Federal Reserve Bank of New York
Staff Reports

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Staff Report no. 349
October 2008
Revised November 2010

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JEL classification: G12, D40

Abstract

This paper studies the relationship between the endogenous arrival of investors to a market and liquidity in a search-based model of asset trading. Entry of investors causes two contradictory effects. First, it reduces trading costs, which attracts new investors (the externality effect). But second, as investors concentrate on one side of the market, the market becomes "congested," decreasing the returns to investing and discouraging new investors from entering (the congestion effect). The equilibrium level of liquidity depends on which of the two effects dominates. When congestion is the leading effect, some interesting results arise. In particular, diminishing trading costs can deteriorate liquidity and welfare.

Key words: liquidity, search, congestion, asset pricing

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1 Introduction

Liquidity is sometimes defined as a coordination phenomenon. In financial markets, as investors move into a specific market they facilitate trade for all investors by reducing the cost of participating in this market. At the same time, easier trade and lower trading costs attract potential investors. There is a market externality where new investors provide market liquidity and market liquidity attracts new investors. However, if investors prefer to join one side of a market, i.e. as they become buyers or sellers, this side of the market becomes "congested", hindering trade. Congestion then discourages investors from entering this market.

One-sided markets arise during financial booms and, more drastically, during market crashes. When a market is in distress, liquidity typically vanishes playing a key role in the build-up of one-sided markets. The study of liquidity in one-sided markets is thus vital to understand the response of financial systems to the threat of market disruptions. Recent episodes of market distress include the LTCM crisis[1] in 1998, the September 11, 2001, events[2] and the 2007-08 turbulence in financial markets.[3]

In this paper we present an alternative view of market liquidity. The main difference with the previous literature is that we consider not only a market externality but also a *congestion effect*. In our model, the arrival of new investors causes two opposite effects. First, it diminishes transaction costs and eases trade, which attracts potential investors. But secondly, if investors concentrate on one side of the market, trade becomes more difficult,

[1]For an analysis of the events surrounding the market turbulence in autumn 1998, see BIS (1999) and IMF (1998).

[2]Cohen and Remolona (2001) presents a summary of the September 11, 2001 episode in global financial markets. Also, McAndrews and Potter (2002) gives a detailed account of the consequences of the September 11, 2001, events on the US payment system and of the actions of the Federal Reserve System to provide liquidity to the financial system.

[3]Greenlaw et al. (2008) and Brunnermeier (2008) provide a detailed analysis of this episode.

reducing the returns to participating in this market and discouraging potential investors from entering. Market liquidity thus results from the tradeoff between market externalities and a congestion effect.

We assume an infinite-horizon steady-state market where agents can invest in one asset which can be traded only bilaterally. In this market, investors cannot trade instantaneously but it takes some time to find a trading partner resulting in opportunity and other costs. Once an investor buys the asset, he holds it until his preference for the ownership changes and he prefers to liquidate the investment and exit the market. To model the search process we adopt the framework introduced in Vayanos and Wang (2007). In our setting though, investors are heterogeneous in their investment opportunities in the sense that some investors have access to better investment options than others.

We compute explicitly the unique equilibrium allocations and the price at which investors trade with each other and show how they depend on the flow of new investors entering the market. Prices negotiated between investors are higher in the flow of potential investors. However, investors' entry decision is endogenous and thus depends on market, asset and investors characteristics. A change in investors' search abilities, for instance, affects both the rate of meetings between trading partners and the flow of investors entering the market, which then determines the distribution of potential partners with whom they can meet.

Moreover, the equilibrium flow of investors arises from a tradeoff between market complementarities and a congestion effect. When congestion is the dominating effect some interesting results come to light. First, *diminishing market frictions can deteriorate market liquidity and reduce the welfare of current generations of investors.* The reason for this counterintuitive result is the following. In a one-sided market with more sellers than buyers, introducing a measure that improves the efficiency of the search process makes it easier for one of the few buyers present in the market to acquire the asset. But when the buyer purchases the asset (and a seller exits), the proportion of buyers to sellers falls further and the market becomes more one-sided. As investors cluster on the sell-side of this market, buyers gain a more favorable position in the bargaining process and try to lower the price they pay to acquire the asset. Reducing market frictions in a distressed market thus magnifies the effect of congestion and results in a lower asset price (a higher price discount) and ultimately in a less liquid market. Investors who hold this asset and those trying to sell it are clearly worse-off as the market becomes more one-sided, leading to a decrease in welfare.

From this point of view, this paper provides an example of the Theory of the Second Best. Improving the efficiency of the search process, when there are other imperfections in the market such as the ones arising from the congestion effect, is not necessarily welfare enhancing. Stated differently, the paper offers a rationale for measures such as trading halts and circuit breakers which can halt trading when there is a significant imbalance in the pending buy and sell orders in a security or when markets decline beyond trigger levels respectively. For instance, circuit breakers were adopted by the New York Stock Exchange (NYSE) following the 1987 stock market crash to reduce volatility and promote investor confidence.

Second, market illiquidity measured by the price discount can increase while trading volume rises. Reducing search frictions during downswings amplifies the effect of congestion, resulting in a higher price discount and in a less liquid market. But a more efficient search process also increases the frequency of meetings between the investors already present in the market. More frequent meetings then translates into a higher trading volume. Thereby, a measure intended to shorten the waiting times needed to locate a trading partner in a market experiencing distressed selling can cause both higher price discount and higher trading volume. This second result joins the discussion on the measurement of the effect of liquidity on asset prices and shows how alternative measures capture different dimensions of market liquidity.

The outline of the paper is as follows. In the next section, we discuss the related literature. We introduce a theoretical framework to examine the relationship between market liquidity and the arrival of potential investors to this market in Section 3. Section 4 first determines the population of investors, their expected utilities and the price of the asset, taking as given investors' decision to enter the market. Then, Subsection 4.3 endogenizes the entering rule. Section 5 characterizes the study of the unique market equilibrium and solves the equilibrium in closed-form when search frictions are small. We complement the analysis with a numerical example in Section 6 to illustrate the effect of changes in frictions on liquidity and welfare. Finally, Section 7 concludes. Some proofs and additional results are presented in the appendices.

2 Related Literature

The impact of strategic complementarities on market liquidity has been previously studied in Brunnermeier and Pedersen (2009), Gromb and Vayanos (2002), Pagano (1989), Dow (2004) and Plantin (2004), among others. From a broad perspective, this literature studies liquidity as a self-fulfilling phenomenon where both liquid and illiquid market equilibria may arise. Illiquid markets are thus a consequence of a coordination failure. In our paper, market liquidity results from a trade-off between market externalities and a congestion effect.

Our paper is also related to the search literature. The economics of search have their roots in Phelps (1972). Search-theoretic models such as the frameworks introduced in labor markets[4] by Diamond (1982a), Diamond (1982b), Mortensen (1982) and Pissarides (1985) have been broadly used in different areas of economics. In asset pricing[5], Duffie, Gârleanu and Pedersen introduce search and bargaining in models of asset market equilibrium to study the impact of these sources of illiquidity on asset prices. Our paper is related to Duffie et al. (2005), which presents a theory of asset pricing and marketmaking in over-the-counter markets with search-based inefficiencies. They conclude that risk neutral investors receive narrower bid-ask spreads if they have easier access to other investors and marketmakers. Similarly to Duffie et al. (2005) we consider risk-neutral agents who can only invest in one asset. In our model though, investors can only trade with other investors and our focus, rather than on liquidity and marketmaking, lies on the endogenous relationship between market liquidity and the arrival of potential investors to this market.

Duffie et al. (2007) extends their setting to incorporate risk aversion and risk limits and finds that, under certain conditions, search frictions as well as risk aversion, volatility and hedging demand increase the illiquidity discount. Lagos and Rocheteau (2009) also generalizes Duffie et al. (2005) to allow for general preferences and unrestricted long positions. Unrestricted (positive) asset holdings and free entry of dealers are introduced in Lagos and Rocheteau (2007). Our paper shares with theirs the existence of strategic complementarities and an endogenous entry decision. Lagos and Rocheteau (2007) assumes free entry of dealers and specifies that the contact rate between investors and dealers increases sublinearly in the number of dealers. In our framework, entry is the result of a decision problem where investors compare the benefits derived from investing in this market to their best alternative investment opportunities.

Weill (2008) and Vayanos and Wang (2007) extend the framework of Duffie, Gârleanu and

[4]See Pissarides (2001) for a review of the literature on search in labor markets.
[5]For an excellent review on liquidity and asset prices, see Amihud et al. (2005).

Pedersen to allow investors to trade multiple assets.[6] They show that search frictions lead to cross-sectional variation in asset returns due to illiquidity differences. In Vayanos and Wang (2007) investors are heterogeneous in their trading horizons while in Weill (2008) investors are homogeneous, but there are differences in the assets' number of tradable shares. From a methodological point of view, our paper is closely related to Vayanos and Wang (2007). The main difference with their work is that we consider only one asset and focus on the analysis of the liquidity in the market for this asset rather than on the liquidity across two assets.

This paper also relates to the literature on asset pricing with exogenous trading costs studied in Amihud and Mendelson (1986), Vayanos (1998) and Acharya and Pedersen (2005), among others. We complement this literature by endogenizing transaction costs.

3 The Model

Time is continuous and goes from zero to infinity. There is only one asset traded in the market with a total supply S. This asset pays a dividend flow d.

Investors are risk-neutral, infinitely lived and have time preferences determined by a constant discount rate equal to $r > 0$. Investors can hold at most one unit of the asset and cannot shortsell.[7]

In this economy, there are outside investors and three types of inside investors: potential buyers, non-searchers and potential sellers. For ease of exposition, we will refer to potential buyers and potential sellers simply as buyers and sellers respectively. At some random time,

[6] See also Vayanos and Weill (2008) for an application to the on-the-run phenomenon, by which recently issued bonds have higher prices than older ones with the same cash flows. They develop a multi-asset model where both the spot market and the repo market operate through search.

[7] Investors are risk neutral and thus have linear utility over the dividend flow d. Consequently, they optimally prefer to hold a maximum long position in the asset (which we can normalize to 1) or zero units of the asset (once they seek to exit the market).

outside investors can choose to participate in the market for this asset aiming to buy one unit of the asset. They then become buyers. Once they purchase the asset, buyers become non-searcher investors. Non-searchers hold the asset and enjoy the full value d of its dividend flow until they receive a liquidity shock which makes them want to liquidate their portfolio and leave the market. At that time, non-searcher investors become sellers and seek to sell. Upon selling, sellers exit the market and join the initial group of outside investors. Also, a buyer, who receives a liquidity shock before purchasing the asset, simply exits the market.

Liquidity shocks arrive with a Poisson rate γ and reduce investors' valuation to a lower level $d-x$ of flow utility, where $x > 0$ captures the notion of a liquidity shock to the investors, for example, a sudden need for cash or the arrival of a good investment opportunity in another market. x could also be understood as the holding cost borne by the investor who receives a liquidity shock and is aiming to exit the market.

The flow of investors entering the economy is defined by a function f. Investors are heterogeneous in their investment opportunities κ, i.e. they differ on their outside options as some investors enjoy better investment possibilities than others. We assume f is a continuous and strictly positive function of the investor's investment opportunity class κ, such that the total flow of investors entering the economy is given by $\int_{\underline{\kappa}}^{\overline{\kappa}} f(\kappa)d\kappa$, where $[\underline{\kappa}, \overline{\kappa}]$ is the support of $f(\kappa)$. Only a fraction $\nu(\kappa)$ of the flow of investors entering the economy chooses to invest in the market for this asset. At any point in time there is a non-negative flow of every class of investor from the outside investors' group into the market, and hence the total flow of investors entering the market is defined by $g = \int_{\underline{\kappa}}^{\overline{\kappa}} \nu(\kappa)f(\kappa)d\kappa$.

We assume markets operate through search, with buyers and sellers matched randomly over time in pairs. Search is characteristic of over-the-counter markets where investors need to locate trading partners and then bargain over prices. There is a cost associated to this search process. In a market where it is more likely to find a counterpart in a short time, the search cost is smaller and liquidity, measured by search costs, is higher. But we could think of a broader interpretation of the search friction. In a centralized market, it represents the cost of being forced to trade with an outside investor who does not understand the full value of the asset and requires an additional compensation for trading. These investors only buy the asset at a discount and sell it at a premium. This transaction cost decreases in the abundance of investors. In the market of a frequently traded asset, it is less likely that it is necessary to trade with an outside investor who "mis-values" the asset and hence the transaction cost linked to this asset is smaller and its liquidity higher. In this paper, we use the first intuition because of its more transparent interpretation.

We adopt the search framework presented in Vayanos and Wang (2007). To define the search process, we first need to describe the rate at which investors willing to buy meet those willing to sell and once they meet we need to specify how the asset price is determined. The ease in finding a trading partner depends on the availability of potential partners. Let us consider that an investor seeking to buy or sell meets other investors according to a Poisson process with a fixed intensity. Thus, for each investor the arrival of a trading partner occurs at a Poisson rate proportional to the measure of the partner's group. Denote by η_b the measure of buyers and by η_s the measure of investors seeking to sell (sellers). Then, a buyer meets sellers with a Poisson intensity $\lambda\eta_s$ and a seller meets buyers at a rate $\lambda\eta_b$, where λ measures the efficiency of the search and a high λ represents an efficient search process. The overall flow of meetings[8] between trading partners is then given by $\lambda\eta_b\eta_s$.

Once investors meet they bargain over the price p of the asset. These meetings always

[8]See Duffie and Sun (2007) for a formal proof of this result. This application of the exact law of large numbers for random search and matching has previously been used in Duffie et al. (2005), Duffie et al. (2007) and Vayanos and Wang (2007) among others.

result in trade as Proposition 3 shows. For simplicity we assume that either the investor willing to buy or the one willing to sell is chosen randomly to make a take-it-or-leave-it offer to his trading partner. Denoting by $\frac{z}{1+z}$ the probability of the buyer being selected to make the offer and thus by $\frac{1}{1+z}$ the probability that the seller makes the offer, $z \in (0, \infty)$ captures the buyer's bargaining power.

Figure 1 describes this market, specifying the different types of investors and the flows between types. η_0 denotes the measure of non-searcher investors.

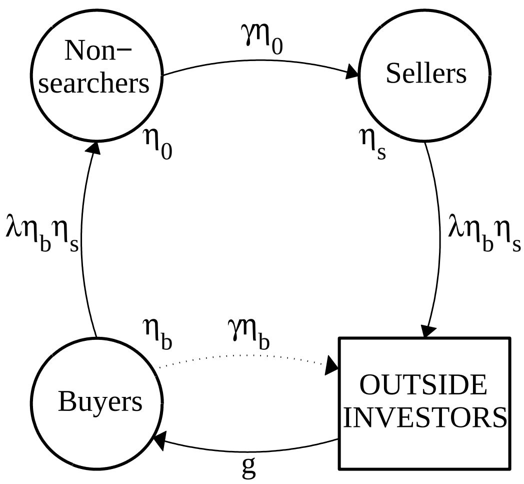

Figure 1: An outside investor enters the market and becomes a buyer aiming to meet a seller. If he suffers a liquidity shock before meeting a trading partner, he exits the market. On the contrary, if he meets a seller, he bargains over the price, buys the asset (pays p) and becomes a non-searcher. A non-searcher holds the asset until he receives a liquidity shock. At that time, he becomes a seller seeking a buyer. When he meets a buyer, he bargains over the price, sells the asset (receives p) and exits the market returning to the group of outside investors.

4 Steady-State Analysis

We organize the analysis in three steps. We first solve for the steady-state measure of every type of inside investor in Step 1 (Subsection 4.1). Next, in Step 2, we determine inside investors' flow utilities (Subsection 4.2). Both Steps 1 and 2 take as given investors' decision to participate in the market for this asset. Then, in Step 3 (Subsection 4.3), we endogenize the entering rule.

4.1 Step 1: Measure of Investors

In this subsection we determine the measure of buyers (η_b), non-searcher investors (η_0) and sellers (η_s). Although investors are heterogeneous in their investment opportunities κ, once they enter the market they all behave in the same way. Investors develop sudden needs for cash at the same Poisson rate γ, independently of their outside investment opportunities κ. In consequence, we do not need to consider the distribution of investment opportunities within each population but the aggregate measure of buyers, non-searcher investors and sellers.[9]

In equilibrium, the market needs to clear and thus the supply of the asset equals the measure of investors holding the asset, each of whom holds one unit of the asset. Specifically, the sum of the measures of non-searchers and sellers is equal to the total supply of the asset:

$$\eta_0 + \eta_s = S \quad \Rightarrow \quad \eta_s = S - \eta_0 \tag{1}$$

[9]This assumption could be generalized by considering γ a function of the outside option κ. The analysis would be similar but the notation more complicated, as we would need to take into account the distribution of investment opportunities κ within each group of investors rather than the aggregate measures. See Section 3 in Vayanos and Wang (2007) for a particular case.

In a steady state, the inflow of investors joining a group matches the outflow such that the rate of change of the group's population is zero. The inflow and outflow of the different types of investors are summarized in Figure 1. Let us first consider the non-searcher investors. In this case, inflows are given by the buyers who meet a trading partner and buy the asset ($\lambda\eta_b\eta_s$), while non-searchers receiving a liquidity shock constitute the outflow ($\gamma\eta_0$). Setting inflow equal to outflow and using equation (1) yields:

$$\eta_b = \frac{\gamma}{\lambda}\frac{\eta_0}{S-\eta_0} \tag{2}$$

We now analyze the population of buyers. The flows of investors coming from the outside group are defined by g. The outflow is comprised of the buyers who receive a liquidity shock before meeting a trading partner ($\gamma\eta_b$) and of those who meet sellers and buy the asset ($\lambda\eta_b\eta_s$). Then,

$$g = \gamma\eta_b + \lambda\eta_b\eta_s$$

Using equations (1) and (2) we can rewrite the previous equation as:

$$g = \gamma\left(1 + \frac{\gamma}{\lambda}\frac{1}{S-\eta_0}\right)\eta_0 \tag{3}$$

Equation (3) determines η_0 as a function of g. Then, substituting η_0 in equations (1) and (2) specifies η_s and η_b respectively. Let us first assume the flow of investors entering the market g is constant. We generalize our results in Subsection 4.3.

