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Abstract

We present a dynamic over-the-counter model of the fed funds market, and use it to study the determination of the fed funds rate, the volume of loans traded, and the intraday evolution of the distribution of reserve balances across banks. We also investigate the implications of changes in the market structure, as well as the effects of central bank policy instruments such as open market operations, the Discount Window lending rate, and the interest rate on bank reserves.

Key words: Fed funds market, search, bargaining, over-the-counter

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1 Introduction

In the United States, financial institutions keep reserve balances at the Federal Reserve Banks to meet requirements, earn interest, or to clear financial transactions. The *fed funds market* is an interbank over-the-counter market for unsecured, mostly overnight loans of dollar reserves held at Federal Reserve Banks. The *fed funds rate* is an average measure of the market interest rate on these loans. The fed funds market acts primarily as a mechanism that reallocates reserves among banks: it allows institutions with excess balances to lend to institutions with deficiencies, and therefore helps participants to manage their reserves and offset liquidity or payment shocks. The fed funds market plays a prominent role in policy, as it has traditionally being the epicenter of monetary policy implementation: The Federal Open Market Committee (FOMC) periodically chooses a fed funds rate target, and implements monetary policy by instructing the trading desk at the Federal Reserve Bank of New York to create conditions in reserve markets that will encourage fed funds to trade at the target level.

In this paper we pry into the micro structure of the fed funds market in order to explain the process of reallocation of liquidity among banks, and the determination of the market price for this liquidity provision—the fed funds rate. Specifically, we study a small-scale version of the fed funds market model developed in Afonso and Lagos (2013) that admits analytical solutions. Relative to the existing literature on the fed funds market, our contribution is to model the intraday allocation of reserves and pricing of overnight loans using a dynamic equilibrium search-theoretic framework that captures the salient features of the decentralized interbank market in which these loans are traded.[1]

The model is presented in Section 2. The equilibrium is then characterized in Section 3. We study the normative properties of the equilibrium in Section 4, and the frictionless competitive limit in Section 5. Section 6 describes the positive implications of the theory for trade volume, the determination of the fed funds rate, and the intraday evolution of the distribution of reserve

[1] Early research on the fed funds market includes the theoretical work of Poole (1968), Ho and Saunders (1985) and Coleman et al (1996), and the empirical work of Hamilton (1996) and Hamilton and Jordà (2002). Gaspar et al (2008) simulate a three-period model where banks trade in exogenously segmented competitive markets, and use it to motivate and interpret a regression analysis of the overnight interbank interest rates in the euro area. The over-the-counter nature of the fed funds market was stressed by Ashcraft and Duffie (2007) in their empirical investigation, and used by Bech and Klee (2011), Ennis and Weinberg (2009), and Furfine (2003), to explain certain aspects of interbank markets such as apparent limits to arbitrage, stigma, and banks' decisions to borrow from standing facilities. Recent theoretical work on financial over-the-counter markets includes Duffie et al (2005) and Lagos and Rocheteau (2009).

balances across banks, i.e., three key variables that summarize how effectively the fed funds market reallocates reserve balances across banks. In Section 8 we show how the equilibrium intraday time path for the fed funds rate depends on the market structure (such as the rate at which transactions take place, or the bargaining power of borrowers) and policy parameters (such as the cost of ending the trading day with excess or defficient reserve balances). Section 7 explains how the model can be used to study some elementary policy questions in central banking, and discusses the relationship with more traditional approaches that abstract from the dynamic and over-the-counter aspects of the fed funds market. Section 9 concludes. The appendix contains all proofs.

