\xi_n/a_n\Big)=o(1).$$

Then,

$$\begin{aligned}\mathbb{P}\Big(\sup_{x\in\mathcal{X}}|T_{\mu^{(v)},p}(x)|\leq u\Big)&\leq\mathbb{P}\Big(\Big\{\sup_{x\in\mathcal{X}}|T_{\mu^{(v)},p}(x)|\leq u\Big\}\cap\Big\{\Big|\sup_{x\in\mathcal{X}}|T_{\mu^{(v)},p}(x)|-\sup_{x\in\mathcal{X}}|Z_{\mu^{(v)},p}(x)|\Big|\leq\xi_n/a_n\Big\}\Big)+o(1)\\&\leq\mathbb{P}\Big(\sup_{x\in\mathcal{X}}|Z_{\mu^{(v)},p}(x)|\leq u+\xi_n/a_n\Big)+o(1)\\&\leq\mathbb{P}\Big(\sup_{x\in\mathcal{X}}|Z_{\mu^{(v)},p}(x)|\leq u\Big)+\sup_{u\in\mathbb{R}}\mathbb{E}\Big[\mathbb{P}\Big(\Big|\sup_{x\in\mathcal{X}}|Z_{\mu^{(v)},p}(x)|-u\Big|\leq\xi_n/a_n\Big|\widehat{\Delta}\Big)\Big]\\&\leq\mathbb{P}\Big(\sup_{x\in\mathcal{X}}|Z_{\mu^{(v)},p}(x)|\leq u\Big)+\mathbb{E}\Big[\sup_{u\in\mathbb{R}}\mathbb{P}\Big(\Big|\sup_{x\in\mathcal{X}}|Z_{\mu^{(v)},p}(x)|-u\Big|\leq\xi_n/a_n\Big|\widehat{\Delta}\Big)\Big]+o(1).\end{aligned}$$

Apply the Anti-Concentration Inequality conditional on $\widehat{\Delta}$ (Chernozhukov et al., 2014) to the second term:

$$\begin{aligned}\sup_{u\in\mathbb{R}}\mathbb{P}\Big(\Big|\sup_{x\in\mathcal{X}}|Z_{\mu^{(v)},p}(x)|-u\Big|\leq\xi_n/a_n\Big|\widehat{\Delta}\Big)&\leq 4\xi_n a_n^{-1}\mathbb{E}\Big[\sup_{x\in\mathcal{X}}|Z_{\mu^{(v)},p}(x)|\Big|\widehat{\Delta}\Big]+o(1)\\&\lesssim_{\mathbb{P}}\xi_n a_n^{-1}\sqrt{\log J}+o(1)\to 0\end{aligned}$$

where the last step uses Gaussian Maximal Inequality (see van der Vaart and Wellner, 1996, Corollary 2.2.8). By Dominated Convergence Theorem,

$$\mathbb{E}\Big[\sup_{u\in\mathbb{R}}\mathbb{P}\Big(\Big|\sup_{x\in\mathcal{X}}|Z_{\mu^{(v)},p}(x)|-u\Big|\leq\xi_n/a_n\Big|\widehat{\Delta}\Big)\Big]=o(1).$$

The other side of the inequality follows similarly.

By similar argument, using Theorem SA-3.6, we have

$$\sup_{u\in\mathbb{R}}\left|\mathbb{P}\Big(\sup_{x\in\mathcal{X}}|\widehat{Z}_{\mu^{(v)},p}(x)|\leq u\Big|\mathbf{D},\widehat{\Delta}\Big)-\mathbb{P}\Big(\sup_{x\in\mathcal{X}}|Z_{\mu^{(v)},p}(x)|\leq u\Big|\widehat{\Delta}\Big)\right|=o_{\mathbb{P}}(1).$$