Proposition 1. *Given g constant, there is a unique solution to the system (1) - (3) given by:*

$$\eta_0 = \frac{1}{2\gamma}A \tag{4}$$

$$\eta_s = S - \frac{1}{2\gamma}A \tag{5}$$

$$\eta_b = \frac{\gamma}{\lambda}\frac{A}{2\gamma S - A} \tag{6}$$

where $A = (g + \gamma S + \frac{\gamma^2}{\lambda}) - \sqrt{(g + \gamma S + \frac{\gamma^2}{\lambda})^2 - 4\gamma g S}$.

The proof is presented in Appendix A. It is interesting to note how the different measures of investors respond to changes in the parameters of our mode. Corollaries 1 and 2 summarizes some relevant comparative statics:

Corollary 1. *A rise in the flow of investors entering the market, increases the measure of buyers and non-searchers and decreases the measure of sellers.*

As the flow of investors g entering the market rises, the measures of investors willing to buy (buyers) and of those passively holding the asset (non-searchers) increase ($\frac{\partial \eta_b}{\partial g}, \frac{\partial \eta_0}{\partial g} > 0$). However, given that there are more investors seeking to buy the asset, it is now easier for a seller to find a trading partner and hence the measure of investors seeking to sell falls ($\frac{\partial \eta_s}{\partial g} < 0$). This is proven in Appendix B.1.

Corollary 2. *For a given flow of investors, increasing the efficiency of the search process reduces the measure of buyers and sellers and increases the measure of non-searchers.*

Given the measures of investors η_b seeking to buy and those η_s seeking to sell, the efficiency of the search process λ defines the overall flow of meetings: $\lambda \eta_b \eta_s$ (and the flow of transactions, according to Proposition 3). However, the measures of the different types of investors (η_b, η_0, η_s) also depend on the efficiency of the search process λ. In particular, for the same level of investors entering the market, if the search process is more efficient, there will be a lower measure of investors "waiting" to meet a potential seller ($\frac{\partial \eta_b}{\partial \lambda} < 0$). Thus, outside investors, who enter the market, meet a trading partner and become non-searcher investors at a faster rate if the search process is more efficient ($\frac{\partial \eta_0}{\partial \lambda} > 0$). A proportion of non-searcher investors then joins the pool of sellers and hence there is a higher flow of investors coming from the non-searchers to the group of sellers. And, although there are more inflows of investors and less investors seeking to buy, if the search process is more efficient, the measure of sellers "waiting" to sell is reduced ($\frac{\partial \eta_s}{\partial \lambda} < 0$). The proof is in Appendix B.2.

4.2 Step 2: Expected Utilities and Price

We now determine the expected utility of the buyers (v_b), the non-searcher investors (v_0) and the sellers (v_s), as well as the price p. Investors exit this market because of a need for cash. We assume that the expected utility of outside investors is zero. Once they are out of the market, investors have different investment opportunities and decide where to invest next. They could even choose to re-enter this market again.

To derive the expected utility of every type of investor we analyze the possible transitions between types. For example, a buyer can leave the market if he receives a liquidity shock, remain a potential buyer or meet a seller and become a non-searcher. This is summarized in Figure 2.

The utility flow rv_b of buyers is thus equal to the expected flow of exiting the market and becoming an outside investor ($(0 - v_b)\gamma$) plus the expected flow derived from meeting a trading partner seeking to sell (which occurs at rate $\lambda\eta_s$), buying the asset (paying p) and

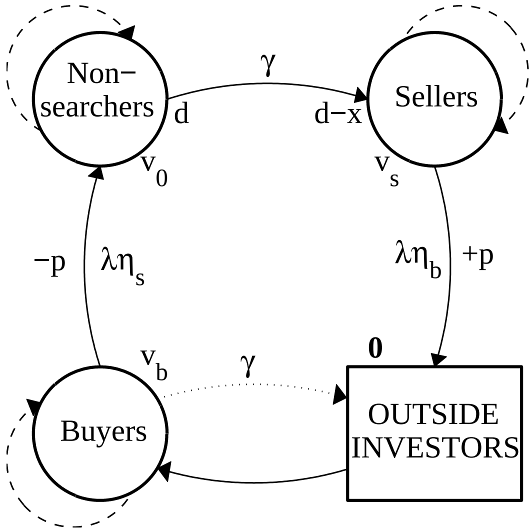

Figure 2: Types of investors and transitions between types.

becoming a non-searcher investor $(\lambda\eta_s(v_0 - v_b - p))$. Then,

$$rv_b = -\gamma v_b + \lambda\eta_s(v_0 - v_b - p) \tag{7}$$

Non-searcher investors can either remain non-searchers enjoying the full value d of the asset's dividend flow or receive a liquidity shock with probability γ and become a seller. In this case, the flow of utility of being a non-searcher is

$$rv_0 = d + \gamma(v_s - v_0) \tag{8}$$

Sellers exit the market as soon as they meet a trading partner, i.e., with intensity $\lambda\eta_b$ they sell the asset (receiving p) and become outside investors with zero expected utility.

Meanwhile, they enjoy a low level $d - x$ of utility. Thus,

$$rv_s = (d - x) + \lambda\eta_b(p + 0 - v_s) \tag{9}$$

The asset price is determined by bilateral bargaining between a buyer and a seller. We have assumed that with probability $\frac{z}{1+z}$ the buyer makes a take-it-or-leave-it offer to his trading partner and offers him his reservation value v_s. With probability $\frac{1}{1+z}$, the seller is chosen to offer the buyer his reservation value $v_0 - v_b$. As a result,

$$p = \frac{z}{1+z}v_s + \frac{1}{1+z}(v_0 - v_b) \tag{10}$$

where z measures the buyer's bargaining power which we treat as exogenous. Proposition 2 summarizes this subsection's main result. The proof is in Appendix A.

Proposition 2. *Given g constant, the system of equations (7)-(10) has a unique solution given by:*

$$\begin{aligned}
v_b &= k\frac{x}{(r+\gamma+\lambda\eta_s)z+\gamma}\frac{\lambda\eta_s z}{r+\gamma} && (11)\\
v_0 &= \frac{d}{r} - k\Big(\frac{x}{r} + \frac{x}{(r+\gamma+\lambda\eta_s)z+\gamma}\Big)\frac{\gamma}{r+\gamma} && (12)\\
v_s &= \frac{d}{r} - k\Big(\frac{x}{r} + \frac{x}{(r+\gamma+\lambda\eta_s)z+\gamma}\Big) && (13)\\
p &= \frac{d}{r} - k\frac{x}{r} && (14)
\end{aligned}$$

where $k = \dfrac{(r+\gamma+\lambda\eta_s)z+\gamma}{(r+\gamma+\lambda\eta_s)z+(r+\gamma+\lambda\eta_b)}$.

The price of the asset as given by equation (14) is thus equal to the present value of all future dividend flows d, discounted at the rate r, minus a price discount due to illiquidity.

The second term is the product of the present value of the holding cost x borne by investors seeking to exit the market and a function k. $k \in (0, 1)$ measures the severity or intensity of the illiquidity discount.

It is interesting to highlight that the asset price will be higher when fundamentals are stronger (i.e. if the asset pays a higher dividend flow d) and whenever the demand for the asset increases ($\frac{\partial p}{\partial d}, \frac{\partial p}{\partial \eta_b} > 0$). On the contrary, the price decreases with investors trying to sell the asset and in the buyer's bargaining power ($\frac{\partial p}{\partial \eta_s}, \frac{\partial p}{\partial z} < 0$). If during the bargaining process the buyer holds a more favorable position, he would try to lower the price paid to acquire the asset. Corollary 3 summarizes these comparative statics results:

Corollary 3. *The price of the asset*

i. increases in the dividend flow and in the measure of buyers.

ii. decreases in the buyer's bargaining power and in the measure of sellers.

The proof of this set of comparative statics is presented in Appendix B.3.

4.2.1 Trade among Investors

In this subsection we prove a result we have assumed so far in our analysis:

Proposition 3. *All meetings between buyers and sellers result in trade.*

Proof. Trade between buyers and sellers occurs if the gain from trade is strictly positive, i.e., if the buyers' reservation value $v_0 - v_b$ exceeds the sellers' reservation value v_s. Let us see if $(v_0 - v_b) - v_s > 0$. Subtracting equations (13) and (11) from (12), we get:

$$(v_0 - v_b) - v_s = \frac{x(1+z)}{(r+\gamma)(1+z) + \lambda\eta_s z + \lambda\eta_b}$$

which is always strictly greater than zero since $x, r, \gamma, \lambda, z > 0$.

Therefore, once investors meet, trade among partners always occurs. $\square$

4.3 Step 3: Entering Rule

We now endogenize the entering rule. In our framework, outside investors can choose between entering the market for this asset and investing in an alternative market. Investors are heterogeneous in their outside investment opportunities κ, i.e. each class of investor has access to different investment opportunities. However, once they enter a market, their type no longer influences their decisions in the sense that every buyer, for instance, enjoys the same expected utility independently of his original outside opportunity. Interestingly, a buyer's expected utility does depend on the flow of investors who entered this market before him.

Let us refer to the investor who is deciding between moving or not into a market as the marginal investor. And, let us denote by κ' and by $v_{alt}(\kappa')$, respectively, the best outside investment opportunity of the marginal investor and his expected utility from investing in that alternative market. For simplicity, we assume $v_{alt}(\kappa') = \kappa'$, such that an investor with a better outside option (higher κ) enjoys a higher level of expected utility (higher v_{alt}).

When an investor faces the decision to choose a market, he prefers to enter and invest in a specific market if the expected utility v_b of being a buyer in that market is higher than the expected utility v_{alt} derived from his best outside option. Then, if a market represents the best opportunity for the marginal investor, it is also preferred by any other investor with a worse investment opportunity, i.e. any investor with type $\kappa < \kappa'$ moves into the market too. As a result, when a market is chosen by a marginal investor with a high type, a high flow of investors enters that market. A high flow of investors implies an increase in the measure of buyers, which then affects the expected utility of being a buyer. Thus, even though each investor's type does not alter his expected utility, the type of the last investor who enters does. The type of this last investor defines the total flow who invests in this market and hence determines how concentrated the population of buyers is.

Let us define the fraction $\nu(\kappa)$ of investors with outside investment opportunity κ who enters the market as follows:

$$\nu(\kappa) = \begin{cases} 0 & \text{if} \quad \kappa > \kappa' \\ [0,1] & \text{if} \quad \kappa = \kappa' \\ 1 & \text{if} \quad \kappa < \kappa' \end{cases}$$

where $1 - \nu(\kappa)$ represents the fraction of investors with outside option κ who invests in alternative markets. The total flow of investors moving into this market is thus given by: $g(\kappa') = \int_{\underline{\kappa}}^{\overline{\kappa}} \nu(\kappa) f(\kappa) d\kappa$, where f defines the total flow of investors entering the economy and $[\underline{\kappa}, \overline{\kappa}]$ is the support of $f(\kappa)$. In equilibrium, as we discuss in more detail in the next section, the total flow g^* depends on the equilibrium fraction of investors ν^* entering the market. But the equilibrium fraction of investors is determined by the marginal investor who is indifferent between this market and his best outside option. We refer to this investor as the indifferent investor. For the indifferent investor, the expected utility of being a buyer equals the expected utility of his best outside option:

$$\begin{cases} v_b(g^*(\underline{\kappa})) \leq v_{alt}(\underline{\kappa}) & \text{if} \quad \kappa^* \leq \underline{\kappa} \\ v_b(g^*(\kappa^*)) = v_{alt}(\kappa^*) & \text{if} \quad \kappa^* \in [\underline{\kappa}, \overline{\kappa}] \\ v_b(g^*(\overline{\kappa})) \geq v_{alt}(\overline{\kappa}) & \text{if} \quad \kappa^* \geq \overline{\kappa} \end{cases} \tag{15}$$

where $g^* = \int_{\underline{\kappa}}^{\overline{\kappa}} \nu^*(\kappa) f(\kappa) d\kappa$. In equilibrium, there are two types of possible scenarios depending on the behavior of the expected utility v_{alt} of investing in an alternative market and the expected utility v_b of being a buyer. There is an equilibrium where all investors clearly prefer one market (either all enter (when $\kappa^* \geq \overline{\kappa}$ or equivalently when $v_b(g^*(\overline{\kappa})) \geq v_{alt}(\overline{\kappa})$) or no one enters (if $\kappa^* \leq \underline{\kappa}$, or when $v_b(g^*(\underline{\kappa})) \leq v_{alt}(\underline{\kappa})$) or an equilibrium where a fraction of investors is better-off by investing in this market while others prefer not to enter. Before we proceed, let us introduce the formal definition of market equilibrium.

5 Equilibrium

5.1 Equilibrium Definition

Definition 1. *A* ***market equilibrium*** *consists of a fraction $\nu(\kappa)$ of investors entering the market, measures $(\eta_s,\ \eta_b,\ \eta_0)$ of investors and expected utilities and prices $(v_b,\ v_0,\ v_s,\ p)$ such that:*

- *$(\eta_s^*,\ \eta_b^*,\ \eta_0^*)$ solve the market-clearing condition and inflow-outflow equations given by the system (1) - (3),*
- *$(v_b^*,\ v_0^*,\ v_s^*,\ p^*)$ solve the flow-value equations for the expected utilities and the pricing condition given by the system (7) - (10),*
- *$\nu^*(\kappa)$ solves the entering condition given by equation (15).*

5.2 Equilibrium Characterization

To analyze the set of equilibria in this market, we need to solve for the fixed points of the system of equations (1) - (3), (7) - (10) and (15). To gain some intuition, let us introduce

Figure 3. Figure 3 represents the expected utility of being a buyer, v_b, and the expected utility v_{alt} derived from investing in an alternative market as a function of the outside investment opportunity κ' of the marginal investor who is deciding between entering or not this market.

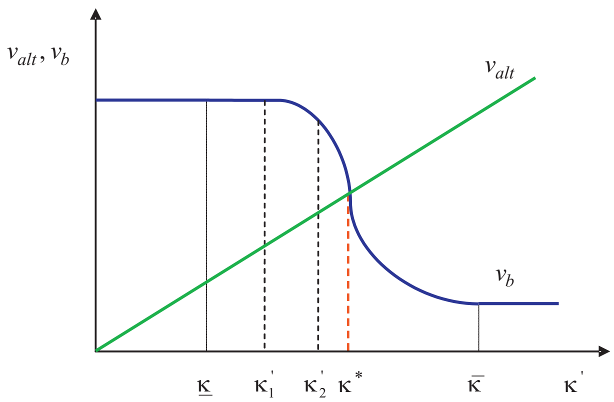

Figure 3: Investors compare expected utilities v_b and v_{alt} and decide to participate in this market if $v_b > v_{alt}$. κ^* defines the outside investment opportunity which makes investors indifferent between entering or not this market. $\underline{\kappa}$ and $\overline{\kappa}$ determine the support of the flow of investors who enter the economy.

Consider, for example, the marginal investor with outside investment opportunity κ_1'. He compares the utility of his outside option, $v_{alt}(\kappa_1') = \kappa_1'$, to the utility of being a buyer, $v_b(g(\kappa_1'))$, given that investors with outside opportunities $\kappa < \kappa_1'$ have already entered. He decides to enter since $v_b(g(\kappa_1')) > v_{alt}(\kappa_1')$ as shown in Figure 3. Similarly, the marginal investor with investment opportunity κ_2' prefers to enter too.[10] Suppose marginal investor κ^* is now facing the entry decision. For him, $v_b(g(\kappa^*)) = v_{alt}(\kappa^*)$ and he is indifferent between markets. Any investor with a better outside opportunity prefers not to enter.

[10]Although the expected utility of being a buyer has decreased because now all investors with $\kappa < \kappa_2'$ are in the market, he is still better-off by moving into this market ($v_b(g(\kappa_2')) > v_{alt}(\kappa_2')$).

Figure 3 depicts the case where some investors prefer to enter and some choose the alternative investment. There are two other possible scenarios, one in which all investors enter (when $\kappa^* \geq \overline{\kappa}$) and one in which every investor prefers the alternative investment ($\kappa^* \leq \underline{\kappa}$). In either scenario, the market equilibrium is unique. Theorem 1 summarizes a key result:

Theorem 1. *There is a unique market equilibrium.*

The complete proof is in Appendix C but let us provide some guidelines for why the equilibrium is unique. Given non-negative expected utilities, if $v_b(\kappa' = 0) > v_{alt}(\kappa' = 0)$ and v_b decreases in κ' while v_{alt} is strictly increasing, then by continuity there exists a unique threshold κ^* such that expected utilities are equal and investors indifferent between markets. A unique threshold κ^* then defines a unique flow of investors $g^* \equiv g(\kappa' = \kappa^*)$ into this market. And given a unique flow of investors g^*, steady-state measures, expected utilities and the asset price can be determined uniquely as stated in Propositions 1 and 2. Consequently, market equilibrium is unique. It is interesting to note that the expected utility of buyers decreases as more investors enter this market.[11]

[11] An increase in the flow of investors g affects differently the steady-state measures of investors. Specifically, the measures of buyers and non-searchers increase in the flow of investors while the measure of those seeking to sell is reduced. Buyers are worse-off as more investors decide to move into a market because their arrival makes it more difficult for them to meet a seller and purchase the asset. Buyers thus suffer from a 'congestion effect' in the sense that as new investors move in, their side of the market becomes more crowded and there is increasing competition among buyers to meet one of the fewer sellers. Similarly, if investors preferred to invest in alternative markets and there were a reduction in the flow of investors into this market, sellers would be worse-off. It would be more difficult for a seller to meet a buyer and sell and exit the market. Sellers would experience a 'congestion effect' as the market becomes more one-sided and their side of the market gets crowded with sellers looking for a trading partner.