2 The model

There is a large population of agents that we refer to as *banks*, each represented by a point in the interval $[0, 1]$. Banks hold an asset that we interpret as *reserve balances*, and can negotiate these balances during a trading session set in continuous time that starts at time 0 and ends at time T. Let τ denote the time remaining until the end of the trading session, so $\tau = T - t$ if the current time is $t \in [0, T]$. The reserve balance that a bank holds at time $T - \tau$ is denoted by $k(\tau) \in \mathbb{K}$, with $\mathbb{K} = \{0, 1, 2\}$. The measure of banks with balance k at time $T - \tau$ is denoted $n_k(\tau)$. Each bank starts the trading session with some balance $k(T) \in \mathbb{K}$; the initial distribution of balances, $\{n_k(T)\}_{k \in \mathbb{K}}$, is given. Let $U_k \in \mathbb{R}$ be the payoff from holding k reserve balances at the end of the trading session. All banks discount payoffs at rate r.

Banks can trade balances with each other in an over-the-counter market where trading opportunities are bilateral and random, and represented by a Poisson process with arrival rate $\alpha > 0$. We model these bilateral transactions as loans of reserve balances. Once two banks have made contact, the size of the loan and the repayment are determined by Nash bargaining where $\theta \in [0, 1]$ denotes the bargaining power of the borrower. After the terms of the transaction have been agreed upon, the banks part ways. We assume that every loan gets repayed at time $T + \Delta$ in the following trading day, where $\Delta \in \mathbb{R}_+$. Let $x \in \mathbb{R}$ denote the net credit position (of reserves due at $T + \Delta$) that has resulted from some history of trades. We assume that the payoff to a bank with a net credit position x that makes a new loan at time $T - \tau$ with repayment R at time $T + \Delta$, is equal to the post-transaction discounted net credit position, $e^{-r(\tau+\Delta)}(x + R)$.

3 Equilibrium

Let $J_k(x,\tau) : \mathbb{K} \times \mathbb{R} \times [0,T] \to \mathbb{R}$ be the maximum attainable payoff of a bank that holds k units of reserve balances and whose net credit position is x, when the time until the end of the trading session is τ. In Afonso and Lagos (2013) we show that $J_k(x,\tau) = V_k(\tau) + e^{-r(\tau+\Delta)}x$, where $\{V_k(\tau)\}_{(k,\tau)\in\mathbb{K}\times[0,T]}$ satisfies

$$rV_0(\tau) + \dot{V}_0(\tau) = \alpha n_2(\tau) \max\left\{V_1(\tau) - V_0(\tau) - e^{-r(\tau+\Delta)}R(\tau), 0\right\} \tag{1}$$

$$rV_1(\tau) + \dot{V}_1(\tau) = 0 \tag{2}$$

$$rV_2(\tau) + \dot{V}_2(\tau) = \alpha n_0(\tau) \max\left\{V_1(\tau) - V_2(\tau) + e^{-r(\tau+\Delta)}R(\tau), 0\right\} \tag{3}$$

with

$$V_k(0) = U_k \text{ for all } k \in \mathbb{K} \tag{4}$$

and

$$\begin{aligned} e^{-r(\tau+\Delta)}R(\tau) &= \arg\max_R \left[V_1(\tau) - R - V_0(\tau)\right]^\theta \left[V_1(\tau) + R - V_2(\tau)\right]^{1-\theta} \\ &= \theta\left[V_2(\tau) - V_1(\tau)\right] + (1-\theta)\left[V_1(\tau) - V_0(\tau)\right]. \end{aligned} \tag{5}$$

The value function $V_k(\tau)$ represents the maximum attainable expected discounted payoff to a bank that holds k units of reserve balances at time $t = T - \tau$. The Bellman equations (1)-(3) are standard: $rV_k(\tau)$, the "flow value" of a bank that holds reserve balance k at time $T - \tau$, is composed of two capital gain terms. The first, $\dot{V}_k(\tau)$, is associated with the passage of time (since the problem is nonstationary given that the trading day ends at time T). The other capital gain term reflects the gains from trading reserves with another bank: it consists of the rate at which the bank contacts a counterparty with balance k, $\alpha n_k(\tau)$, times the gain from trade that the bank obtains in the bilateral transaction. Notice that the joint gain from trade, and therefore each bank's individual gain from trade is necessarily nil whenever either of the banks in the bilateral meeting has a reserve balance equal to 1. Thus the only payoff-relevant bargaining situation results when a bank with $k = 0$ meets a bank with $k' = 2$. In this case there are two relevant outcomes: the bank with $k' = 2$ lends one unit of reserves to the bank with $k = 0$, or there is no loan. In the former case, the borrower and the lender reap gains from trade equal to $V_1(\tau) - V_0(\tau) - e^{-r(\tau+\Delta)}R(\tau)$ and $V_1(\tau) - V_2(\tau) + e^{-r(\tau+\Delta)}R(\tau)$, respectively, with the present value of the loan repayment given by (5). The capital gain to each bank is 0 if there is no loan.