Then, it remains to show that

$$\sup_{u\in\mathbb{R}}\left|\mathbb{P}\Big(\sup_{x\in\mathcal{X}}|Z_{\mu^{(v)},p}(x)|\leq u\Big)-\mathbb{P}\Big(\sup_{x\in\mathcal{X}}|Z_{\mu^{(v)},p}(x)|\leq u|\widehat{\Delta}\Big)\right|=o_{\mathbb{P}}(1). \tag{SA-5.1}$$

We can write

$$Z_{\mu^{(v)},p}(x)=\frac{\widehat{\mathbf{b}}^{(v)}_{p,0}(x)'}{\sqrt{\widehat{\mathbf{b}}^{(v)}_{p,0}(x)'\mathbf{V}_0\widehat{\mathbf{b}}^{(v)}_{p,0}(x)}}\breve{\mathbf{N}}_{K_{p,0}}$$

where $\mathbf{V}_0=\mathbf{T}_s'\mathbf{Q}_0^{-1}\mathbf{\Sigma}_0\mathbf{Q}_0^{-1}\mathbf{T}_s$ and $\breve{\mathbf{N}}_{K_{p,0}}:=\mathbf{T}_s'\mathbf{Q}_0^{-1}\mathbf{\Sigma}_0^{1/2}\mathbf{N}_{K_{p,s}}$ is a $K_{p,0}$-dimensional Gaussian random vector. Importantly, by this construction, $\breve{\mathbf{N}}_{K_{p,0}}$ and $\mathbf{V}_0$ do not depend on $\widehat{\Delta}$ and x, and they are only determined by the deterministic partition Δ_0.

First consider $v=0$. For any two partitions $\Delta_1,\Delta_2\in\Pi$, for any $x\in\mathcal{X}$, there exists $\breve{x}\in\mathcal{X}$ such that

$$\mathbf{b}^{(0)}_{p,0}(x;\Delta_1)=\mathbf{b}^{(0)}_{p,0}(\breve{x};\Delta_2),$$

and vice versa. Therefore, the following two events are equivalent: $\{\omega:\sup_{x\in\mathcal{X}}|Z_p(x;\Delta_1)|\leq u\}=\{\omega:\sup_{x\in\mathcal{X}}|Z_p(x;\Delta_2)|\leq u\}$ for any u. Thus,

$$\mathbb{E}\Big[\mathbb{P}\Big(\sup_{x\in\mathcal{X}}|Z_{\mu^{(v)},p}(x)|\leq u\Big|\widehat{\Delta}\Big)\Big]=\mathbb{P}\Big(\sup_{x\in\mathcal{X}}|Z_{\mu^{(v)},p}(x)|\leq u\Big|\widehat{\Delta}\Big)+o_{\mathbb{P}}(1).$$

Then for $v=0$, the desired result follows.

For $v>0$, simply notice that $\widehat{\mathbf{b}}^{(v)}_{p,0}(x)=\widehat{\mathfrak{T}}_v\widehat{\mathbf{b}}_{p,0}(x)$ for some transformation matrix $\widehat{\mathfrak{T}}_v$. Clearly, $\widehat{\mathfrak{T}}_v$ takes a similar structure as $\widehat{\mathbf{T}}_s$: each row and each column only have a finite number of nonzeros. Each nonzero element is simply $\hat{h}_j^{-v}$ up to some constants. By Lemma SA-5.2, it can be shown that $\|\widehat{\mathfrak{T}}_v-\mathfrak{T}_v\|\lesssim J^v\sqrt{J\log J/n}$ where $\mathfrak{T}_v$ is the population analogue ($\hat{h}_j$ replaced by h_j). Repeating the argument given in the proof of Theorems SA-3.5 and SA-3.6, we can replace $\widehat{\mathfrak{T}}_v$ in $Z_{\mu^{(v)},p}(x)$ by $\mathfrak{T}_v$ without affecting the approximation rate. Then the desired result for $T_{\mu^{(v)},p}(x)$ follows by repeating the argument given for $v=0$ above.