5.3 Market Equilibrium

In general, the market equilibrium can only be computed numerically. To gain some intuition we solve the equilibrium in closed-form when search frictions are small.[12] Small search frictions correspond to a large value of the parameter λ that measures the efficiency of the search process. Before we introduce the general case of endogenous entry, let us first discuss what happens when the flow of investors entering the market g is given. This allows us to explicitly analyze the additional effect on liquidity and welfare of an endogenous flow of investors.

5.3.1 Exogenous entry

For a given flow of investors g, when search frictions are small, the market converges to the Walrasian Equilibrium where the price of the asset is determined by supply and demand for the asset. Let us consider three cases:

- Case 1: $g < \gamma S$ & $\lambda \to \infty$ - If the flow of investors g is lower than γS, buyers are the short side of the market and there is "excess supply" of this asset.[13] Sellers are thus marginal and the price is equal to their valuation: $p = \frac{d-x}{r}$.

- Case 2: $g = \gamma S$ - If the flow of investors g equals γS, there is no excess demand or excess supply.[14] As $\lambda \to \infty$ the price of the asset is given by $p = \frac{d}{r} - \frac{z}{z+1}\frac{x}{r}$.

- Case 3: $g > \gamma S$ & $\lambda \to \infty$ - If the flow of investors g exceeds γS, sellers are now the short side of the market and there is "excess demand" for this asset.[15] Buyers are thus

[12]We complement the analysis with a numerical example in Section 6.

[13]The measure of sellers is of order 1 while the measure of buyers is of order $\frac{1}{\lambda}$.

[14]The measure of sellers equals the measure of buyers ($\frac{\eta_b}{\eta_s} = 1\ \forall \lambda$). See Proposition 5.

[15]The measure of sellers is of order $\frac{1}{\lambda}$ while the measure of buyers is of order 1.

marginal and the price is equal to their valuation: $p = \frac{d}{r}$.

This is summarized in the next proposition, which is proved in Appendix D.

Proposition 4. *For a given g, when $\lambda \to \infty$, the price of the asset is given by*

$$p = \begin{cases} \dfrac{d-x}{r} & \text{if} \quad g < \gamma S \\ \dfrac{d}{r} - \dfrac{z}{z+1}\dfrac{x}{r} & \text{if} \quad g = \gamma S \\ \dfrac{d}{r} & \text{if} \quad g > \gamma S \end{cases} \tag{16}$$

Let us briefly highlight the case where the flow of investors g equals γS (Case 2).

Proposition 5. *When $g = \gamma S$ and λ goes to infinity the following asymptotics hold:*

$$\eta_s = \eta_b = \frac{\sqrt{g}}{\sqrt{\lambda}} + o\Big(\frac{1}{\sqrt{\lambda}}\Big) \tag{17}$$

$$p = \frac{d}{r} - \frac{z}{z+1}\frac{x}{r} - \frac{\gamma}{z+1}\frac{x}{r}\frac{1}{\sqrt{g}}\frac{1}{\sqrt{\lambda}} + o\Big(\frac{1}{\sqrt{\lambda}}\Big) \tag{18}$$

where $o\Big(\frac{1}{\sqrt{\lambda}}\Big)$ denotes terms of order smaller than $\frac{1}{\sqrt{\lambda}}$.

When $g = \gamma S$, the measure of buyers and sellers are of the same order and the ratio of buyers to sellers equals 1 for any level of search frictions. The ratio $\frac{\eta_b}{\eta_s}$ constitutes an important element in our analysis as it captures how "congested" a market is. $\frac{\eta_b}{\eta_s} = 1$ represents a "balanced" market where for every investor willing to buy there is a potential seller.

Corollary 4. *When $g = \gamma S$ and $\lambda \to \infty$, illiquidity (measured by price discount) increases when the buyer's bargaining power rises.*

When search frictions are small and $\frac{\eta_b}{\eta_s} = 1$, i.e. when $\lambda \to \infty$ and $g = \gamma S$, the illiquidity discount depends on the buyer's bargaining power. As buyers gain a stronger position, they try to lower the price they pay to acquire the asset. The proofs of Proposition 5 and Corollary 4 are presented in Appendix D.

5.3.2 Endogenous entry

Let us now discuss the more general case where investors can decide whether or not to enter this market and thus the flow of new investors responds to changes in market and asset characteristics. The equilibrium flow is defined in Proposition 6.

Proposition 6. *When $\lambda \to \infty$, the equilibrium flow of investors g^* is given by*

$$g^* = \begin{cases} F\left(\dfrac{x}{r+\gamma}\right) & \textit{if} \quad F(\frac{x}{r+\gamma}) < \gamma S \\ \gamma S & \textit{if} \quad F(\frac{x}{r+\gamma}) \geq \gamma S \end{cases} \tag{19}$$

When search frictions are small and investors can choose between investment opportunities, i.e. when $\lambda \to \infty$ and g is endogenous, investors prefer not to enter a market where "too many" investors have already entered. Put differently,

Corollary 5. *In equilibrium, when $\lambda \to \infty$, the flow of investors never exceeds γS.*

To give some intuition for why this is the case, let us discuss the different scenarios. $F(\frac{x}{r+\gamma}) < \gamma S$ corresponds to a market where as frictions are attenuated, the population of investors becomes unbalanced and there are many sellers per buyer in the market, i.e. when $g^* = F(\frac{x}{r+\gamma}) < \gamma S$ and λ tends to infinity, the flow of investors into the market g^* is not large enough to build up a stock of buyers capable of absorbing the asset supply S.[16] However, when $F(\frac{x}{r+\gamma}) \geq \gamma S$ and frictions are small, the market does not become unbalanced in the opposite direction (there will not be too many buyers) because investors can choose between investment opportunities and will not enter a market where there are too many buyers competing to meet a seller. This is a key difference with respect to a market where the flow of investors is assumed to be constant. As frictions go to zero, the flow of investors does not exceed γS because investors will only enter this market until the expected utility of becoming a buyer equals the utility derived from outside investments ($F^{-1}(\gamma S)$). Investors enter until the market is balanced and they are indifferent between investments. The price of the asset when entry is endogenous is summarized in Proposition 7:

Proposition 7. *When $\lambda \to \infty$, the price of the asset is given by*

$$p = \begin{cases} \dfrac{d-x}{r} & \text{if} \quad F(\frac{x}{r+\gamma}) < \gamma S \\ \dfrac{d}{r} - \dfrac{r+\gamma}{x} F^{-1}(\gamma S) \dfrac{x}{r} & \text{if} \quad F(\frac{x}{r+\gamma}) \geq \gamma S \end{cases} \tag{20}$$

Compared to the case where the flow of investors is fixed, investors avoid participating in a market where they would face excess demand for the asset and where they would have to pay the highest possible price to acquire the asset. Proofs are presented in Appendix D.

Figure 4 highlights the differences in the price of the asset when entry is exogenous (Proposition 4) and in the case of endogenous entry of investors (Proposition 7).

Result 1. *Diminishing market frictions does not necessarily lead to a more liquid market.*

[16]When $F(\frac{x}{r+\gamma}) < \gamma S$ and $\lambda \to \infty$, the measure of sellers tends to a positive constant while the measure of buyers tends to zero. See Lemma 1 in Appendix D.

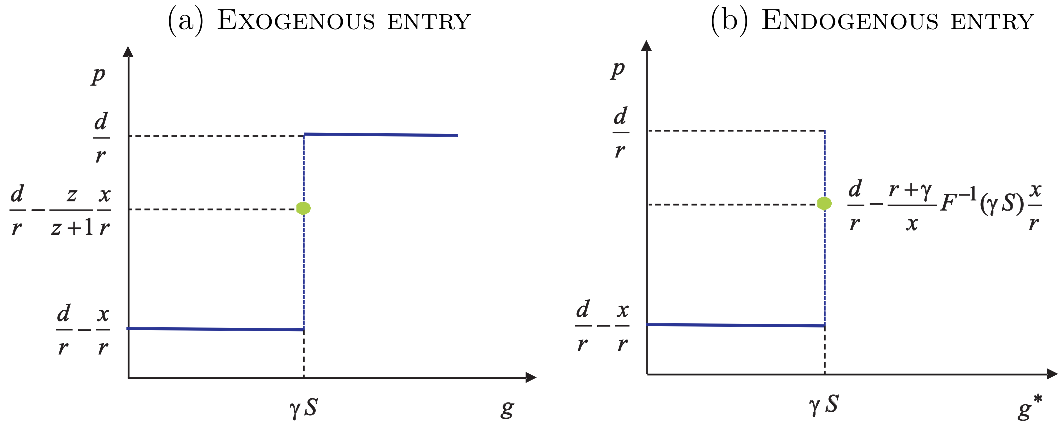

Figure 4: Price of the asset when search frictions are small, i.e. $\lambda \to \infty$, as a function of the flow of investors entering the market, when the flow is given (a) and when investors can choose whether or not to enter (b).

Even in the limiting case where search frictions tend to zero, the market for this asset might be illiquid.

5.3.3 Welfare

We measure welfare by the weighted sum of investors' expected utilities. Weights are determined by the measure of every type of investor in our economy, including future generations of investors. We can decompose the welfare into two components: the welfare of current generations and the welfare of future generations of investors:

$$W = W_{\text{current investors}} + W_{\text{future investors}} \tag{21}$$

The welfare of current generations of investors is comprised of the welfare of investors who prefer to enter the market, i.e. buyers, non-searchers and sellers (inside investors), and the welfare of investors who prefer alternative investment opportunities (outside investors):

$$W_{\text{current investors}} = W_{\text{inside investors}} + W_{\text{outside investors}} =$$
$$= \eta_b v_b + \eta_0 v_0 + \eta_s v_s + \int_{\kappa^*}^{\overline{\kappa}} v_{alt} f(\kappa) d\kappa \tag{22}$$

where the first three terms represent the welfare of inside investors ($W_{\text{inside investors}}$) and the last term reflects the welfare of outside investors ($W_{\text{outside investors}}$). Outside investors (those with investment opportunities above the threshold value κ^*) enjoy the expected utility derived from investing in an alternative market v_{alt}, which for simplicity we assume equal to their outside investment opportunity κ. Substituting equations (11)-(13) and $v_{alt} = \kappa$ into equation (22) we get:

$$W_{\text{inside investors}} = \frac{d}{r}S - \frac{x}{r+\gamma}\frac{\frac{\gamma}{r}S\big[(r+\gamma)+\gamma z\big] + \frac{\gamma}{r}Sz\lambda\eta_s + \big[(r+\gamma)(z+1)+\gamma z + z\lambda\eta_s\big]\eta_s}{(r+\gamma)(z+1)+z\lambda\eta_s+\lambda\eta_b} \tag{23}$$

$$W_{\text{outside investors}} = \int_{\kappa^*}^{\overline{\kappa}} \kappa f(\kappa) d\kappa \tag{24}$$

The welfare of future generations is determined by the welfare of new entrants to this market (future entrants) and by the welfare of new entrants to alternative markets (future outside investors).

$$W_{\text{future investors}} = W_{\text{future entrants}} + W_{\text{future outside investors}}$$

where the welfare of the stream of future investors is captured by the welfare of outside investors whose utility is the discounted value of entering this market or of investing in alternative markets:

$$W_{\text{future entrants}} = \frac{1}{r}\int_{\underline{\kappa}}^{\kappa^*} v_b f(\kappa)d\kappa = \frac{1}{r}v_b F(\kappa^*) \tag{25}$$

$$W_{\text{future outside investors}} = \frac{1}{r}\int_{\kappa^*}^{\overline{\kappa}} v_{alt} f(\kappa)d\kappa = \frac{1}{r}\int_{\kappa^*}^{\overline{\kappa}} \kappa f(\kappa)d\kappa \tag{26}$$

The analysis of welfare for any level of market frictions requires (numerically) solving for the equilibrium. To provide some motivation, we briefly introduce the scenario where search frictions are small before we present a numerical example in Section 6. Proposition 8 summarizes the main result:

Proposition 8. *When $\lambda \to \infty$, welfare is given by*

$$W = \begin{cases} \frac{d}{r}S - \frac{x}{r}\left[S - \frac{1}{\gamma}F\left(\frac{x}{r+\gamma}\right)\right] + \frac{1+r}{r}\int_{\frac{x}{r+\gamma}}^{\overline{\kappa}} \kappa f(\kappa)d\kappa & \textit{if} \quad F(\frac{x}{r+\gamma}) < \gamma S \\ \frac{d}{r}S + \frac{1+r}{r}\int_{F^{-1}(\gamma S)}^{\overline{\kappa}} \kappa f(\kappa)d\kappa & \textit{if} \quad F(\frac{x}{r+\gamma}) \geq \gamma S \end{cases} \tag{27}$$

where η_s tends to $S - \frac{1}{\gamma}F(\frac{x}{r+\gamma})$ when $F(\frac{x}{r+\gamma}) < \gamma S$ and to 0 when $F(\frac{x}{r+\gamma}) \geq \gamma S$ as λ goes to infinity.

Result 2. *Diminishing market frictions is not necessarily welfare enhancing.*

Even when search frictions tend to zero, welfare might be at a low level. Proofs are in Appendix D. To address how changes in frictions affect liquidity and welfare for any level of search frictions we need to study the general case of the equilibrium.

6 General Case: An Example

In this section we present a numerical example to show how diminishing frictions can have an adverse effect on liquidity and on some components of welfare. We assume the flow of investors f entering the economy follows a beta distribution[17] with support $[\underline{\kappa} \equiv 0, \overline{\kappa} \equiv 5]$ and parameters $a = 8$ and $b = 2$, which is a left-skewed hump-shaped density function. Investors have time preferences with discount rate equal to 1% ($r = 0.01$). The asset pays a dividend flow $d = 2$ and is in total supply $S = 2$. The holding cost is defined as a 40% of the dividend flow to indicate that once an investor receives a liquidity shock his valuation of the asset drops to a 60% of the initial value. Liquidity shocks arrive at a Poisson rate $\gamma = 0.2$ and hence the expected time between shocks is 5. The value of z is chosen such that buyers and sellers have the same bargaining power, i.e. $z = 1$. Table I summarizes the set of exogenous parameters:

$f \sim f_{beta(8,2)}$	$d = 2$	$S = 2$	$z = 1$
$\underline{\kappa} = 0$, $\overline{\kappa} = 5$	$x = 0.8$	$\gamma = 0.2$	$r = 0.01$

Table I: Parameter Values.

Market equilibrium is the solution to the system of equations (1)-(3), (7)-(10) and (15). However, for a generic level of search frictions computing the equilibrium can only be done numerically.[18] The set of parameter values in Table I corresponds to a case where $g^* < \gamma S$ and there are many investors willing to sell per buyer present in the market: $\frac{\eta_b}{\eta_s} << 1$ as shown in Figure 5(a) and (b) respectively. This scenario represents a one-sided market where there is congestion on the sell-side of the market.

[17]The beta distribution is a flexible class of distributions defined on the unit interval $[0, 1]$, whose density function may take on different shapes depending on the choice of the two parameters. These include the uniform density function and hump-shaped densities (See Evans et al. (1993)).

[18]See Appendix E.

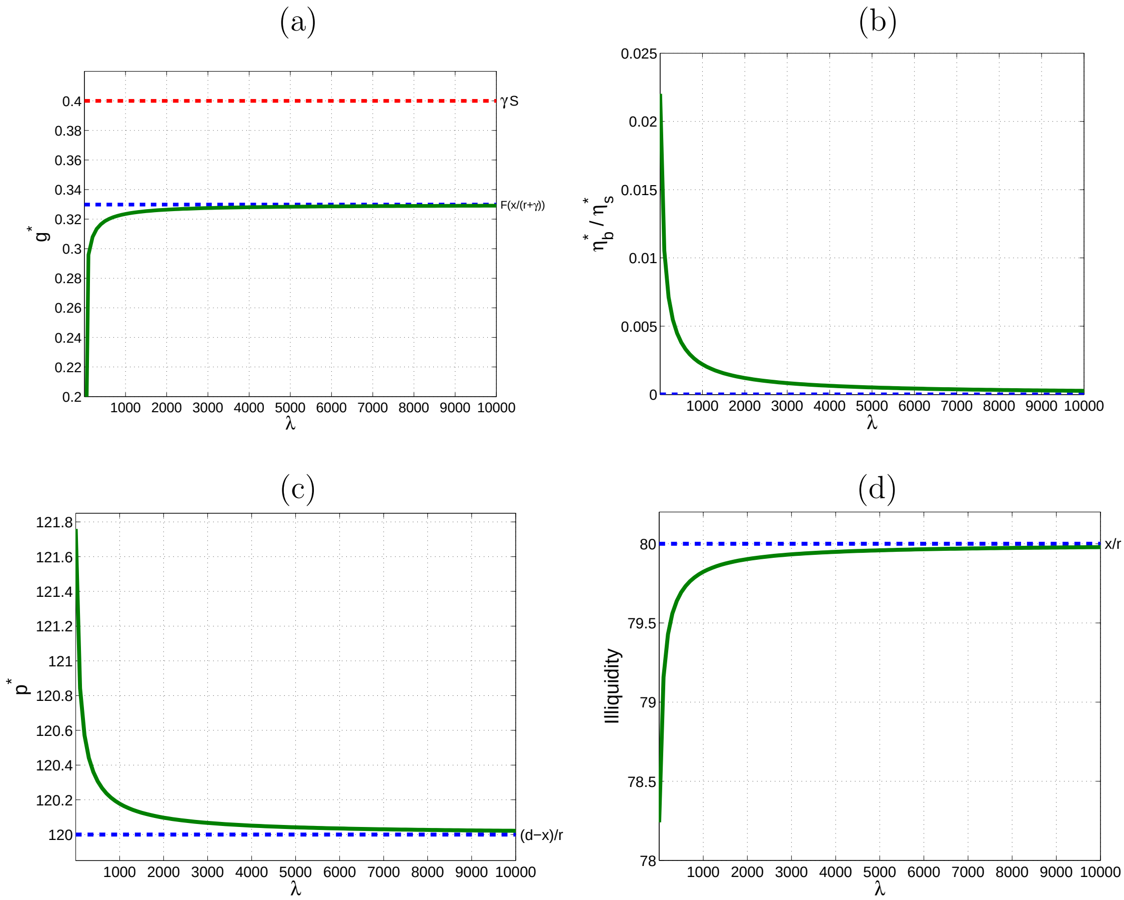

Figure 5: Equilibrium flow of investors g^* entering the market (a), ratio of buyers to sellers (b), asset price (c) and illiquidity (measured by price discount) as a function of the efficiency of the search process λ.