With the bargaining outcome (5), the Bellman equations (1)-(3) become

$$rV_0(\tau) + \dot{V}_0(\tau) = \alpha n_2(\tau)\,\phi(\tau)\,\theta S(\tau) \tag{6}$$

$$rV_1(\tau) + \dot{V}_1(\tau) = 0 \tag{7}$$

$$rV_2(\tau) + \dot{V}_2(\tau) = \alpha n_0(\tau)\,\phi(\tau)\,(1-\theta)\,S(\tau) \tag{8}$$

where

$$S(\tau) \equiv 2V_1(\tau) - V_0(\tau) - V_2(\tau)$$

and

$$\phi(\tau) = \begin{cases} 1 & \text{if } 0 < S(\tau) \\ 0 & \text{if } S(\tau) \leq 0. \end{cases} \tag{9}$$

Intuitively, $S(\tau)$ is the value of executing a trade when the remaining time is τ, i.e., the total gain from trade or "surplus" that can be jointly achieved in a bilateral trade at time $t = T - \tau$ between a bank with 2 units of reserve balances, and a bank with 0 balance. The function $\phi(\tau) : [0,T] \to \{0,1\}$ represents the probability that the outcome of the bargaining is that the former lends one unit of reserves to the latter (i.e., the probability that trade is mutually beneficial). From the Bellman equations (6)-(8) it is clear that conditional on trade, the borrower and the lender of reserves get a fraction θ and $1-\theta$ of the total gain from trade, $S(\tau)$, respectively.

Given the initial condition $\{n_k(T)\}_{k\in\mathbb{K}}$, the distribution of balances at time $T-\tau$, i.e., $\{n_k(\tau)\}_{k\in\mathbb{K}}$, evolves according to

$$\dot{n}_0(\tau) = \alpha\phi(\tau)\,n_2(\tau)\,n_0(\tau) \tag{10}$$

$$\dot{n}_1(\tau) = -2\alpha\phi(\tau)\,n_2(\tau)\,n_0(\tau) \tag{11}$$

$$\dot{n}_2(\tau) = \alpha\phi(\tau)\,n_2(\tau)\,n_0(\tau)\,. \tag{12}$$

Definition 1 *An equilibrium is a time path for the value function, $\{V_k(\tau)\}_{(k,\tau)\in\mathbb{K}\times[0,T]}$, a time path for the distribution of reserve balances, $\{n_k(\tau)\}_{(k,\tau)\in\mathbb{K}\times[0,T]}$, and a time path for the trading probability, $\{\phi(\tau)\}_{\tau\in[0,T]}$, such that: (a) given time the time paths of the value function and the trading probability, the time path of the distribution of balances satisfies (10)-(12) with initial condition $\{n_k(T)\}_{k\in\mathbb{K}}$; and (b) given the time path for the distribution of balances, the time paths of the value function and the trading probability satisfy (4), (6)-(8), and (9).*

In order to characterize an equilibrium, it is useful to combine (6), (7) and (8) to obtain

$$\dot{S}(\tau) + \delta(\tau) S(\tau) = 0, \tag{13}$$

where

$$\delta(\tau) \equiv \{r + \alpha\phi(\tau)[\theta n_2(\tau) + (1-\theta) n_0(\tau)]\}.$$