Finally, the result for $T_{\vartheta,p}(x)$ ($T_{\zeta,p}(x)$) follows by the fact that $Z_{\vartheta,p}(x)$ and $\widehat{Z}_{\vartheta,p}(x)$ ($Z_{\zeta,p}(x)$ and $\widehat{Z}_{\zeta,p}(x)$) differ from $Z_{\mu^{(v)},p}(x)$ and $\widehat{Z}_{\mu^{(v)},p}(x)$ up to a sign only. $\square$

SA-5.14 Proof of Theorem SA-3.8

Proof. We only consider $\widehat{I}_{\mu^{(v)},p}(x)$. The results in part (ii) and part (iii) follow similarly.

Let $\xi_{1,n}=o(1)$, $\xi_{2,n}=o(1)$ and $\xi_{3,n}=o(1)$. Then,

$$\begin{aligned}\mathbb{P}\left[\sup_{x\in\mathcal{X}}|T_{\mu^{(v)},p}(x)|\leq\mathfrak{c}_{\mu^{(v)}}\right]&\leq\mathbb{P}\left[\sup_{x\in\mathcal{X}}|\bar{Z}_{\mu^{(v)},p}(x)|\leq\mathfrak{c}_{\mu^{(v)},p}+\xi_{1,n}/a_n\right]+o(1)\\&\leq\mathbb{P}\left[\sup_{x\in\mathcal{X}}|\bar{Z}_{\mu^{(v)},p}(x)|\leq c^0(1-\alpha+\xi_{3,n})+(\xi_{1,n}+\xi_{2,n})/a_n\right]+o(1)\\&\leq\mathbb{P}\left[\sup_{x\in\mathcal{X}}|\bar{Z}_{\mu^{(v)},p}(x)|\leq c^0(1-\alpha+\xi_{3,n})\right]+o(1)\to1-\alpha,\end{aligned}$$

where $c^0(1-\alpha+\xi_{3,n})$ denotes the $(1-\alpha+\xi_{3,n})$-quantile of $\sup_{x\in\mathcal{X}}|\bar{Z}_{\mu^{(v)},p}(x)|$ conditional on $\mathscr{F}_{XW\Delta}$ (the σ-field generated by $\mathbf{X}$, $\mathbf{W}$ and the partition $\widehat{\Delta}$), the first inequality holds by Theorem SA-3.5, the second by Lemma A.1 of Belloni et al. (2015), and the third by Anti-Concentration Inequality in Chernozhukov et al. (2014). The other side of the bound follows similarly. $\square$

SA-5.15 Proof of Theorem SA-3.9

Proof. We only consider the proof for part (i). The results in part (ii) and part (iii) follow similarly.

Throughout this proof, we let $\xi_{1,n}=o(1)$, $\xi_{2,n}=o(1)$ and $\xi_{3,n}=o(1)$ be sequences of vanishing constants. Moreover, let A_n be a sequence of diverging constants such that $\sqrt{\log J}A_n\lesssim\sqrt{\frac{n}{J^{1+2v}}}$. Note that

$$\sup_{x\in\mathcal{X}}|\dot{T}_{\mu^{(v)},p}(x)|\leq\sup_{x\in\mathcal{X}}\left|\frac{\widehat{\mu}(x)-\mu_0^{(v)}(x)}{\sqrt{\widehat{\Omega}_{\mu^{(v)}}(x)/n}}\right|+\sup_{x\in\mathcal{X}}\left|\frac{\mu_0^{(v)}(x)-m^{(v)}(x;\widetilde{\boldsymbol{\theta}})}{\sqrt{\widehat{\Omega}_{\mu^{(v)}}(x)/n}}\right|.$$