Note that when search frictions are small (λ large), the equilibrium flow of investors g^* tends to $F(\frac{x}{r+\gamma})$ as proved in Proposition 6 and shown in Figure 5(a), and buyers are the short side of the market, i.e. $\frac{\eta_b}{\eta_s} \to 0$ (Figure 5(b)). Also, the price of the asset, as depicted in Figure 5(c), decreases as search frictions are attenuated (for larger values of λ) and converges to $\frac{d-x}{r}$ as stated in Proposition 7. As a result, illiquidity (measured by the illiquidity discount introduced in Subsection 4.2) increases as search frictions are smaller (Figure 5(d)). Put differently,

Result 3. *Diminishing frictions can deteriorate liquidity.*

In this distressed market, where there are many investors seeking to sell per potential buyer, reducing frictions would amplify the effect of congestion. Lesser frictions allow potential buyers to acquire the asset faster leading to a more unbalanced distribution of investors. As there are even fewer investors left willing to buy, the price of the asset falls and illiquidity rises.

Figure 6 presents the equilibrium expected utilities and measures of inside investors as well as the different components of the welfare of current generations of investors, i.e. welfare of inside investors (buyers, non-searchers and sellers) and outside investors. As shown in the top panel of Figure 6(a), investors holding the asset (non-searchers) and those trying to sell it (sellers) are worse-off when the distribution becomes more unbalanced as search frictions get smaller. And, although the expected utility of buyers increases in search efficiency (bottom panel of Figure 6(a)), in equilibrium there are very few potential buyers present in this market (bottom panel of Figure 6(b)). Consequently, the welfare of inside investors diminishes in λ as illustrated in the bottom panel of Figure 6(d).

Figure 6(e) depicts how the welfare of outside investors decreases when search frictions are attenuated as the result of a raise in the equilibrium flow of investors g^* who gives up outside investment opportunities to participate in this market.

Overall, the welfare of current generations of investors decreases as search frictions are attenuated as shown in Figure 6(f). This leads to a key result:

Result 4. *Diminishing frictions can reduce the welfare of current generations of investors.*

Let us now consider the welfare of future generations of investors. Figure 7 presents the welfare of future entrants to this market (top panel of Figure 7(a)), of future outside

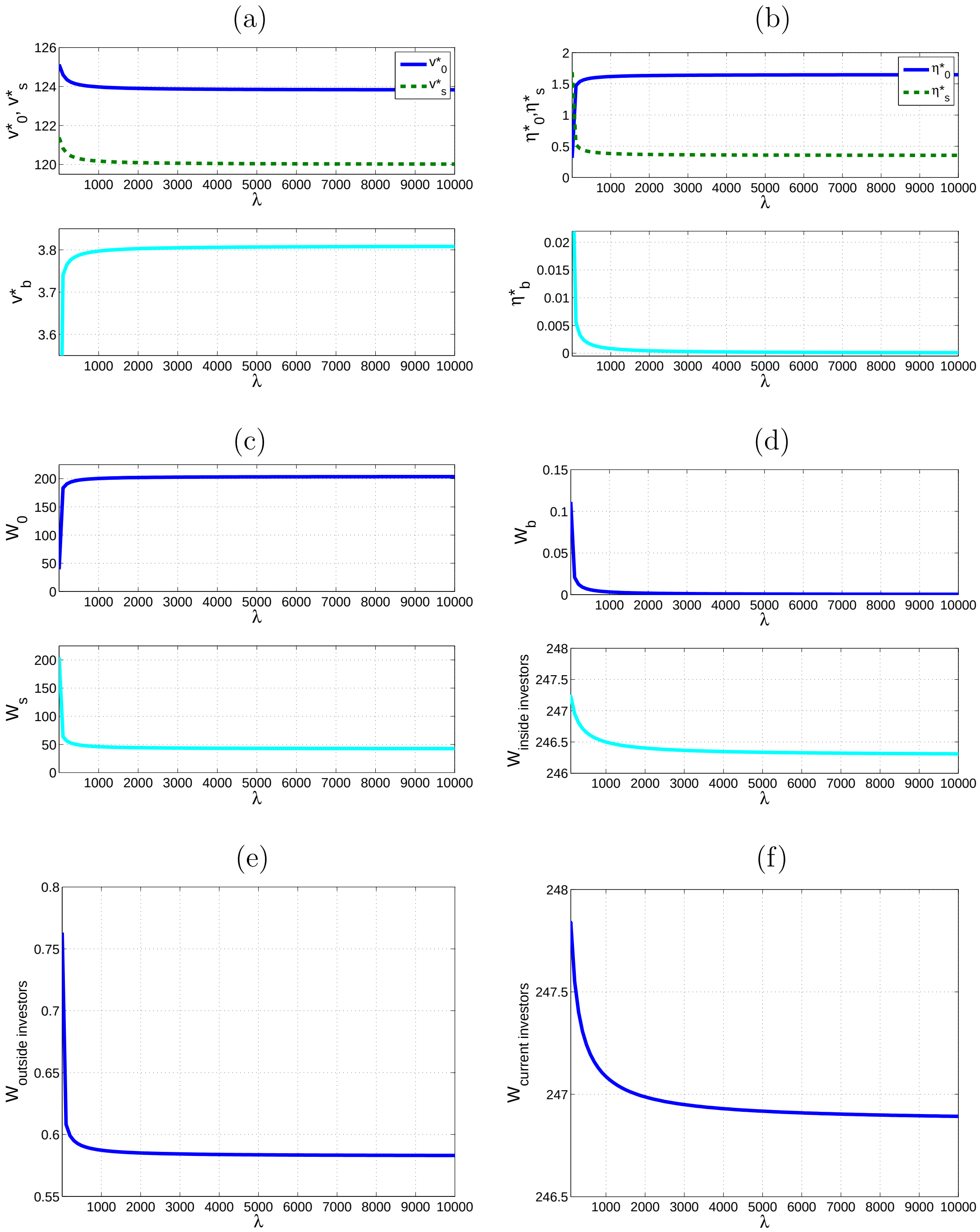

Figure 6: Equilibrium expected utilities of inside investors (a), measures of inside investors (b), welfare of non-searchers(c)(top), of sellers (c)(bottom), of buyers (d)(top), of inside investors (d)(bottom), of outside investors (e) and of current investors (f) as a function of the efficiency of the search process λ.

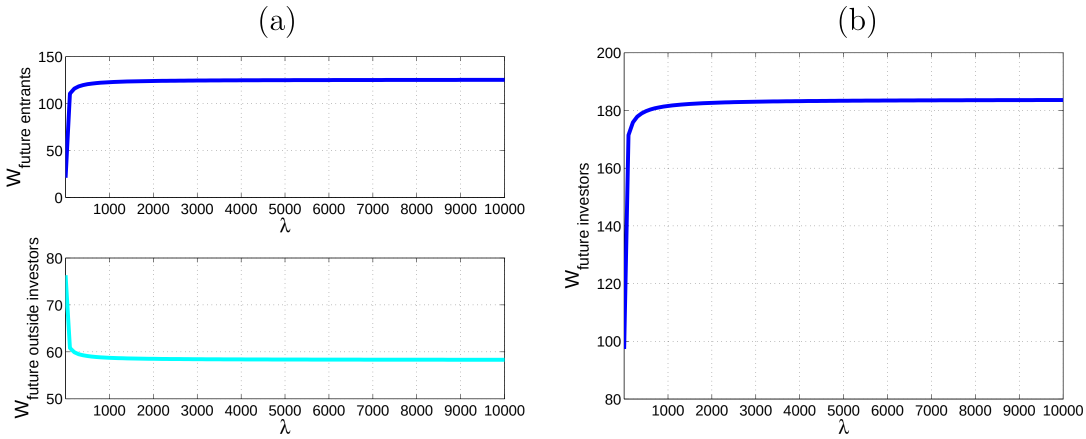

Figure 7: Welfare of future entrants (a)(top), of future outside investors (a)(bottom) and total welfare of future investors (b) as a function of the efficiency of the search process λ.

investors (bottom panel of Figure 7(a)) as well as the total welfare of future generations of investors (Figure 7(b)).

In this example, welfare of future generations of investors increases as search frictions are attenuated as illustrated in Figure 7(b). This is driven by an increasing welfare of future entrants (top panel of Figure 7(a)) and, more specifically, by an increasing expected utility of becoming a buyer (bottom panel of Figure 6(a)) in a market where as search frictions are mitigated, the distribution of investors in the market becomes more unbalanced and there are many investors seeking to sell per potential buyer (Figure 5(b)). As a result, total welfare, depicted in Figure 8, also increases as λ tends to infinity. For different parameter values, it is possible to find an example where total welfare decreases as frictions tend to zero. In an exercise similar to the one discussed in this section, where both the welfare of current generations and of future outside investors diminish as frictions tend to zero, total welfare will be decreasing in λ when the contribution of the welfare of future entrants to total welfare is attenuated. This can be achieved, for example, by reducing the value of x,

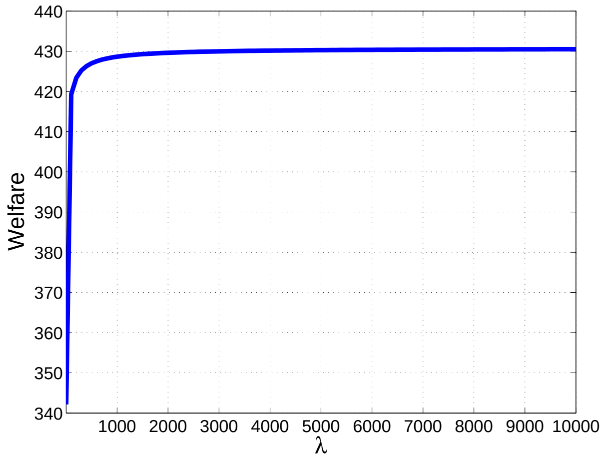

Figure 8: Welfare as a function of the efficiency of the search process λ.

the holding cost borne by sellers trying to exit the market. For a similar specification as the one used in this section and values of x close to zero, total welfare decreases as search frictions are attenuated.[19]

Results 3 and 4 provide a rationale for measures that would delay, rather than facilitate, trading in a market like the one considered in this section where there are many sellers per investor interested in purchasing the asset. A potential extension of this study could analyze the benefits (and costs) of trading halts, circuit breakers and other mechanisms to slow down trading in wildly swinging securities. Rules such as the ones proposed by the Securities and Exchange Commission (SEC) last May 18, 2010 in response to the market disruption of May 6 would enhance market liquidity in a market with many more sellers than buyers.[20] However, a rigorous study of these circuit breaker rules would require an out-of-steady state

[19] These results are available upon request.

[20] On May 18, 2010, the SEC proposed rules to pause trading in individual securities in the S&P 500 Index if, in a five-minute period, the price drops 10 percent or more (See SEC (2010)).

analysis of the model, which is out of the scope of the paper.

6.1 Alternative Measures of Liquidity

In this paper we measured liquidity by price discount. Liquidity could also be measured by trading volume: $V = \lambda\eta_b\eta_s$ or by search times, among other measures. Search times capture the expected time it takes to find a trading partner and can hence differ for buyers and sellers. Let us denote by $\tau_b \equiv \frac{1}{\lambda\eta_s}$ the expected time it takes for buyer to meet a seller and by $\tau_s \equiv \frac{1}{\lambda\eta_b}$ the expected time necessary to find a buyer.

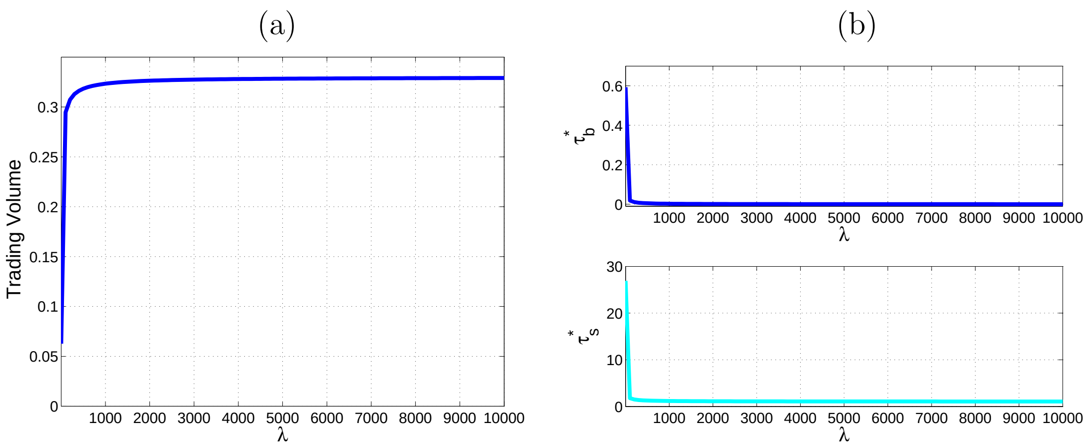

Figure 9: Trading volume (a) and expected trading time for a buyer (b)(top) and expected trading time for a seller (b)(bottom) as a function of the efficiency of the search process λ.

An interesting result is presented in Figure 9. As search frictions are attenuated and liquidity dries up, trading volume rises and the expected time needed to buy and sell in this market diminish. The reason for this counterintuitive result is the following. Facilitating search by reducing frictions yields two consequences. First, it magnifies the effect of congestion leading to a lower price and thus to a less liquid market (as measured by price discount).

This is depicted in Figure 5(d). Second, it raises the frequency of meetings between trading partners. Investors meet at a faster rate which translates into an increasing volume and a decreasing trading delay. Consequently, even though the market is less liquid, investors meet faster and hence expected trading times fall and volume rises as shown in Figure 9(a). This is our last result:

Result 5. *Illiquidity (measured by price discount) can increase while trading volume rises.*

This result highlights that alternative measures capture different dimensions of liquidity. An example of a dramatic drop in share price and heavy volume followed Lehman Brothers' bankruptcy filing on Monday, September 15, 2008. On Monday, its share price fell over 94% from \$3.65 to \$0.21. Trading volume, however, reached historical levels exceeding thirty times the average daily volume for the last five years.

7 Conclusions

This paper proposes a search-based model to study the relationship between market liquidity and the endogenous arrival of potential investors to a specific market. As investors enter a market, they make trade easier, attracting new investors. This gives rise to a market externality. Interestingly, as investors get attracted to a market, the market can become crowded and thus congestion would reduce the returns to investing. This paper aims to complement the literature on self-fulfilling liquidity by incorporating a second effect: the congestion effect.

In this market traders can choose to invest in one asset which can be traded only when a pair of investors meet and bargain over the terms of trade. Finding a trading partner takes time and introduces opportunity and other costs. Investors' decision to participate in this market and their ability to trade thus affect the illiquidity discount and ultimately, the equilibrium price. We first solve the equilibrium in closed-form when search frictions are small and then introduce a numerical example to complement the analysis of the implications of changes in frictions on liquidity and welfare.

We find that diminishing frictions in a market with many sellers and too few buyers, as it is the case in one-sided markets with sell-side congestion, induces an adverse effect on both liquidity and welfare of current generations of investors. Improving search efficiency leads to a more unbalanced distribution of investors (even less potential buyers per seller trying to exit) magnifying the effect of congestion to the detriment of the overall level of market liquidity and social welfare. From this perspective, this paper presents an example of the Theory of the Second Best, where eliminating one but not all market imperfections does not necessary increase efficiency as it may amplify the effect of the remaining distortions, and provides motivation for measures such as trading halts and circuit breakers.