Given (4), we have $S(0) = 2U_1 - U_2 - U_0$, so (13) can be solved for

$$S(\tau) = e^{-\bar{\delta}(\tau)} S(0), \tag{14}$$

where $\bar{\delta}(\tau) \equiv \int_0^\tau \delta(x)\,dx$.[2] Hereafter we specialize the analysis to the case in which the vector of terminal payoffs $\{U_k\}_{k\in\mathbb{K}}$ satisfies the following assumption:

$$S(0) > 0. \tag{A}$$

Assumption (A) ensures that the vector $\{U_k\}_{k\in\mathbb{K}}$ has, loosely speaking, a "strict concavity property" in the sense that $U_1 - U_0 > U_2 - U_1$. In reality, central banks that pay interest in reserves typically do not implement compensation schemes that are convex in the level of reserves, so Assumption (A) seems reasonable in the context of our application. Under Assumption (A), (14) implies $S(\tau) > 0$ for all τ, and therefore (9) implies $\phi(\tau) = 1$ for all τ. Then together with the initial condition $\{n_k(T)\}_{k\in\mathbb{K}}$, the law of motion for the distribution of reserve balances, (10)-(12), allows us to solve for the cross-sectional distribution of reserve balances across banks at each time τ, i.e.,

$$n_0(\tau) = \begin{cases} \frac{[n_2(T)-n_0(T)]n_0(T)}{e^{\alpha[n_2(T)-n_0(T)](T-\tau)}n_2(T)-n_0(T)} & \text{if } n_2(T) \neq n_0(T) \\ \frac{n_0(T)}{1+\alpha n_0(T)(T-\tau)} & \text{if } n_2(T) = n_0(T) \end{cases} \tag{15}$$

$$n_1(\tau) = 1 - 2n_0(\tau) + n_0(T) - n_2(T) \tag{16}$$

$$n_2(\tau) = n_0(\tau) + n_2(T) - n_0(T). \tag{17}$$

At this point it is clear that an equilibrium exists and that it is unique: given $\phi(\tau) = 1$ and (15)-(17), the unique path for the surplus $S(\tau)$ is given explicitly by (14), and given $S(\tau)$ and

[2] According to (14), $S(\tau)$ is a discounted version of $S(0)$ with effective discount rate given by $\bar{\delta}(\tau)$. Intuitively, the actual payoffs from the various reserve balances accrue at the end of the trading session, so $S(0)$ is discounted by the pure rate of time preference, r. The value $S(0)$ is discounted further when the remaining time is $\tau > 0$, because banks might still meet alternative trading partners before the end of the session, and this possibility increases their outside options. The borrower's outside option, $V_0(\tau)$, is increasing in the average rate at which he is able to contact a lender and reap gains from trade between time $T-\tau$ and T, i.e., $\alpha\theta\int_0^\tau n_2(s)\,ds$. Similarly, the lender's outside option, $V_2(\tau)$, is increasing in the average rate at which he is able to contact a borrower and reap gains from trade between time $T-\tau$ and T, i.e., $\alpha(1-\theta)\int_0^\tau n_0(s)\,ds$.

the terminal condition (4), the linear system of ordinary differential equations (6)-(8) can be solved for the time path of the value function, i.e.,

$$V_0(\tau) = e^{-r\tau}U_0 + \int_0^\tau e^{-r(\tau-z)}\alpha n_2(z)\,\theta S(z)\,dz \tag{18}$$

$$V_1(\tau) = e^{-r\tau}U_1 \tag{19}$$

$$V_2(\tau) = e^{-r\tau}U_2 + \int_0^\tau e^{-r(\tau-z)}\alpha n_0(z)\,(1-\theta)\,S(z)\,dz. \tag{20}$$

The present value of the repayment (given in (5)) can be written as

$$e^{-r(\tau+\Delta)}R(\tau) = V_1(\tau) - V_0(\tau) - \theta S(\tau). \tag{21}$$

Let $\rho(\tau)$ denote the interest rate implicit in a loan that promises to repay $R(\tau)$ at time $\tau+\Delta$ for one unit of reserve balances borrowed at time $T-\tau$. According to the conventional calculations of fed funds market analysts,

$$1+\rho(\tau) \equiv R(\tau). \tag{22}$$

Some properties of the path for the equilibrium surplus are immediate from (14). For example, $\dot{S}(\tau) < 0$ (the gain from trade is increasing in chronological time, i.e., as t approaches T). The following proposition reports analytical expressions for the equilibrium surplus and the interest rate.