Therefore, under $\dot{\mathsf{H}}_0^{\mu^{(v)}}$,

$$\begin{aligned}\mathbb{P}\Big[\sup_{x\in\mathcal{X}}|\dot{T}_{\mu^{(v)},p}(x)|>\mathfrak{c}_{\mu^{(v)}}\Big]&\leq\mathbb{P}\Big[\sup_{x\in\mathcal{X}}|T_{\mu^{(v)},p}(x)|>\mathfrak{c}_{\mu^{(v)}}-\sup_{x\in\mathcal{X}}\left|\frac{\mu_0^{(v)}(x)-m^{(v)}(x;\widetilde{\boldsymbol{\theta}})}{\sqrt{\widehat{\Omega}_{\mu^{(v)}}(x)/n}}\right|\Big]\\&\leq\mathbb{P}\Big[\sup_{x\in\mathcal{X}}|\bar{Z}_{\mu^{(v)},p}(x)|>\mathfrak{c}_{\mu^{(v)}}-\xi_{1,n}/a_n-\sup_{x\in\mathcal{X}}\left|\frac{\mu_0^{(v)}(x)-m^{(v)}(x;\widetilde{\boldsymbol{\theta}})}{\sqrt{\widehat{\Omega}_{\mu^{(v)}}(x)/n}}\right|\Big]+o(1)\end{aligned}$$

$$
\begin{aligned}
&\leq \mathbb{P}\Big[\sup_{x\in\mathcal{X}}|\bar{Z}_{\mu^{(v)},p}(x)| > c^0(1-\alpha-\xi_{3,n}) - (\xi_{1,n}+\xi_{2,n})/a_n - \\
&\qquad \sup_{x\in\mathcal{X}}\Big|\frac{\mu_0^{(v)}(x) - m^{(v)}(x;\widetilde{\boldsymbol{\theta}})}{\sqrt{\widehat{\Omega}_{\mu^{(v)}}(x)/n}}\Big|\Big] + o(1) \\
&\leq \mathbb{P}\Big[\sup_{x\in\mathcal{X}}|\bar{Z}_{\mu^{(v)},p}(x)| > c^0(1-\alpha-\xi_{3,n})\Big] + o(1) \\
&= \alpha + o(1)
\end{aligned}
$$

where $c^0(1-\alpha-\xi_{3,n})$ denotes the $(1-\alpha-\xi_{3,n})$-quantile of $\sup_{x\in\mathcal{X}}|\bar{Z}_{\mu^{(v)},p}(x)|$ conditional on $\mathscr{F}_{XW\Delta}$ (the σ-field generated by $\mathbf{X}$, $\mathbf{W}$ and $\widehat{\Delta}$), the second inequality holds by Theorem SA-3.5, the third by Lemma A.1 of Belloni et al. (2015), the fourth by the fact that $\sup_{x\in\mathcal{X}}\big|\frac{\mu_0^{(v)}(x)-m^{(v)}(x;\widetilde{\boldsymbol{\theta}})}{\sqrt{\widehat{\Omega}_{\mu^{(v)}}(x)/n}}\big| = o_{\mathbb{P}}(\frac{1}{\sqrt{\log J}})$ and Anti-Concentration Inequality in Chernozhukov et al. (2014). The other side of the bound follows similarly.