Appendix

A Proofs of Propositions 1 - 2

Proof of Proposition 1

Proof. Rearranging equation (3), we get

$$h(\eta_0) \equiv \gamma\eta_0^2 - \left(g + \gamma S + \frac{\gamma^2}{\lambda}\right)\eta_0 + Sg = 0$$

where $\eta_0 \in \mathbb{R}_+$. This quadratic function takes positive values as $\eta_0 \to \infty$, is non-negative at $\eta_0 = 0$ and negative at $\eta_0 = S$. Then, by continuity, the polynomial equation has a root in the interval

$[0, S)$ and another one in the interval (S, ∞). The two solutions $\eta_0^{(1)}$ and $\eta_0^{(2)}$ are given by:

$$\begin{aligned}
\eta_0^{(1)} &= \frac{1}{2\gamma}\Big[(g + \gamma S + \frac{\gamma^2}{\lambda}) - \sqrt{(g + \gamma S + \frac{\gamma^2}{\lambda})^2 - 4\gamma g S}\Big] \\
\eta_0^{(2)} &= \frac{1}{2\gamma}\Big[(g + \gamma S + \frac{\gamma^2}{\lambda}) + \sqrt{(g + \gamma S + \frac{\gamma^2}{\lambda})^2 - 4\gamma g S}\Big]
\end{aligned}$$

where $0 \leq \eta_0^{(1)} < S < \eta_0^{(2)} < \infty$. $\eta_0^{(2)}$ is thus not a valid solution since the total supply of the asset is held either by the non-searchers or by the sellers and as a result the measure of non-searchers cannot exceed the supply of the asset. Then, there is unique solution to equation (3) given by:

$$\eta_0 = \frac{1}{2\gamma} A \tag{A.1}$$

where $A = (g + \gamma S + \frac{\gamma^2}{\lambda}) - \sqrt{(g + \gamma S + \frac{\gamma^2}{\lambda})^2 - 4\gamma g S}$. Plugging equation (A.1) into equations (1) and (2), we find

$$\begin{aligned}
\eta_s &= S - \frac{1}{2\gamma} A \\
\eta_b &= \frac{\gamma}{\lambda} \frac{A}{2\gamma S - A}
\end{aligned}$$

which proves Proposition 1. □

Proof of Proposition 2

Proof. Using equation (10), we can rewrite equations (7) and (9) as:

$$rv_b = -\gamma v_b + \lambda \eta_s \frac{z}{1+z}(v_0 - v_b - v_s) \tag{A.2}$$

$$rv_s = d - x + \lambda \eta_b \frac{1}{1+z}(v_0 - v_b - v_s) \tag{A.3}$$

Subtracting equation (A.2) from equation (8) yields:

$$\begin{aligned} r(v_0 - v_b) &= d + \gamma(v_s - v_0) - \Big[-\gamma v_b + \lambda\eta_s \frac{z}{1+z}(v_0 - v_b - v_s)\Big] = \\ &= d + \gamma(v_s - v_0 + v_b) - \lambda\eta_s \frac{z}{1+z}(v_0 - v_b - v_s) \quad \Rightarrow \\ \Rightarrow \quad v_0 - v_b &= \frac{d + (\gamma + \lambda\eta_s \frac{z}{1+z})v_s}{r + \gamma + \lambda\eta_s \frac{z}{1+z}} \end{aligned} \tag{A.4}$$

We can solve for v_s by plugging equation (A.4) into equation (A.3):

$$\begin{aligned} rv_s &= d - x + \lambda\eta_b \frac{1}{1+z}\left[\frac{d + \big(\gamma + \lambda\eta_s \frac{z}{1+z}\big)v_s}{r + \gamma + \lambda\eta_s \frac{z}{1+z}} - v_s\right] = \\ &= d - x + \lambda\eta_b \frac{d - rv_s}{(r+\gamma+\lambda\eta_s)z + r + \gamma} \quad \Rightarrow \\ &\Rightarrow \quad \left[1 + \frac{\lambda\eta_b}{(r+\gamma+\lambda\eta_s)z + r + \gamma}\right] rv_s = \left[1 + \frac{\lambda\eta_b}{(r+\gamma+\lambda\eta_s)z + r + \gamma}\right] d - x \quad \Rightarrow \\ \Rightarrow \quad v_s &= \frac{d}{r} - k\frac{x}{r} - k\frac{x}{(r+\gamma+\lambda\eta_s)z + \gamma} \end{aligned} \tag{A.5}$$

where

$$k \equiv \frac{(r+\gamma+\lambda\eta_s)z + \gamma}{(r+\gamma+\lambda\eta_s)z + (r+\gamma+\lambda\eta_b)}$$

Given v_s, we can determined v_0, v_b and p uniquely from equations (8), (A.2) and (10) respectively. Let us compute them. We can solve for v_0 by plugging equation (A.5) into equation (8):

$$\begin{aligned} rv_0 &= d + \gamma\left[\frac{d}{r} - k\frac{x}{r} - k\frac{x}{(r+\gamma+\lambda\eta_s)z + \gamma}\right] - \gamma v_0 \quad \Rightarrow \\ \Rightarrow \quad v_0 &= \frac{d}{r} - k\frac{x}{r}\frac{\gamma}{r+\gamma} - k\frac{\gamma}{r+\gamma}\frac{x}{(r+\gamma+\lambda\eta_s)z + \gamma} \end{aligned} \tag{A.6}$$

We now compute v_b by substituting equations (A.5) and (A.6) into equation (A.2):

$$
\begin{aligned}
rv_b &= -\gamma v_b + \lambda\eta_s \frac{z}{1+z}\Big[\frac{d}{r} - k\frac{x}{r}\frac{\gamma}{r+\gamma} - k\frac{\gamma}{r+\gamma}\frac{x}{(r+\gamma+\lambda\eta_s)z+\gamma} - \\
&- v_b - \frac{d}{r} + k\frac{x}{r} + k\frac{x}{(r+\gamma+\lambda\eta_s)z+\gamma}\Big] \quad \Rightarrow \\
&\Rightarrow \quad \left[r+\gamma+\lambda\eta_s\frac{z}{1+z}\right] v_b = \lambda\eta_s\frac{z}{1+z}\left[k\frac{x}{r+\gamma} + k\frac{x}{(r+\gamma+\lambda\eta_s)z+\gamma}\frac{r}{r+\gamma}\right] \quad \Rightarrow
\end{aligned}
$$

$$
\Rightarrow \quad v_b = k\frac{x}{r+\gamma}\frac{\lambda\eta_s z}{(r+\gamma+\lambda\eta_s)z+\gamma} \tag{A.7}
$$

We now solve for the price. Plugging equations (A.5) - (A.7) into equation (10) we get:

$$
\begin{aligned}
p &= \frac{1}{1+z}\Bigg[\Big(\frac{d}{r} - k\frac{x}{r} - k\frac{x}{(r+\gamma+\lambda\eta_s)z+\gamma}\Big)z + \\
&+ \frac{d}{r} - k\frac{x}{r}\frac{\gamma}{r+\gamma} - k\frac{\gamma}{r+\gamma}\frac{x}{(r+\gamma+\lambda\eta_s)z+\gamma} - k\frac{x}{r+\gamma}\frac{\lambda\eta_s z}{(r+\gamma+\lambda\eta_s)z+\gamma}\Bigg] = \\
&= \frac{d}{r} - \frac{1}{1+z}\left[k\frac{x}{r}\Big(\frac{\gamma}{r+\gamma} + z\Big) + k\frac{x}{r+\gamma}\right] \quad \Rightarrow
\end{aligned}
$$

$$
\Rightarrow \quad p = \frac{d}{r} - k\frac{x}{r} \tag{A.8}
$$

This concludes the proof of Proposition 2. □

B Additional Proofs

B.1 Proof of Corollary 1: $\frac{\partial\eta_b}{\partial g}, \frac{\partial\eta_0}{\partial g} > 0$ and $\frac{\partial\eta_s}{\partial g} < 0$

Proof. Let us compute the partial derivatives of the measures given by the system of equations (4) - (6) with respect to g:

$$
\frac{\partial\eta_0}{\partial g} = \frac{\partial\eta_0}{\partial A}\frac{\partial A}{\partial g} = \frac{1}{2\gamma}\frac{\partial A}{\partial g} \tag{B.1}
$$

$$
\frac{\partial\eta_s}{\partial g} = \frac{\partial\eta_s}{\partial A}\frac{\partial A}{\partial g} = -\frac{1}{2\gamma}\frac{\partial A}{\partial g} \tag{B.2}
$$

$$
\frac{\partial\eta_b}{\partial g} = \frac{\partial\eta_b}{\partial A}\frac{\partial A}{\partial g} = \frac{2\gamma^2}{\lambda}\frac{S}{\big(2\gamma S - A\big)^2}\frac{\partial A}{\partial g} \tag{B.3}
$$

where

$$\frac{\partial A}{\partial g} = 1 - \frac{(g + \gamma S + \frac{\gamma^2}{\lambda}) - 2\gamma S}{\sqrt{(g + \gamma S + \frac{\gamma^2}{\lambda})^2 - 4\gamma g S}} \tag{B.4}$$

To determine the sign of $\frac{\partial A}{\partial g}$, we check if the second term on the right-hand-side of equation (B.4) is greater than 1:

$$\begin{aligned} \frac{\left(g + \gamma S + \frac{\gamma^2}{\lambda}\right) - 2\gamma S}{\sqrt{\left(g + \gamma S + \frac{\gamma^2}{\lambda}\right)^2 - 4\gamma g S}} &> 1 \quad ; \\ \left(g + \gamma S + \frac{\gamma^2}{\lambda}\right) - 2\gamma S &> \sqrt{\left(g + \gamma S + \frac{\gamma^2}{\lambda}\right)^2 - 4\gamma g S} \quad ; \end{aligned} \tag{B.5}$$

where the right-hand-side of equation (B.5) is strictly positive since

$$\sqrt{(g + \gamma S + \frac{\gamma^2}{\lambda})^2 - 4\gamma g S} = \sqrt{(g - \gamma S)^2 + 2(g + \gamma S)\frac{\gamma^2}{\lambda} + \frac{\gamma^4}{\lambda^2}} > 0 \tag{B.6}$$

We analyze two cases. If $(g + \gamma S + \frac{\gamma^2}{\lambda}) - 2\gamma S \leq 0$, then equation (B.5) is not satisfied. On the contrary, if $(g + \gamma S + \frac{\gamma^2}{\lambda}) - 2\gamma S > 0$,

$$\begin{aligned} \left[\left(g + \gamma S + \frac{\gamma^2}{\lambda}\right) - 2\gamma S\right]^2 &> \left[\sqrt{\left(g + \gamma S + \frac{\gamma^2}{\lambda}\right)^2 - 4\gamma g S}\right]^2; \\ \left(g - \gamma S + \frac{\gamma^2}{\lambda}\right)^2 &> \left(g + \gamma S + \frac{\gamma^2}{\lambda}\right)^2 - 4\gamma g S; \end{aligned}$$

Simplifying we arrive to:

$$4\frac{\gamma^3}{\lambda}S < 0$$

a contradiction, since γ, λ and $S > 0$. Therefore, the second term in equation (B.4) is strictly lower than 1 and as a result:

$$\frac{\partial A}{\partial g} > 0 \tag{B.7}$$

Thus, substituting the previous equation into equations (B.1) - (B.3) yields:

$$\frac{\partial \eta_0}{\partial g} > 0$$
$$\frac{\partial \eta_s}{\partial g} < 0$$
$$\frac{\partial \eta_b}{\partial g} > 0$$

since γ, λ and $S > 0$. $\square$

B.2 Proof of Corollary 2: $\frac{\partial \eta_b}{\partial \lambda}, \frac{\partial \eta_s}{\partial \lambda} < 0$ and $\frac{\partial \eta_0}{\partial \lambda} > 0$

Proof. Using equations (4) - (6) we can compute the partial derivatives of the measures of every type of investor with respect to the efficiency of the search process λ:

$$\frac{\partial \eta_0}{\partial \lambda} = \frac{\partial \eta_0}{\partial A}\frac{\partial A}{\partial \lambda} = \frac{1}{2\gamma}\frac{\partial A}{\partial \lambda} \tag{B.8}$$
$$\frac{\partial \eta_s}{\partial \lambda} = \frac{\partial \eta_s}{\partial A}\frac{\partial A}{\partial \lambda} = -\frac{1}{2\gamma}\frac{\partial A}{\partial \lambda} \tag{B.9}$$
$$\frac{\partial \eta_b}{\partial \lambda} = \frac{\gamma}{\lambda}\frac{1}{2\gamma S - A}\Big[\frac{2\gamma S}{2\gamma S - A}\frac{\partial A}{\partial \lambda} - \frac{1}{\lambda}A\Big] \tag{B.10}$$

where

$$\frac{\partial A}{\partial \lambda} = -\frac{\gamma^2}{\lambda^2}\left(1 - \frac{g + \gamma S + \frac{\gamma^2}{\lambda}}{\sqrt{(g + \gamma S + \frac{\gamma^2}{\lambda})^2 - 4\gamma g S}}\right) \tag{B.11}$$

We verify whether the second term in the expression in parenthesis is greater than 1 to determine the sign of $\frac{\partial A}{\partial \lambda}$,

$$\frac{(g + \gamma S + \frac{\gamma^2}{\lambda})}{\sqrt{(g + \gamma S + \frac{\gamma^2}{\lambda})^2 - 4\gamma g S}} > 1;$$
$$\left(g + \gamma S + \frac{\gamma^2}{\lambda}\right)^2 > \Big[\sqrt{(g + \gamma S + \frac{\gamma^2}{\lambda})^2 - 4\gamma g S}\Big]^2;$$
$$\left(g + \gamma S + \frac{\gamma^2}{\lambda}\right)^2 > \left(g + \gamma S + \frac{\gamma^2}{\lambda}\right)^2 - 4\gamma g S; \tag{B.12}$$

where we can square both sides of the expression because, using equation (B.6) and g, γ, S and

$\lambda > 0$, the numerator and denominator are strictly positive. Rearranging equation (B.12) we get:

$$4\gamma g S > 0$$

which is true since γ, g and $S > 0$. As a result, the second term in the expression in parenthesis in equation (B.11) is strictly greater than 1 and

$$\frac{\partial A}{\partial \lambda} > 0 \tag{B.13}$$

Thus, plugging the previous equation into equations (B.8) - (B.9) we find:

$$\begin{aligned} \frac{\partial \eta_0}{\partial \lambda} &> 0 \\ \frac{\partial \eta_s}{\partial \lambda} &< 0 \end{aligned}$$

The proof that $\frac{\partial \eta_b}{\partial \lambda} < 0$ is not so straightforward. Let us first rearrange equation (B.11) as follows

$$\frac{\partial A}{\partial \lambda} = \frac{\gamma^2}{\lambda^2} \frac{A}{\sqrt{(g + \gamma S + \frac{\gamma^2}{\lambda})^2 - 4\gamma g S}} \tag{B.14}$$

Now, substituting equation (B.14) in equation (B.10) we get:

$$\frac{\partial \eta_b}{\partial \lambda} = \frac{\gamma}{\lambda^2} \frac{A}{2\gamma S - A} \left[\frac{2\gamma S}{2\gamma S - A} \frac{\gamma^2}{\lambda} \frac{1}{\sqrt{(g + \gamma S + \frac{\gamma^2}{\lambda})^2 - 4\gamma g S}} - 1 \right] \tag{B.15}$$

where we need to derive the sign of the expression in brackets to determine the sign of $\frac{\partial \eta_b}{\partial \lambda}$. Let us then verify if the first term of the expression in brackets in equation (B.15) is strictly lower than 1:

$$\begin{aligned} \frac{2\gamma S}{2\gamma S - A} \frac{\gamma^2}{\lambda} \frac{1}{\sqrt{(g + \gamma S + \frac{\gamma^2}{\lambda})^2 - 4\gamma g S}} &< 1; \\ (2\gamma S - A)\lambda \sqrt{\left(g + \gamma S + \frac{\gamma^2}{\lambda}\right)^2 - 4\gamma g S} - 2\gamma^3 S &> 0; \\ \lambda \left[(2\gamma S - A) \sqrt{\left(g + \gamma S + \frac{\gamma^2}{\lambda}\right)^2 - 4\gamma g S} - \frac{2\gamma^3 S}{\lambda} \right] &> 0; \end{aligned}$$

Given that $\lambda > 0$ and $A = (g + \gamma S + \frac{\gamma^2}{\lambda}) - \sqrt{(g + \gamma S + \frac{\gamma^2}{\lambda})^2 - 4\gamma g S}$, then

$$\begin{aligned} \Big[2\gamma S - \big(g + \gamma S + \frac{\gamma^2}{\lambda}\big)\Big]\sqrt{(g + \gamma S + \frac{\gamma^2}{\lambda})^2 - 4\gamma g S} + \Big[\sqrt{(g + \gamma S + \frac{\gamma^2}{\lambda})^2 - 4\gamma g S}\Big]^2 - \frac{2\gamma^3 S}{\lambda} &> 0; \\ \Big(g - \gamma S + \frac{\gamma^2}{\lambda}\Big)^2 - \Big(g - \gamma S + \frac{\gamma^2}{\lambda}\Big)\sqrt{(g - \gamma S + \frac{\gamma^2}{\lambda})^2 + \frac{4\gamma^3 S}{\lambda}} + \frac{2\gamma^3 S}{\lambda} &> 0; \end{aligned}$$

To simplify the exposition of the proof, let us define $D \equiv g - \gamma S + \frac{\gamma^2}{\lambda}$. Therefore,

$$D^2 - D\sqrt{D^2 + \frac{4\gamma^3 S}{\lambda}} + \frac{2\gamma^3 S}{\lambda} > 0 \tag{B.16}$$

We consider two possible scenarios. If $D \leq 0$, then equation (B.16) is satisfied since λ, γ and $S > 0$. On the contrary, if $D > 0$, then we need to prove that

$$D^2 + \frac{2\gamma^3 S}{\lambda} > D\sqrt{D^2 + \frac{4\gamma^3 S}{\lambda}}$$

Squaring both sides and rearranging, we find

$$D^4 + \frac{4\gamma^3 S}{\lambda}D^2 + \frac{4\gamma^6 S^2}{\lambda^2} > D^2\Big(D^2 + \frac{4\gamma^3 S}{\lambda}\Big)$$