Proposition 1 *Suppose (A) holds. The surplus of a match at time $T-\tau$ between a bank with balance $k=2$ and a bank with balance $k'=0$ is*

$$S(\tau) = \begin{cases} \frac{n_2(T)-e^{-\alpha[n_2(T)-n_0(T)](T-\tau)}n_0(T)}{n_2(T)-e^{-\alpha[n_2(T)-n_0(T)]T}n_0(T)}e^{-\{r+\alpha\theta[n_2(T)-n_0(T)]\}\tau}S(0) & \text{if } n_2(T) \neq n_0(T) \\ \frac{1+\alpha n_0(T)(T-\tau)}{1+\alpha n_0(T)T}e^{-r\tau}S(0) & \text{if } n_2(T) = n_0(T). \end{cases}$$

The equilibrium loan repayment is

$$R(\tau) = e^{r\Delta}\left\{\Theta(\tau)(U_2-U_1) + [1-\Theta(\tau)](U_1-U_0)\right\}$$

and the interest paid on the loan is $\rho(\tau) = R(\tau)-1$, where

$$\Theta(\tau) = \begin{cases} \beta(\tau) & \text{if } n_2(T) \neq n_0(T) \\ \theta & \text{if } n_2(T) = n_0(T) \end{cases}$$

with

$$\beta(\tau) \equiv \frac{\theta\left[n_2(T)-e^{-\alpha[n_2(T)-n_0(T)](T-\tau)}n_0(T)\right]e^{-\alpha\theta[n_2(T)-n_0(T)]\tau}+\left[1-e^{-\alpha\theta[n_2(T)-n_0(T)]\tau}\right]n_2(T)}{n_2(T)-e^{-\alpha[n_2(T)-n_0(T)]T}n_0(T)}.$$

To interpret the expression for the interest rate, notice that $\beta(\tau) \in [0,1]$, with $\beta(0) = \theta$, and $\partial\beta(\tau)/\partial\theta > 0$ for all τ. It is possible to show (see Lemma 1 in the appendix) that if $n_2(T) < n_0(T)$, then $0 \leq \beta(\tau) \leq \theta$ with $\beta'(\tau) < 0$, and conversely, if $n_0(T) < n_2(T)$, then $\theta \leq \beta(\tau) \leq 1$ with $\beta'(\tau) > 0$. Thus $\beta(\tau)$ can be thought of as a borrower's *effective bargaining power* at time $T - \tau$, determined by the borrower's fundamental bargaining power, θ, as well as his ability to realize gains from trade in the time remaining until the end of the trading session, which depends on the evolution of the endogenous distribution of balances across banks. For example, if $n_0(T) < n_2(T)$, it is relatively difficult for banks with excess balances to find potential borrowers, and $\beta(\tau)$ is larger than θ throughout the trading session. In this case the lenders' effective bargaining power, $1 - \beta(\tau)$, increases toward their fundamental bargaining power, $1 - \theta$, as the trading session progresses, reflecting the fact that although borrowers face a favorable distribution of potential trading partners throughout the session, their chances to execute the desired trade diminish as the end of the session draws closer.