On the other hand, under $\dot{\mathsf{H}}_{\mathrm{A}}^{\mu^{(v)}}$,

$$
\begin{aligned}
&\mathbb{P}\Big[\sup_{x\in\mathcal{X}}|\dot{T}_{\mu^{(v)},p}(x)| > \mathfrak{c}_{\mu^{(v)}}\Big] \\
=&\,\mathbb{P}\Big[\sup_{x\in\mathcal{X}}\Big|T_{\mu^{(v)},p}(x) + \frac{\mu_0^{(v)}(x) - m^{(v)}(x;\bar{\boldsymbol{\theta}})}{\sqrt{\widehat{\Omega}_{\mu^{(v)}}(x)/n}} + \frac{m^{(v)}(x;\bar{\boldsymbol{\theta}}) - m^{(v)}(x;\widetilde{\boldsymbol{\theta}})}{\sqrt{\widehat{\Omega}_{\mu^{(v)}}(x)/n}}\Big| > \mathfrak{c}_{\mu^{(v)}}\Big] \\
\geq&\,\mathbb{P}\Big[\sup_{x\in\mathcal{X}}|T_{\mu^{(v)},p}(x)| < \sup_{x\in\mathcal{X}}\Big|\frac{\mu_0^{(v)}(x) - m^{(v)}(x;\bar{\boldsymbol{\theta}})}{\sqrt{\widehat{\Omega}_{\mu^{(v)}}(x)/n}} + \frac{m^{(v)}(x;\bar{\boldsymbol{\theta}}) - m^{(v)}(x;\widetilde{\boldsymbol{\theta}})}{\sqrt{\widehat{\Omega}_{\mu^{(v)}}(x)/n}}\Big| - \mathfrak{c}_{\mu^{(v)}}\Big] \\
\geq&\,\mathbb{P}\Big[\sup_{x\in\mathcal{X}}|\bar{Z}_{\mu^{(v)},p}(x)| \leq \sqrt{\log J}A_n - \xi_{1,n}/a_n\Big] - o(1) \\
\geq&\,1 - o(1).
\end{aligned}
$$

where the fourth line holds by Lemma SA-3.2, Theorem SA-3.2, Theorem SA-3.5, the condition that $J^v\sqrt{J\log J/n} = o(1)$ and the definition of A_n, and the last by the Talagrand-Samorodnitsky Concentration Inequality (van der Vaart and Wellner, 1996, Proposition A.2.7). □

SA-5.16 Proof of Theorem SA-3.10

Proof. We only consider the proof for part (i). The results in part (ii) and part (iii) follow similarly.

Throughout this proof, the definitions of A_n, $\xi_{1,n}, \xi_{2,n}$ and $\xi_{3,n}$ are the same as in the proof of

Theorem SA-3.9. Under $\ddot{\mathsf{H}}_0^{\mu^{(v)}}$,

$$\sup_{x\in\mathcal{X}} \ddot{T}_{\mu^{(v)},p}(x) \leq \sup_{x\in\mathcal{X}} T_{\mu^{(v)},p}(x) + \sup_{x\in\mathcal{X}} \frac{|m^{(v)}(x;\bar{\boldsymbol{\theta}}) - m^{(v)}(x;\widetilde{\boldsymbol{\theta}})|}{\sqrt{\widehat{\Omega}_{\mu^{(v)}}(x)/n}}.$$

Then,

$$\begin{aligned}
\mathbb{P}\Big[\sup_{x\in\mathcal{X}} \ddot{T}_{\mu^{(v)},p}(x) > \mathfrak{c}_{\mu^{(v)}}\Big] &\leq \mathbb{P}\Big[\sup_{x\in\mathcal{X}} T_{\mu^{(v)},p}(x) > \mathfrak{c}_{\mu^{(v)}} - \sup_{x\in\mathcal{X}} \frac{|m^{(v)}(x;\bar{\boldsymbol{\theta}}) - m^{(v)}(x;\widetilde{\boldsymbol{\theta}})|}{\sqrt{\widehat{\Omega}_{\mu^{(v)}}(x)/n}}\Big] \\
&\leq \mathbb{P}\Big[\sup_{x\in\mathcal{X}} \bar{Z}_{\mu^{(v)},p}(x) > \mathfrak{c}_{\mu^{(v)}} - \xi_{1,n}/a_n\Big] + o(1) \\
&\leq \mathbb{P}\Big[\sup_{x\in\mathcal{X}} \bar{Z}_{\mu^{(v)},p}(x) > c^0(1-\alpha-\xi_{3,n}) - (\xi_{1,n}+\xi_{2,n})/a_n\Big] + o(1) \\
&\leq \mathbb{P}\Big[\sup_{x\in\mathcal{X}} \bar{Z}_{\mu^{(v)},p}(x) > c^0(1-\alpha-\xi_{3,n})\Big] + o(1) \\
&= \alpha + o(1)
\end{aligned}$$