Simplifying,

$$\frac{4\gamma^6 S^2}{\lambda^2} > 0$$

and this is always satisfied. Then, we have shown that the first term in the expression in brackets in equation (B.15) is strictly lower than 1 and as a result

$$\frac{\partial \eta_b}{\partial \lambda} < 0$$

which completes the proof. □

B.3 Proof of Corollary 3: $\frac{\partial p}{\partial d}, \frac{\partial p}{\partial \eta_b} > 0$ and $\frac{\partial p}{\partial z}, \frac{\partial p}{\partial \eta_s} < 0$

Proof. Using equation (14), the partial derivative of the price with respect to the dividend flow d is

$$\frac{\partial p}{\partial d} = \frac{1}{r} > 0 \quad \Rightarrow \quad \frac{\partial p}{\partial d} > 0 \quad \forall d$$

Let us now compute the partial derivative of the price with respect to the measure of buyers

η_b:

$$\frac{\partial p}{\partial \eta_b} = \frac{\partial p}{\partial k}\frac{\partial k}{\partial \eta_b} = -\frac{x}{r}\frac{\partial k}{\partial \eta_b}$$

where:

$$\frac{\partial k}{\partial \eta_b} = -\lambda \frac{(r+\gamma+\lambda\eta_s)z+\gamma}{[(r+\gamma+\lambda\eta_s)z+(r+\gamma+\lambda\eta_b)]^2}$$

which is strictly lower than zero since $r, \gamma, \lambda, \eta_s, z > 0$. Therefore,

$$\frac{\partial p}{\partial \eta_b} > 0 \quad \forall \eta_b$$

Next, we obtain the partial derivative of the price with respect to the buyer's bargaining power z:

$$\frac{\partial p}{\partial z} = \frac{\partial p}{\partial k}\frac{\partial k}{\partial z} = -\frac{x}{r}\frac{\partial k}{\partial z}$$

where:

$$\frac{\partial k}{\partial z} = \frac{r\lambda\eta_s\eta_b}{[(r+\gamma+\lambda\eta_s)z+(r+\gamma+\lambda\eta_b)]^2}$$

which is strictly greater than zero since $r, \lambda, \eta_s, \eta_b > 0$. Then,

$$\frac{\partial p}{\partial z} < 0 \quad \forall z$$

To complete the proof, we calculate the partial derivative of the asset price with respect to the measure of sellers:

$$\frac{\partial p}{\partial \eta_s} = \frac{\partial p}{\partial k}\frac{\partial k}{\partial \eta_s} = -\frac{x}{r}\frac{\partial k}{\partial \eta_s}$$

where:

$$\frac{\partial k}{\partial \eta_s} = \frac{\lambda z(r+\lambda\eta_b)}{[(r+\gamma+\lambda\eta_s)z+(r+\gamma+\lambda\eta_b)]^2}$$

which is strictly greater than zero since $r, \lambda, \eta_b, z > 0$. Thus,

$$\frac{\partial p}{\partial \eta_s} < 0 \quad \forall \eta_s$$

$\square$

B.4 Proof of $\frac{\partial p}{\partial g} > 0$

Proof. Using equation (14), the partial derivative of the asset price with respect to the flow of investors g entering the market is

$$\frac{\partial p}{\partial g} = -\frac{x}{r}\frac{\partial k}{\partial g}$$

where $k = \frac{(r+\gamma+\lambda\eta_s)z+\gamma}{(r+\gamma+\lambda\eta_s)z+(r+\gamma+\lambda\eta_b)}$. Let us derive the partial derivative of k with respect to the flow of investors g:

$$\frac{\partial k}{\partial g} = \frac{1}{[(r+\gamma+\lambda\eta_s)z+(r+\gamma+\lambda\eta_b)]^2}\left\{\Big(r+\lambda\eta_b\Big)\lambda z\frac{\partial \eta_s}{\partial g} - \Big[\Big(r+\gamma+\lambda\eta_s\Big)z+\gamma\Big]\lambda\frac{\partial \eta_b}{\partial g}\right\}$$

which is strictly lower than zero since $r, \gamma, \lambda, \eta_s, \eta_b, z > 0$ and $\frac{\partial \eta_s}{\partial g} < 0$ and $\frac{\partial \eta_b}{\partial g} > 0$ as shown Appendix B.1. Then,

$$\frac{\partial p}{\partial g} = -\frac{x}{r}\frac{\partial k}{\partial g} > 0$$

which proves the price increases in the flow of investors entering the market. □

C Proof of Theorem 1

Proof. In our framework, the marginal investor decides whether to enter or not after comparing the expected utility v_{alt} of investing in an alternative market to the expected utility v_b of a buyer in the search market. The expected utility of the marginal investor $v_{alt} = \kappa'$ is a non-negative and strictly increasing function of his outside investment opportunity κ'. Also, $v_b(\kappa' = 0) > v_{alt}(\kappa' = 0) = 0$. Hence, if v_b were decreasing in the outside investment opportunity of the marginal investor, κ', then there would be a unique threshold κ^* satisfying the indifference condition $v_b(g(\kappa^*)) = v_{alt}(\kappa^*)$. Let us show this is the case.

The expected utility v_b of a buyer is a function of the flow of investors g entering the market. Let us compute the partial derivative of v_b, defined in equation (11), with respect to g:

$$\frac{\partial v_b}{\partial g} = \frac{\lambda z x}{r+\gamma}\frac{1}{[(1+z)(r+\gamma)+\lambda(z\eta_s+\eta_b)]^2}\left\{\Big[(1+z)(r+\gamma)+\lambda\eta_b\Big]\frac{\partial \eta_s}{\partial g} - \lambda\eta_s\frac{\partial \eta_b}{\partial g}\right\}$$

which is strictly negative since $r, \gamma, x, z, \lambda, \eta_b, \eta_s > 0$ and $\frac{\partial \eta_s}{\partial g} < 0$ and $\frac{\partial \eta_b}{\partial g} > 0$ as shown Appendix B.1. Hence, the expected utility v_b of a buyer strictly decreases in the flow of investors g entering the market. However g, as given by $g(\kappa') = \int_{\underline{\kappa}}^{\overline{\kappa}} \nu(\kappa)f(\kappa)d\kappa = \int_{\underline{\kappa}}^{\kappa'} f(\kappa)d\kappa$, is increasing in κ'. As a result,

$$\frac{\partial v_b}{\partial \kappa'} = \frac{\partial v_b}{\partial g}\frac{\partial g}{\partial \kappa'} \leq 0$$

where $\frac{\partial v_b}{\partial g} < 0$ and $\frac{\partial g}{\partial \kappa'} \geq 0$.

Then, by continuity, there exists a unique value of κ' satisfying the indifference condition: $v_b(g(\kappa^*)) = v_{alt}(\kappa^*)$. A unique threshold κ^* thus defines a unique flow of investors $g^* = g(\kappa^*)$ entering the market. But given a flow of investors entering the market, there exists unique equilibrium measures $(\eta_b^*, \eta_0^*, \eta_s^*)$ of each type of investor, expected utilities (v_b^*, v_0^*, v_s^*) and price of the asset, p^*, as proved in Propositions 1 and 2. Consequently, market equilibrium, as presented in Definition 1, is unique. This proves Theorem 1. □

D Proof of Propositions 4-8, Corollary 4 and Results 1 and 2

Proof of Proposition 4

Proof. The price is given by $p = \frac{d}{r} - k\frac{x}{r}$, where

$$k = \frac{(r+\gamma)z + \gamma + z\lambda\eta_s}{(r+\gamma)(z+1) + z\lambda\eta_s + \lambda\eta_b}$$

and the measures of sellers and buyers are given by equations (5) and (6) respectively. To prove Proposition 4 we calculate $\lim\limits_{\lambda\to\infty} k$. Multiplying equations (5) and (6) by λ and taking limits as λ tends to infinity gives

$$\begin{aligned}
\lim_{\lambda\to\infty} \lambda\eta_s &= \frac{1}{2\gamma}\lim_{\lambda\to\infty}\left[2\gamma S\lambda - A\lambda\right] = \\
&= \frac{1}{2\gamma}\lim_{\lambda\to\infty}\left[2\gamma S\lambda - (g+\gamma S)\lambda - \gamma^2 + \sqrt{(g-\gamma S)^2\lambda^2 + 2(g+\gamma S)\gamma^2\lambda + \gamma^4}\right] = \\
&= \frac{1}{2\gamma}\lim_{\lambda\to\infty}\left[-(g-\gamma S)\lambda + |g-\gamma S|\lambda\sqrt{1 + \frac{2(g+\gamma S)\gamma^2\lambda + \gamma^4}{(g-\gamma S)^2\lambda^2}}\right] - \frac{\gamma}{2} \approx \\
&\approx \frac{1}{2\gamma}\lim_{\lambda\to\infty}\left[-(g-\gamma S)\lambda + |g-\gamma S|\lambda + \frac{1}{2}\frac{2(g+\gamma S)\gamma^2\lambda + \gamma^4}{|g-\gamma S|\lambda}\right] - \frac{\gamma}{2}
\end{aligned}$$

$$
\begin{aligned}
\lim_{\lambda\to\infty} \lambda\eta_b &= \gamma \lim_{\lambda\to\infty} \frac{A}{2\gamma S - A} \\
&= \gamma \lim_{\lambda\to\infty} \frac{g+\gamma S+\frac{\gamma^2}{\lambda} - \sqrt{(g-\gamma S)^2 + 2(g+\gamma S)\frac{\gamma^2}{\lambda} + \frac{\gamma^4}{\lambda^2}}}{-(g-\gamma S) - \frac{\gamma^2}{\lambda} + \sqrt{(g-\gamma S)^2 + 2(g+\gamma S)\frac{\gamma^2}{\lambda} + \frac{\gamma^4}{\lambda^2}}}
\end{aligned}
$$

Let us consider three alternative cases:

Case (i): $g < \gamma S$. Then, $\lim_{\lambda\to\infty} \lambda\eta_s = \infty$, $\lim_{\lambda\to\infty} \lambda\eta_b = \gamma\frac{g}{\gamma S - g}$. Therefore, $\lim_{\lambda\to\infty} k = 1$ and thus $p = \frac{d-x}{r}$.

Case (ii): $g = \gamma S$. Then, $\lim_{\lambda\to\infty} \lambda\eta_s = \infty$, $\lim_{\lambda\to\infty} \lambda\eta_b = \infty$. Therefore, $\lim_{\lambda\to\infty} k = \frac{z}{z+1}$ and thus $p = \frac{d}{r} - \frac{z}{z+1}\frac{x}{r}$.

Case (iii): $g > \gamma S$. Then, $\lim_{\lambda\to\infty} \lambda\eta_s = \gamma\frac{\gamma S}{g-\gamma S}$, $\lim_{\lambda\to\infty} \lambda\eta_b = \infty$. Therefore, $\lim_{\lambda\to\infty} k = 0$ and thus $p = \frac{d}{r}$. □

Proof of Proposition 5

Proof. The measures of sellers and buyers are given by equations (5) and (6) respectively, where $A = (g+\gamma S+\frac{\gamma^2}{\lambda}) - \sqrt{(g+\gamma S+\frac{\gamma^2}{\lambda})^2 - 4\gamma g S}$. Substituting $g = \gamma S$, we can write A as:

$$
A = 2g + 2\gamma\sqrt{g}\frac{1}{\sqrt{\lambda}} + o\Big(\frac{1}{\sqrt{\lambda}}\Big) \tag{D.1}
$$

where $\sqrt{1+\frac{1}{4g}\frac{\gamma^2}{\lambda}} \approx 1 + \frac{1}{4g}\frac{\gamma^2}{\lambda}$ and $o\Big(\frac{1}{\sqrt{\lambda}}\Big)$ denotes terms of order smaller than $\frac{1}{\sqrt{\lambda}}$.

Plugging equation (D.1) into equations (5) and (6) we find:

$$
\begin{aligned}
\eta_s &= \frac{\sqrt{g}}{\sqrt{\lambda}} + o\Big(\frac{1}{\sqrt{\lambda}}\Big) \\
\eta_b &= \frac{\sqrt{g}}{\sqrt{\lambda}} + o\Big(\frac{1}{\sqrt{\lambda}}\Big)
\end{aligned}
$$

Therefore,

$$
\eta_s = \eta_b = \frac{\sqrt{g}}{\sqrt{\lambda}} + o\Big(\frac{1}{\sqrt{\lambda}}\Big) \tag{D.2}
$$

Let us know derive the expression for the asset price when $g = \gamma S$. The price is given by $p = \frac{d}{r} - k\frac{x}{r}$, where

$$k = \frac{(r+\gamma)z + \gamma + z\lambda\eta_s}{(r+\gamma)(z+1) + z\lambda\eta_s + \lambda\eta_b} \tag{D.3}$$

Multiplying both sides of equation (D.2) by λ and substituting into equation (D.3) we get:

$$k = \frac{z}{z+1} + \frac{\gamma}{z+1}\frac{1}{\sqrt{g}}\frac{1}{\sqrt{\lambda}} + o\Big(\frac{1}{\sqrt{\lambda}}\Big) \tag{D.4}$$

Then,

$$p = \frac{d}{r} - \frac{z}{z+1}\frac{x}{r} - \frac{\gamma}{z+1}\frac{x}{r}\frac{1}{\sqrt{g}}\frac{1}{\sqrt{\lambda}} + o\Big(\frac{1}{\sqrt{\lambda}}\Big)$$

which completes the proof of Proposition 5. $\square$

Proof of Corollary 4

Proof. Illiquidity, measured by price discount, is defined as $k\frac{x}{r}$. In the case of $g = \gamma S$, k is given by equation (D.4). Then, the illiquidity discount is

$$\text{Illiquidity} \equiv k\frac{x}{r} = \Big[\frac{z}{z+1} + \frac{\gamma}{z+1}\frac{1}{\sqrt{g}}\frac{1}{\sqrt{\lambda}} + o\Big(\frac{1}{\sqrt{\lambda}}\Big)\Big]\frac{x}{r} \tag{D.5}$$

Taking limits in (D.5) as λ goes to infinity

$$\lim_{\lambda\to\infty} \text{Illiquidity} \equiv \lim_{\lambda\to\infty} k\frac{x}{r} = \frac{z}{z+1}\frac{x}{r}$$

which is increasing in z. As a result, illiquidity increases in z. $\square$

Proof of Proposition 6

Before we prove Proposition 6 let us introduce Lemmas 1 and 2.

Lemma 1. *If* $F(\frac{x}{r+\gamma}) < \gamma S$ *, then*

i. $\lim\limits_{\lambda\to\infty} \eta_s = S - \frac{1}{\gamma}F\left(\frac{x}{r+\gamma}\right)$

ii. $\lim\limits_{\lambda\to\infty} \eta_b = 0$

iii. $\lim\limits_{\lambda\to\infty} \lambda\eta_s = \infty$

iv. $\lim\limits_{\lambda\to\infty} \lambda\eta_b = \gamma\frac{F(\frac{x}{r+\gamma})}{\gamma S - F(\frac{x}{r+\gamma})}$

Proof of Lemma 1

Proof. Let $F(\frac{x}{r+\gamma}) < \gamma S$ and assume $g^* = F(\frac{x}{r+\gamma}) < \gamma S$ (we prove this result after the proof of Lemma 2). Also, let the measure of sellers and buyers be given by equations (5) and (6). Then, taking limits as λ tends to infinity gives:

$$\begin{aligned}
i. \quad \lim_{\lambda\to\infty} \eta_s &= \frac{1}{2\gamma}\lim_{\lambda\to\infty}[2\gamma S - A] = \\
&= S - \frac{1}{2\gamma}\lim_{\lambda\to\infty}\left[(g+\gamma S) + \frac{\gamma^2}{\lambda} - \sqrt{(g-\gamma S)^2 + 2(g+\gamma S)\frac{\gamma^2}{\lambda} + \frac{\gamma^4}{\lambda^2}}\right] \approx \\
&\approx S - \frac{1}{2\gamma}\lim_{\lambda\to\infty}\left[(g+\gamma S) + \frac{\gamma^2}{\lambda} - (\gamma S - g) - \frac{1}{2}\frac{2(g+\gamma S)\frac{\gamma^2}{\lambda} + \frac{\gamma^4}{\lambda^2}}{(\gamma S - g)}\right] = \\
&= S - \frac{1}{\gamma}F\left(\frac{x}{r+\gamma}\right)
\end{aligned}$$

$$\begin{aligned}
ii. \quad \lim_{\lambda\to\infty} \eta_b &= \lim_{\lambda\to\infty}\frac{\gamma}{\lambda}\frac{A}{2\gamma S - A} \\
&= \gamma\lim_{\lambda\to\infty}\frac{g+\gamma S + \frac{\gamma^2}{\lambda} - \sqrt{(g-\gamma S)^2 + 2(g+\gamma S)\frac{\gamma^2}{\lambda} + \frac{\gamma^4}{\lambda^2}}}{(\gamma S - g)\lambda - \gamma^2 + \sqrt{(g-\gamma S)^2\lambda^2 + 2(g+\gamma S)\gamma^2\lambda + \gamma^4}} \approx \\
&\approx \gamma\lim_{\lambda\to\infty}\frac{(g+\gamma S) + \frac{\gamma^2}{\lambda} - (\gamma S - g) - \frac{1}{2}\frac{2(g+\gamma S)\frac{\gamma^2}{\lambda} + \frac{\gamma^4}{\lambda^2}}{(\gamma S - g)}}{(\gamma S - g)\lambda - \gamma^2 + (\gamma S - g)\lambda + \frac{1}{2}\frac{2(g+\gamma S)\frac{\gamma^2}{\lambda} + \frac{\gamma^4}{\lambda^2}}{(\gamma S - g)}} = 0
\end{aligned}$$

$$\begin{aligned}
iii. \quad \lim_{\lambda\to\infty} \lambda\eta_s &= \frac{1}{2\gamma}\lim_{\lambda\to\infty}[2\gamma S\lambda - A\lambda] = \\
&= \frac{1}{2\gamma}\lim_{\lambda\to\infty}\left[-(g-\gamma S)\lambda + |g-\gamma S|\lambda\sqrt{1 + \frac{2(g+\gamma S)\gamma^2\lambda + \gamma^4}{(g-\gamma S)^2\lambda^2}}\right] - \frac{\gamma}{2} \approx \\
&\approx \frac{1}{2\gamma}\lim_{\lambda\to\infty}\left[-(g-\gamma S)\lambda + (\gamma S - g)\lambda + \frac{1}{2}\frac{2(g+\gamma S)\gamma^2\lambda + \gamma^4}{(\gamma S - g)\lambda}\right] - \frac{\gamma}{2} = \infty
\end{aligned}$$

$$\begin{aligned}
iv. \quad \lim_{\lambda\to\infty} \lambda\eta_b &= \gamma\lim_{\lambda\to\infty}\frac{A}{2\gamma S - A} \\
&= \gamma\lim_{\lambda\to\infty}\frac{g+\gamma S + \frac{\gamma^2}{\lambda} - \sqrt{(g-\gamma S)^2 + 2(g+\gamma S)\frac{\gamma^2}{\lambda} + \frac{\gamma^4}{\lambda^2}}}{-(g-\gamma S) - \frac{\gamma^2}{\lambda} + \sqrt{(g-\gamma S)^2 + 2(g+\gamma S)\frac{\gamma^2}{\lambda} + \frac{\gamma^4}{\lambda^2}}} \approx \\
&\approx \gamma\lim_{\lambda\to\infty}\frac{(g+\gamma S) + \frac{\gamma^2}{\lambda} - (\gamma S - g) - \frac{1}{2}\frac{2(g+\gamma S)\frac{\gamma^2}{\lambda} + \frac{\gamma^4}{\lambda^2}}{(\gamma S - g)}}{(\gamma S - g) - \frac{\gamma^2}{\lambda} + (\gamma S - g) + \frac{1}{2}\frac{2(g+\gamma S)\frac{\gamma^2}{\lambda} + \frac{\gamma^4}{\lambda^2}}{(\gamma S - g)}} = \frac{F\left(\frac{x}{r+\gamma}\right)}{S - \frac{1}{\gamma}F\left(\frac{x}{r+\gamma}\right)} > 0
\end{aligned}$$