4 Efficiency

In this section we use the theory to characterize the optimal process of reallocation of reserve balances. The spirit of the exercise is to take as given the market structure, including α, θ and $\{U_k\}_{k\in\mathbb{K}}$, and ask whether the equilibrium reallocates reserve balances efficiently given these institutions. This leads to the following social planning problem:

$$\max_{[\chi(t)]_{t=0}^{T}} \left[e^{-rT} \sum_{k\in\mathbb{K}} m_k(T)\, U_k \right]$$

subject to $\chi(t) \in [0,1]$ and

$$\dot{m}_0(t) = -\alpha\chi(t)\, m_2(t)\, m_0(t) \tag{23}$$

$$\dot{m}_1(t) = 2\alpha\chi(t)\, m_2(t)\, m_0(t) \tag{24}$$

$$\dot{m}_2(t) = -\alpha\chi(t)\, m_2(t)\, m_0(t) \tag{25}$$

for all $t \in [0,T]$, where $m_k(t)$ denotes the measure of banks with balance k at time t. Since $\tau \equiv T - t$, $m_k(t) = m_k(T - \tau) \equiv n_k(\tau)$, and therefore $\dot{m}_k(t) = -\dot{n}_k(\tau)$. The control variable, $\chi(t)$, represents the planner's choice of reallocation of balances between a bank with zero balance and a bank with 2 units of reserve balances who contact each other at time t. Use $\{\mu_k(t)\}_{k\in\mathbb{K}}$ to denote the vector of co-states associated with the law of motion for the distribution of balances

(23)-(25) in the (current-value) Hamiltonian formulation of the planner's problem. Then let $\lambda_i(\tau) \equiv \mu_i(T-\tau) = \mu_i(t)$, and define

$$S^*(\tau) \equiv 2\lambda_1(\tau) - \lambda_0(\tau) - \lambda_2(\tau).$$

In Afonso and Lagos (2013) we show that the necessary conditions for optimality are:

$$r\lambda_0(\tau) + \dot{\lambda}_0(\tau) = \alpha\psi(\tau) n_2(\tau) S^*(\tau) \tag{26}$$

$$r\lambda_1(\tau) + \dot{\lambda}_1(\tau) = 0 \tag{27}$$

$$r\lambda_2(\tau) + \dot{\lambda}_2(\tau) = \alpha\psi(\tau) n_0(\tau) S^*(\tau), \tag{28}$$

where $\lambda_k(0) = U_k$ for all $k \in \mathbb{K}$,

$$\psi(\tau) = \begin{cases} 1 & \text{if } 0 < S^*(\tau) \\ 0 & \text{if } S^*(\tau) \leq 0, \end{cases} \tag{29}$$

and $\{n_k(\tau)\}_{k\in\mathbb{K}}$ satisfies

$$\dot{n}_0(\tau) = \alpha\psi(\tau) n_2(\tau) n_0(\tau) \tag{30}$$

$$\dot{n}_1(\tau) = -2\alpha\psi(\tau) n_2(\tau) n_0(\tau) \tag{31}$$

$$\dot{n}_2(\tau) = \alpha\psi(\tau) n_2(\tau) n_0(\tau) \tag{32}$$

given $\{n_k(T)\}_{k\in\mathbb{K}}$.

The Euler equations (26)-(28) imply

$$\dot{S}^*(\tau) + \delta^*(\tau) S^*(\tau) = 0, \tag{33}$$

with

$$\delta^*(\tau) \equiv \{r + \alpha[n_2(\tau) + n_0(\tau)]\}.$$

Given the boundary condition $S^*(0) = 2U_1 - U_2 - U_0$, the solution to (33) is

$$S^*(\tau) = e^{-\bar{\delta}^*(\tau)} S(0), \tag{34}$$

where $\bar{\delta}^*(\tau) \equiv \int_0^\tau \delta^*(x)\,dx$.