where $c^0(1-\alpha-\xi_{3,n})$ denotes the $(1-\alpha-\xi_{3,n})$-quantile of $\sup_{x\in\mathcal{X}} \bar{Z}_{\mu^{(v)},p}(x)$ conditional on $\mathscr{F}_{XW\Delta}$ (the σ-field generated by $\mathbf{X}$, $\mathbf{W}$ and $\widehat{\Delta}$), the second line holds by Theorem SA-3.5, the third by Lemma A.1 of Belloni et al. (2015), the fourth by Anti-Concentration Inequality in Chernozhukov et al. (2014).

On the other hand, under $\ddot{\mathsf{H}}_{\mathrm{A}}^{\mu^{(v)}}$,

$$\begin{aligned}
\mathbb{P}\Big[\sup_{x\in\mathcal{X}} \ddot{T}_{\mu^{(v)},p}(x) > \mathfrak{c}_{\mu^{(v)}}\Big] &= \mathbb{P}\Big[\sup_{x\in\mathcal{X}} \Big(T_{\mu^{(v)},p}(x) + \frac{\mu_0^{(v)}(x) - m^{(v)}(x;\widetilde{\boldsymbol{\theta}})}{\sqrt{\widehat{\Omega}_{\mu^{(v)}}(x)/n}} - \mathfrak{c}_{\mu^{(v)}}\Big) > 0\Big] \\
&\geq \mathbb{P}\Big[\sup_{x\in\mathcal{X}} |T_{\mu^{(v)},p}(x)| < \sup_{x\in\mathcal{X}} \frac{\mu_0^{(v)}(x) - m^{(v)}(x;\widetilde{\boldsymbol{\theta}})}{\sqrt{\widehat{\Omega}_{\mu^{(v)}}(x)/n}} - \mathfrak{c}_{\mu^{(v)}}, \\
&\qquad \sup_{x\in\mathcal{X}} \frac{\mu_0^{(v)}(x) - m^{(v)}(x;\widetilde{\boldsymbol{\theta}})}{\sqrt{\widehat{\Omega}_{\mu^{(v)}}(x)/n}} > \mathfrak{c}_{\mu^{(v)}}\Big] \\
&\geq \mathbb{P}\Big[\sup_{x\in\mathcal{X}} |T_{\mu^{(v)},p}(x)| < \sup_{x\in\mathcal{X}} \frac{\mu_0^{(v)}(x) - m^{(v)}(x;\widetilde{\boldsymbol{\theta}})}{\sqrt{\widehat{\Omega}_{\mu^{(v)}}(x)/n}} - \mathfrak{c}_{\mu^{(v)}}\Big] - o(1) \\
&\geq \mathbb{P}\Big[\sup_{x\in\mathcal{X}} |T_{\mu^{(v)},p}(x)| < \sqrt{\log J} A_n\Big] - o(1) \\
&\geq \mathbb{P}\Big[\sup_{x\in\mathcal{X}} |\bar{Z}_{\mu^{(v)},p}(x)| < \sqrt{\log J} A_n - \xi_{1,n}/a_n\Big] - o(1)
\end{aligned}$$

$$\geq 1 - o(1)$$

where the third line holds by Lemma SA-3.2, Theorem SA-3.2, Lemma A.1 of Belloni et al. (2015), the assumption that $\sup_{x\in\mathcal{X}} |m^{(v)}(x; \widetilde{\boldsymbol{\theta}}) - m^{(v)}(x; \bar{\boldsymbol{\theta}})| = o_{\mathbb{P}}(1)$ and $J^v\sqrt{J \log J/n} = o(1)$, the fourth by definition of A_n, and the fifth by Theorem SA-3.5, and the last by Proposition A.2.7 in van der Vaart and Wellner (1996).

$\square$

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