$\square$

Lemma 2. *If* $F(\frac{x}{r+\gamma}) \geq \gamma S$*, then*

i. $\lim\limits_{\lambda\to\infty} \eta_s = 0$

ii. $\lim\limits_{\lambda\to\infty} \eta_b = 0$

iii. $\lim\limits_{\lambda\to\infty} \lambda\eta_s = \infty$

iv. $\lim\limits_{\lambda\to\infty} \lambda\eta_b = \infty$

Proof of Lemma 2

Proof. Let $F(\frac{x}{r+\gamma}) \geq \gamma S$ and assume $g^* = \gamma S$ (we will next prove this result). Also, let the measure of sellers and buyers be given by equations (5) and (6). We consider the different rates of convergence of g to γS before taking limits as λ tends to infinity:

$$\begin{aligned}
\lim_{\substack{\lambda\to\infty\\ g\to\gamma S}} \eta_s &= S - \frac{1}{2\gamma}\lim_{\substack{\lambda\to\infty\\ g\to\gamma S}}\left[(g+\gamma S) + \frac{\gamma^2}{\lambda} - \sqrt{(g-\gamma S)^2 + 2(g+\gamma S)\frac{\gamma^2}{\lambda} + \frac{\gamma^4}{\lambda^2}}\right]\\
\lim_{\substack{\lambda\to\infty\\ g\to\gamma S}} \eta_b &= \gamma\lim_{\substack{\lambda\to\infty\\ g\to\gamma S}} \frac{g+\gamma S + \frac{\gamma^2}{\lambda} - \sqrt{(g-\gamma S)^2 + 2(g+\gamma S)\frac{\gamma^2}{\lambda} + \frac{\gamma^4}{\lambda^2}}}{-(g-\gamma S)\lambda - \gamma^2 + \sqrt{(g-\gamma S)^2\lambda^2 + 2(g+\gamma S)\gamma^2\lambda + \gamma^4}}\\
\lim_{\substack{\lambda\to\infty\\ g\to\gamma S}} \lambda\eta_s &= \frac{1}{2\gamma}\lim_{\substack{\lambda\to\infty\\ g\to\gamma S}}\left[-(g-\gamma S)\lambda - |g-\gamma S|\lambda\sqrt{1 + \frac{2(g+\gamma S)\gamma^2\lambda + \gamma^4}{(g-\gamma S)^2\lambda^2}}\right] - \frac{\gamma}{2}\\
\lim_{\substack{\lambda\to\infty\\ g\to\gamma S}} \lambda\eta_b &= \gamma\lim_{\substack{\lambda\to\infty\\ g\to\gamma S}} \frac{g+\gamma S + \frac{\gamma^2}{\lambda} - \sqrt{(g-\gamma S)^2 + 2(g+\gamma S)\frac{\gamma^2}{\lambda} + \frac{\gamma^4}{\lambda^2}}}{-(g-\gamma S) - \frac{\gamma^2}{\lambda} + \sqrt{(g-\gamma S)^2 + 2(g+\gamma S)\frac{\gamma^2}{\lambda} + \frac{\gamma^4}{\lambda^2}}}
\end{aligned}$$

We study six possible cases:

i. Linear convergence from above, i.e. $(g-\gamma S) \sim \frac{c}{\lambda}$, $c > 0$. Then, $\lim\limits_{\substack{\lambda\to\infty\\ g\to\gamma S}} \eta_s = 0$, $\lim\limits_{\substack{\lambda\to\infty\\ g\to\gamma S}} \eta_b = 0$, $\lim\limits_{\substack{\lambda\to\infty\\ g\to\gamma S}} \lambda\eta_s = \infty$ and $\lim\limits_{\substack{\lambda\to\infty\\ g\to\gamma S}} \lambda\eta_b = \infty$.

ii. Linear convergence from below, i.e. $(g-\gamma S) \sim -\frac{c}{\lambda}$, $c > 0$. Then, $\lim\limits_{\substack{\lambda\to\infty\\ g\to\gamma S}} \eta_s = 0$, $\lim\limits_{\substack{\lambda\to\infty\\ g\to\gamma S}} \eta_b = 0$, $\lim\limits_{\substack{\lambda\to\infty\\ g\to\gamma S}} \lambda\eta_s = \infty$ and $\lim\limits_{\substack{\lambda\to\infty\\ g\to\gamma S}} \lambda\eta_b = \infty$.

iii. Superlinear convergence from above, i.e. $(g - \gamma S) \sim \frac{c}{\lambda^p}$, $c > 0$ and $p > 1$. Then, $\lim\limits_{\substack{\lambda\to\infty \\ g\to\gamma S}} \eta_s = 0$, $\lim\limits_{\substack{\lambda\to\infty \\ g\to\gamma S}} \eta_b = 0$, $\lim\limits_{\substack{\lambda\to\infty \\ g\to\gamma S}} \lambda\eta_s = \infty$ and $\lim\limits_{\substack{\lambda\to\infty \\ g\to\gamma S}} \lambda\eta_b = \infty$.

iv. Superlinear convergence from below, i.e. $(g - \gamma S) \sim -\frac{c}{\lambda^p}$, $c > 0$ and $p > 1$. Then, $\lim\limits_{\substack{\lambda\to\infty \\ g\to\gamma S}} \eta_s = 0$, $\lim\limits_{\substack{\lambda\to\infty \\ g\to\gamma S}} \eta_b = 0$, $\lim\limits_{\substack{\lambda\to\infty \\ g\to\gamma S}} \lambda\eta_s = \infty$ and $\lim\limits_{\substack{\lambda\to\infty \\ g\to\gamma S}} \lambda\eta_b = \infty$.

v. Sublinear convergence from above, i.e. $(g - \gamma S) \sim \frac{c}{\lambda^p}$, $c > 0$ and $p < 1$. Then, $\lim\limits_{\substack{\lambda\to\infty \\ g\to\gamma S}} \eta_s = 0$, $\lim\limits_{\substack{\lambda\to\infty \\ g\to\gamma S}} \eta_b = 0$, $\lim\limits_{\substack{\lambda\to\infty \\ g\to\gamma S}} \lambda\eta_s = \infty$ and $\lim\limits_{\substack{\lambda\to\infty \\ g\to\gamma S}} \lambda\eta_b = \infty$.

vi. Sublinear convergence from below, i.e. $(g - \gamma S) \sim -\frac{c}{\lambda^p}$, $c > 0$ and $p < 1$. Then, $\lim\limits_{\substack{\lambda\to\infty \\ g\to\gamma S}} \eta_s = 0$, $\lim\limits_{\substack{\lambda\to\infty \\ g\to\gamma S}} \eta_b = 0$, $\lim\limits_{\substack{\lambda\to\infty \\ g\to\gamma S}} \lambda\eta_s = \infty$ and $\lim\limits_{\substack{\lambda\to\infty \\ g\to\gamma S}} \lambda\eta_b = \infty$.

Therefore, $\lim\limits_{\lambda\to\infty} \eta_s = 0$, $\lim\limits_{\lambda\to\infty} \eta_b = 0$, $\lim\limits_{\lambda\to\infty} \lambda\eta_s = \infty$ and $\lim\limits_{\lambda\to\infty} \lambda\eta_b = \infty$ for any rate of convergence of g to γS. □

Proof of Proposition 6

Proof. Assume that $\lambda \to \infty$. Equation (15) can be written as

$$v_b(g^*) = \kappa^* \tag{D.6}$$

where $\kappa^* = v_{alt}(\kappa^*)$ without loss of generality and the equilibrium flow of investors g^* is given by

$$g^* = \int_{\underline{\kappa}}^{\kappa^*} f(\kappa)d\kappa = F(\kappa^*) - F(\underline{\kappa}) = F(\kappa^*) \qquad \Rightarrow \qquad g^* = F(\kappa^*) \tag{D.7}$$

where f is a continuous and strictly positive function with finite support $[\underline{\kappa}, \overline{\kappa}]$. Then, from equations (D.6) and (D.7)

$$F(v_b(g^*)) = g^* \tag{D.8}$$

We need to solve for the fixed point of equation (D.8).[21] Let us consider two alternative cases:

$$(i) \quad F\left(\frac{x}{r+\gamma}\right) < \gamma S$$
$$(ii) \quad F\left(\frac{x}{r+\gamma}\right) \geq \gamma S$$

Case (i): $F(\frac{x}{r+\gamma}) < \gamma S$. We will show that if $F(\frac{x}{r+\gamma}) < \gamma S$, then $g^* = F(\frac{x}{r+\gamma})$ as λ goes to infinity using a "guess-and-check" argument. Let us assume that $\tilde{g} = F(\frac{x}{r+\gamma})$ is the fixed point of equation (D.8). Substituting $\tilde{g} = F(\frac{x}{r+\gamma})$ in equation (11) when λ goes to infinity yields:

$$F(v_b(\tilde{g})) = F\left(\frac{x}{r+\gamma}\frac{z\lambda\tilde{\eta}_s}{(r+\gamma)(z+1)+z\lambda\tilde{\eta}_s+\lambda\tilde{\eta}_b}\right) \overset{\lambda\to\infty}{\longrightarrow} F\left(\frac{x}{r+\gamma}\right) = \tilde{g}$$

Note that in case (i), $F(\frac{x}{r+\gamma}) < \gamma S$ and hence by assumption $\tilde{g} = F(\frac{x}{r+\gamma}) < \gamma S$. Also, when $\tilde{g} < \gamma S$, $\lambda\tilde{\eta}_s$ is of order λ and $\lambda\tilde{\eta}_b$ is of order 1. As λ goes to infinity, $\lambda\tilde{\eta}_s$ thus tends to infinity while $\lambda\tilde{\eta}_b$ tends to a constant as shown in Lemma 1. As a result, $g^* = F(\frac{x}{r+\gamma})$ is the fixed point of equation (D.8) when $F(\frac{x}{r+\gamma}) < \gamma S$ and $\lambda \to \infty$.

Case (ii): $F(\frac{x}{r+\gamma}) \geq \gamma S$. Assume that $\tilde{g} = \gamma S$ is the fixed point of equation (D.8). We will prove it by contradiction. Let us first suppose that $\tilde{g} > \gamma S$ is the fixed point of equation (D.8). Substituting $\tilde{g} > \gamma S$ in equation (11) when λ goes to infinity yields:

$$F(v_b(\tilde{g})) = F\left(\frac{x}{r+\gamma}\frac{z\lambda\tilde{\eta}_s}{(r+\gamma)(z+1)+z\lambda\tilde{\eta}_s+\lambda\tilde{\eta}_b}\right) \overset{\lambda\to\infty}{\longrightarrow} F(0) = 0 \neq \tilde{g}$$

If $\tilde{g} > \gamma S$, $\lambda\tilde{\eta}_s$ is of order 1 and $\lambda\tilde{\eta}_b$ is of order λ. Hence, as λ goes to infinity, $\lambda\tilde{\eta}_s$ tends to a constant while $\lambda\tilde{\eta}_b$ goes to infinity. As a result, $F(v_b(\tilde{g}))$ tends to zero as λ goes to infinity. But since $\tilde{g}$ is strictly positive, we reach a contradiction and thus $\tilde{g} > \gamma S$ cannot be a fixed point of equation (D.8).

Now suppose that $\tilde{g} < \gamma S$ is the fixed point of equation (D.8). Substituting $\tilde{g} < \gamma S$ in equation (11) when λ goes to infinity yields:

$$F(v_b(\tilde{g})) = F\left(\frac{x}{r+\gamma}\frac{z\lambda\tilde{\eta}_s}{(r+\gamma)(z+1)+z\lambda\tilde{\eta}_s+\lambda\tilde{\eta}_b}\right) \overset{\lambda\to\infty}{\longrightarrow} F\left(\frac{x}{r+\gamma}\right) \neq \tilde{g}$$

Note that in case (ii), $F(\frac{x}{r+\gamma}) \geq \gamma S$, but we assumed that $\tilde{g} < \gamma S$, then $F(\frac{x}{r+\gamma}) \neq \tilde{g}$. We have reached a contradiction and hence $\tilde{g} < \gamma S$ cannot be a fixed point of equation (D.8). As a result, $\tilde{g} = \gamma S$ is the fixed point of equation (D.8) when $F(\frac{x}{r+\gamma}) \geq \gamma S$ and $\lambda \to \infty$. □

[21] The fixed point is unique since the equilibrium is unique as proved in Theorem 1.

Proof of Corollary 5

Proof. The proof follows directly from Proposition 6. Let us consider two cases. If $F(\frac{x}{r+\gamma}) < \gamma S$, then $g^* = F(\frac{x}{r+\gamma}) < \gamma S$. Alternatively, if $F(\frac{x}{r+\gamma}) \geq \gamma S$, then $g^* = \gamma S$. Hence, $g^* \leq \gamma S$. $\square$

Proof of Proposition 7

Proof. Substituting equation (11) into (D.8) yields:

$$F\left(k\frac{x}{r+\gamma}\frac{z\lambda\eta_s}{(r+\gamma)z+\gamma+z\lambda\eta_s}\right) = g^* \Rightarrow k = F^{-1}(g^*)\frac{r+\gamma}{x}\frac{(r+\gamma)z+\gamma+z\lambda\eta_s}{z\lambda\eta_s} \tag{D.9}$$

Let us consider two alternative cases:

$$\begin{aligned} &(i) \quad F\left(\frac{x}{r+\gamma}\right) < \gamma S \\ &(ii) \quad F\left(\frac{x}{r+\gamma}\right) \geq \gamma S \end{aligned}$$

Case (i): $F(\frac{x}{r+\gamma}) < \gamma S$. From Proposition 6, $g^* = F(\frac{x}{r+\gamma})$ when $F(\frac{x}{r+\gamma}) < \gamma S$. Then, substituting g^* into equation (D.9) and taking limits as λ tends to infinity gives

$$\lim_{\lambda\to\infty} k = \lim_{\lambda\to\infty} \frac{(r+\gamma)z+\gamma+z\lambda\eta_s}{z\lambda\eta_s} = 1$$

since $\lambda\eta_s$ tends to infinity as λ goes to infinity (Lemma 1). As a result, $p = \frac{d-x}{r}$ when $F(\frac{x}{r+\gamma}) < \gamma S$.
Case (ii): $F(\frac{x}{r+\gamma}) \geq \gamma S$. From Proposition 6, $g^* = \gamma S$ when $F(\frac{x}{r+\gamma}) \geq \gamma S$. Then, substituting g^* into equation (D.9) and taking limits as λ tends to infinity gives

$$\lim_{\lambda\to\infty} k = \lim_{\lambda\to\infty} F^{-1}(\gamma S)\frac{r+\gamma}{x}\frac{(r+\gamma)z+\gamma+z\lambda\eta_s}{z\lambda\eta_s} = \frac{r+\gamma}{x}F^{-1}(\gamma S)$$

since $\lambda\eta_s$ tends to infinity as λ goes to infinity (Lemma 2) for any rate of convergence of g to γS. Note that $0 \leq k \leq 1$ because $k = \frac{r+\gamma}{x}F^{-1}(\gamma S) \geq 0$ since $r, \gamma > 0$ and $\underline{\kappa} \geq 0$, and $k = \frac{r+\gamma}{x}F^{-1}(\gamma S) \leq 1$ since $F^{-1}(\gamma S) \leq F^{-1}(F(\frac{x}{r+\gamma})) = \frac{x}{r+\gamma}$. As a result, $p = \frac{d}{r} - \frac{r+\gamma}{r}F^{-1}(\gamma S)$ when $F(\frac{x}{r+\gamma}) \geq \gamma S$. This concludes the proof of Proposition 7. $\square$

Proof of Result 1

Proof. When $\lambda \to \infty$ and $F\left(\frac{x}{r+\gamma}\right) < \gamma S$, then $g^* = F(\frac{x}{r+\gamma})$ and $k = 1$. As a result, the illiquidity discount equals $\frac{x}{r}$ (its largest value). Hence, in this case, reducing search frictions does not lead to a more liquid market. $\square$

Proof of Proposition 8

Welfare can be decomposed into four components: the welfare of inside investors, outside investors, future entrants and future outside investors. Lemma 3 summarizes the welfare of inside investors when search frictions are small:

Lemma 3. *When $\lambda \to \infty$, the welfare of inside investors is given by*

$$W_{inside\ investors} = \begin{cases} \dfrac{d}{r}S - \dfrac{x}{r+\gamma}\dfrac{\gamma S}{r} - \dfrac{x}{r+\gamma}\left[S - \dfrac{1}{\gamma}F\left(\dfrac{x}{r+\gamma}\right)\right] & \text{if } F(\frac{x}{r+\gamma}) < \gamma S \\ \dfrac{d}{r}S - F^{-1}(\gamma S)\dfrac{\gamma S}{r} & \text{if } F(\frac{x}{r+\gamma}) \geq \gamma S \end{cases} \tag{D.10}$$

Proof of Lemma 3

Proof. The welfare of inside investors is given by:

$$W_{\text{inside investors}} = \eta_b v_b + \eta_0 v_0 + \eta_s v_s \tag{D.11}$$

Substituting equations (11)-(13) into (D.11) yields:

$$\begin{aligned} W_{\text{inside investors}} &= \eta_b \left\{ k\frac{x}{r+\gamma}\frac{z\lambda\eta_s}{(r+\gamma)z+\gamma+z\lambda\eta_s} \right\} + \\ &+ \eta_0 \left\{ \frac{d}{r} - k\left[\frac{x}{r} + \frac{x}{(r+\gamma)z+\gamma+z\lambda\eta_s}\right]\frac{\gamma}{r+\gamma} \right\} + \\ &+ \eta_s \left\{ \frac{d}{r} - k\left[\frac{x}{r} + \frac{x}{(r+\gamma)z+\gamma+z\lambda\eta_s}\right] \right\} \end{aligned} \tag{D.12}$$

Let us consider two alternative cases:

Case (i): $F(\frac{x}{r+\gamma}) < \gamma S$. From Proposition 6, $g^* = F(\frac{x}{r+\gamma})$ when $F(\frac{x}{r+\gamma}) < \gamma S$. Then, taking limits into equation (D.12) as λ tends to infinity gives

$$W_{\text{inside investors}} = \frac{d}{r}S - \frac{x}{r+\gamma}\frac{\gamma S}{r} - \frac{x}{r+\gamma}\left[S - \frac{1}{\gamma}F\left(\frac{x}{r+\gamma}\right)\right]$$

since η_b tends to 0, η_0 tends to $\frac{1}{\gamma}F(\frac{x}{r+\gamma})$ and η_s tends to $S - \frac{1}{\gamma}F(\frac{x}{r+\gamma})$ as λ goes to infinity and, from Proposition 7, $\lim_{\lambda\to\infty} k = 1$ when $F(\frac{x}{r+\gamma}) < \gamma S$.

Case (ii): $F(\frac{x}{r+\gamma}) \geq \gamma S$. From Proposition 6, $g^* = \gamma S$ when $F(\frac{x}{r+\gamma}) \geq \gamma S$. Then, taking limits into equation (D.12) as λ tends to infinity gives

$$W_{\text{inside investors}} = \frac{d}{r}S - F^{-1}(\gamma S)\frac{\gamma S}{r}$$

since η_b and η_s tend to 0 while η_0 tends to S as λ goes to infinity and, from Proposition 7, $\lim_{\lambda\to\infty} k = \frac{r+\gamma}{x}F^{-1}(\gamma S)$ and $\lim_{\lambda\to\infty} \lambda\eta_s = \infty$ for any rate of convergence of g to γS. This concludes the proof of Lemma 3. □

Let us now focus on the welfare of outside investors and future outside investors. From equation (D.7), $\kappa^* = F^{-1}(g^*)$. Then, substituting the equilibrium flow of investors g^* (Proposition 6) into equations (24) and (26) gives:

$$W_{\text{outside investors}} = \begin{cases} \int_{\frac{x}{r+\gamma}}^{\overline{\kappa}} \kappa f(\kappa)d\kappa & \text{if} \quad F(\frac{x}{r+\gamma}) < \gamma S \\ \int_{F^{-1}(\gamma S)}^{\overline{\kappa}} \kappa f(\kappa)d\kappa & \text{if} \quad F(\frac{x}{r+\gamma}) \geq \gamma S \end{cases} \tag{D.13}$$

$$W_{\text{future outside investors}} = \begin{cases} \frac{1}{r}\int_{\frac{x}{r+\gamma}}^{\overline{\kappa}} \kappa f(\kappa)d\kappa & \text{if} \quad F(\frac{x}{r+\gamma}) < \gamma S \\ \frac{1}{r}\int_{F^{-1}(\gamma S)}^{\overline{\kappa}} \kappa f(\kappa)d\kappa & \text{if} \quad F(\frac{x}{r+\gamma}) \geq \gamma S \end{cases} \tag{D.14}$$

Finally, Lemma 3 presents the welfare of future entrants when search frictions are small:

Lemma 4. *When $\lambda \to \infty$, the welfare of future entrants is given by*

$$W_{\textit{future entrants}} = \begin{cases} \frac{1}{r}\frac{x}{r+\gamma}F\left(\frac{x}{r+\gamma}\right) & \textit{if} \quad F(\frac{x}{r+\gamma}) < \gamma S \\ \frac{1}{r}F^{-1}(\gamma S)\gamma S & \textit{if} \quad F(\frac{x}{r+\gamma}) \geq \gamma S \end{cases} \tag{D.15}$$

Proof of Lemma 4

Proof. Substituting equation (11) into (25) yields

$$W_{\text{future entrants}} = \frac{1}{r}F(\kappa^*)k\frac{x}{r+\gamma}\frac{\lambda\eta_s z}{(r+\gamma)z + z\lambda\eta_s + \gamma} \tag{D.16}$$

Let us consider two alternative cases:

Case (i): $F(\frac{x}{r+\gamma}) < \gamma S$. From Proposition 6, $g^* = F(\frac{x}{r+\gamma})$ when $F(\frac{x}{r+\gamma}) < \gamma S$. Also, from equation (D.7), $g^* = F(\kappa^*)$. Then, plugging $F(\kappa^*) = F(\frac{x}{r+\gamma})$ into equation (D.16) and taking limits as λ tends to infinity gives

$$W_{\text{future entrants}} = \frac{1}{r}F\left(\frac{x}{r+\gamma}\right)\frac{x}{r+\gamma}$$

since $\lambda\eta_s$ tends to infinity as λ goes to infinity and, from Proposition 7, $\lim_{\lambda\to\infty} k = 1$ when $F(\frac{x}{r+\gamma}) < \gamma S$.

Case (ii): $F(\frac{x}{r+\gamma}) \geq \gamma S$. From Proposition 6, $g^* = \gamma S$ when $F(\frac{x}{r+\gamma}) \geq \gamma S$. Also, from equation (D.7), $g^* = F(\kappa^*)$. Then, plugging $F(\kappa^*) = \gamma S$ into equation (D.16) and taking limits as λ tends to infinity gives

$$W_{\text{future entrants}} = \frac{1}{r}\gamma S F^{-1}(\gamma S)$$

since, as shown in Proposition 7, $\lim_{\lambda\to\infty} \lambda\eta_s = \infty$ for any rate of convergence of g to γS and $\lim_{\lambda\to\infty} k = \frac{r+\gamma}{x}F^{-1}(\gamma S)$. This concludes the proof of Lemma 4. $\square$

Adding equations (D.10), (D.13)-(D.15) yields the final expression of welfare in equation (27), which concludes the proof of Proposition 8.

Proof of Result 2

Proof. When $\lambda \to \infty$ and $F\left(\frac{x}{r+\gamma}\right) < \gamma S$, total welfare $W(F(\frac{x}{r+\gamma}) < \gamma S)$ is given by:

$$\frac{d}{r}S + \frac{1+r}{r}\int_{\frac{x}{r+\gamma}}^{\overline{\kappa}} \kappa f(\kappa)d\kappa - \frac{x}{r}\left[S - \frac{1}{\gamma}F\left(\frac{x}{r+\gamma}\right)\right] < \frac{d}{r}S + \frac{1+r}{r}\int_{F^{-1}(\gamma S)}^{\overline{\kappa}} \kappa f(\kappa)d\kappa$$

which is the value of welfare when $F(\frac{x}{r+\gamma}) \geq \gamma S$ and λ goes to infinity, since $\frac{x}{r+\gamma} \geq F^{-1}(\gamma S)$ and $S - \frac{1}{\gamma}F\left(\frac{x}{r+\gamma}\right) > 0$. If we are in market where $F(\frac{x}{r+\gamma}) < \gamma S$, then even when search frictions tend to zero welfare will be lower than it would have been in the case where $F(\frac{x}{r+\gamma}) \geq \gamma S$. In this case, reducing frictions would not lead to the highest possible level of welfare. $\square$

E Market Equilibrium: General Case

Market equilibrium is the solution to the system of equations (1)-(3), (7)-(10) and (15). We thus need to solve for the fixed point of this system, which is reduced to solving the indifference condition that defines investors' entry rule. Investors, in our framework, compare the expected utility v_{alt} of investing in an alternative market to the expected utility v_b derived from being a buyer and they decide to move in whenever $v_b > \kappa' \equiv v_{alt}$. For simplicity, we derive the equilibrium measure of sellers (η_s^*) as a function of the efficiency of the search process λ and the other nine parameters of the model (γ, r, S, x, z, a, b, $\underline{\kappa}$ and $\overline{\kappa}$). To present the indifference condition ($v_b = \kappa'$) as a function of the measure of sellers (η_s), let us first redefine the measure of buyers η_b as a function of η_s. Using equations (5) and (6) we find:

$$\eta_b = \frac{\gamma}{\lambda}\frac{A}{2\gamma S - A} = \frac{\gamma}{\lambda}\frac{2\gamma(S-\eta_s)}{2\gamma S - 2\gamma(S-\eta_s)} \qquad \Rightarrow \qquad \eta_b = \frac{\gamma}{\lambda}\frac{S-\eta_s}{\eta_s} \tag{E.1}$$

We can now express the expected utility v_b of buyers as a function of η_s by substituting equation (E.1) into equation (11):

$$v_b = \frac{x}{r+\gamma}\frac{\lambda z \eta_s^2}{\lambda z \eta_s^2 + \left[(r+\gamma)(1+z) - \gamma\right]\eta_s + \gamma S} \tag{E.2}$$

Next we write κ' as a function of η_s. In this model, the flow of investors g who move into the market is determined by the proportion of the total flow of investors f whose expected utility v_b of being a buyer exceeds their best outside option κ'. We assume the flow of investors f follows a beta distribution with support $[\underline{\kappa}, \overline{\kappa}]$ and shape parameters:[22] a and b. For notational convenience

[22] The probability density function of the beta distribution defined over the interval $[0,1]$ with shape parameters a and b is

$$f_{beta}(y;a,b) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} y^{a-1}(1-y)^{b-1}$$

where $a, b > 0$ and $\Gamma(\cdot)$ is the gamma function. For integer values of a and b, the cumulative distribution function of the beta distribution is given by:

$$F_{beta}(y;a,b) = \sum_{j=a}^{a+b-1} \binom{a+b-1}{j} y^j (1-y)^{a+b-1-j}$$

where $\binom{a+b-1}{j} = \frac{(a+b-1)!}{j!(a+b-1-j)!}$.

we omit reference to the shape parameters. Then,

$$g(\kappa') = \int_{\underline{\kappa}}^{\kappa'} f_{beta}(\kappa) d\kappa = F_{beta}(\kappa') \qquad \Rightarrow \qquad \kappa' = F_{beta}^{-1}(g) \tag{E.3}$$

where f_{beta} and F_{beta} denote respectively the probability density function (*pdf*) and the cumulative distribution function (*cdf*) of a beta distribution. F_{beta}^{-1} is the inverse cumulative distribution function. Using equation (5) and the definition of A in Page 12 we can express the flow of investors g as a function of the measure of sellers η_s:

$$g = \gamma \left(1 + \frac{\gamma}{\lambda \eta_s}\right)(S - \eta_s) \tag{E.4}$$

Substituting equation (E.4) in equation (E.3) yields:

$$\kappa' = F_{beta}^{-1}\left(\gamma \left(1 + \frac{\gamma}{\lambda \eta_s}\right)(S - \eta_s)\right) \tag{E.5}$$

The indifference condition results from equating the expected utility v_b of buyers (equation (E.2)) to the marginal investor outside option κ' (equation (E.5)):

$$\frac{x}{r+\gamma} \frac{\lambda z \eta_s^2}{\lambda z \eta_s^2 + [(r+\gamma)(1+z) - \gamma]\eta_s + \gamma S} = F_{beta}^{-1}\left(\gamma \left(1 + \frac{\gamma}{\lambda \eta_s}\right)(S - \eta_s)\right)$$

Rearranging, we get

$$\gamma \left(1 + \frac{\gamma}{\lambda \eta_s}\right)(S - \eta_s) = F_{beta}\left(\frac{x}{r+\gamma} \frac{\lambda z \eta_s^2}{\lambda z \eta_s^2 + [(r+\gamma)(1+z) - \gamma]\eta_s + \gamma S}\right)$$

Then,

$$\begin{aligned}\gamma \left(1 + \frac{\gamma}{\lambda \eta_s}\right)(S - \eta_s) = \sum_{j=a}^{a+b-1} \binom{a+b-1}{j} &\Big(\frac{x}{r+\gamma} \frac{\lambda z \eta_s^2}{\lambda z \eta_s^2 + [(r+\gamma)(1+z) - \gamma]\eta_s + \gamma S}\Big)^j \\ &\Big(1 - \frac{x}{r+\gamma} \frac{\lambda z \eta_s^2}{\lambda z \eta_s^2 + [(r+\gamma)(1+z) - \gamma]\eta_s + \gamma S}\Big)^{a+b-1-j}\end{aligned} \tag{E.6}$$

Equation (E.6) is a polynomial of degree $2(a+b)$ in the measure of sellers.[23] To solve for η_s^*

[23]In the simple case of shape parameters of the beta distribution both equal to 1 ($a = 1 = b$), which

we use the bisection method.[24] Market equilibrium consists of $((\eta_s^*, \eta_b^*, \eta_0^*), (v_b^*, v_0^*, v_s^*, p^*)$ and $\nu^*)$. Once we compute η_s^*, we can determine η_0^* and η_b^* by substituting η_s^* in equations (1) and (E.1). Then, we can derive v_b^*, v_0^*, v_s^* and p^* by plugging equation (E.1) and η_s^* into equations (11) - (14) respectively. Finally, there is a one-to-one relationship between g^* and η_s^*. Given η_s^*, we can determine the equilibrium flow of investors g^* entering market (equation (E.4)). Once we derive g^*, equation (15) specifies a unique κ^* and thus a unique equilibrium fraction ν^* of investors who prefer to invest in this market.

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corresponds to a uniform distribution with support $[\underline{\kappa}, \overline{\kappa}]$, the indifference condition ($v_b = \kappa'$) is:

$$\frac{x}{r+\gamma}\frac{\lambda z \eta_s^2}{\lambda z \eta_s^2 + [(r+\gamma)(1+z)-\gamma]\eta_s + \gamma S} = \underline{\kappa} + (\overline{\kappa} - \underline{\kappa})\gamma\left(1 + \frac{\gamma}{\lambda \eta_s}\right)(S - \eta_s)$$

Reorganizing terms yields the following polynomial of degree four in the measure of sellers η_s:

$$\begin{aligned} \lambda^2 z(\overline{\kappa} - \underline{\kappa})\gamma\eta_s^4 \quad &+ \quad \Big[\lambda(\overline{\kappa} - \underline{\kappa})\gamma C - \lambda z D + \frac{x}{r+\gamma}\lambda^2 z\Big]\eta_s^3 + \Big[\lambda(\overline{\kappa} - \underline{\kappa})\gamma^2 S(1-z) - CD\Big]\eta_s^2 - \\ &- \quad \Big[\gamma S D + (\overline{\kappa} - \underline{\kappa})\gamma^2 S C\Big]\eta_s - (\overline{\kappa} - \underline{\kappa})\gamma^3 S^2 = 0 \end{aligned}$$

where $C = (r+\gamma)(1+z) - \gamma$ and $D = \lambda\underline{\kappa} + \lambda(\overline{\kappa} - \underline{\kappa})\gamma S - (\overline{\kappa} - \underline{\kappa})\gamma^2$. There exists closed-form solution to this equation. In particular, there are at most four solutions but only one, η_s^*, (as proved in Subsection 5.2) lies in the interval $[0, S]$, the set of possible values of the measure of sellers. Unfortunately, the solution is intractable. We use the bisection method over the interval $[0, S]$ to determine the zero of this equation.

[24]The bisection algorithm is a numerical method for finding the root of a function. It recursively divides an interval in half and selects the subinterval containing the root, until the interval is sufficiently small. Burden and Faires (1993) presents a clear description of this algorithm as well as other numerical methods for solving root-finding problems.

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