Under Assumption (A), (34) implies that $S^*(\tau) > 0$ for all τ, and therefore (29) implies $\psi(\tau) = 1$ for all τ. This means that whenever a bank with no reserves contacts a bank with 2 units of reserve balances, the planner makes the latter lend one unit to the former, just as in the decentralized equilibrium, i.e., $\psi(\tau) = \phi(\tau) = 1$ for all τ. Then together with the initial condition, $\{n_k(T)\}_{k\in\mathbb{K}}$, the law of motion for the distribution of reserve balances, (30)-(32), reduces to (10)-(12). Hence along the optimal path, the cross-sectional distribution of reserve balances at each time τ is given by (15)-(17). We summarize this result as follows:

Proposition 2 *Assume $S(0) > 0$. The equilibrium supports an efficient allocation of reserve balances.*

It is instructive to compare the equilibrium conditions with the planner's optimality conditions. The path for the equilibrium values, $\{V_k(\tau)\}_{k\in\mathbb{K}}$, satisfies (6)-(8) and $V_k(0) = U_k$ for all $k \in \mathbb{K}$, while the path for the planner's shadow prices satisfies (26)-(28) and $\lambda_k(0) = U_k$ for all $k \in \mathbb{K}$. These sets of conditions would be identical were it not for the fact that the planner imputes to each bank with zero balance, gains from trade equal to the whole surplus with frequency α, rather $\alpha\theta$. Similarly, the planner imputes to each bank with two units of reserves, gains from trade equal to the whole surplus with frequency α, rather $\alpha(1-\theta)$. This difference between the private and the social value of a borrower and a lender reflects a composition externality typical of random matching environments. The planner's calculation of the value of a marginal agent in state i includes not only the expected gain from trade to this agent, but also the expected gain from trade that having this marginal agent in state k generates for all other agents by increasing their contact rates with agents in state k. In the equilibrium, the individual agent in state k internalizes the former, but not the latter.[3]

Since in this simple model meetings involving at least one bank that holds one unit of reserves never entail gains from trade, (7) and (27) confirm that the equilibrium value of a bank with one unit of reserve balances coincides with the shadow price it is assigned by the planner. In contrast, comparing (6) to (26), and (8) to (28) reveals that the equilibrium gains from trade as perceived by an individual borrower and lender at time $T-\tau$ are $\theta S(\tau)$ and $(1-\theta) S(\tau)$, respectively, while according to the planner each of their marginal contributions equals $S^*(\tau)$. Notice that $\delta^*(\tau) \geq \delta(\tau)$ for all $\tau \in [0, T]$, with "=" only for $\tau = 0$, so the planner effectively "discounts" the end-of-day gain from trade more heavily than the equilibrium. It is immediate that $S^*(\tau) < S(\tau)$ for all $\tau \in (0, 1]$, with $S^*(0) = S(0) = 2U_1 - U_2 - U_0$. In words: due to the matching externality, the social value of a loan (loans are always of size 1 in this

[3] In a labor market context, a similar composition externality arises in the competitive matching equilibrium of Kiyotaki and Lagos (2007).

simple model) is smaller than the joint private value of a loan in equilibrium. Intuitively, the reason is that the planner internalizes the fact that borrowers and lenders who are searching make it easier for other lenders and borrowers to find trading partners, but these "liquidity provision services" to others receive no compensation in the equilibrium, so individual agents ignore them when calculating their equilibrium payoffs. Naturally, depending on the value of θ, the equilibrium payoff to lenders may be too high or too low relative to their shadow price in the planner's problem. It will be high if $S^*(\tau) < (1-\theta) S(\tau)$, as would be the case for example, if the borrower's bargaining power, θ, is small. As these considerations make clear, the efficiency proposition (Proposition 2) would typically become an *inefficiency* proposition in contexts where banks make some additional choices based on their private gains from trade (e.g., entry, search intensity decisions, etc.).

5 Frictionless limit

In this section we characterize the limit of the equilibrium as the contact rate, α, becomes arbitrarily large so the OTC trading delays vanish. From (15), (16) and (17),

$$\begin{aligned}
\lim_{\alpha\to\infty} n_0(\tau) &= \max\{n_0(T) - n_2(T), 0\} \\
\lim_{\alpha\to\infty} n_1(\tau) &= 1 - \max\{n_0(T) - n_2(T), n_2(T) - n_0(T)\} \\
\lim_{\alpha\to\infty} n_2(\tau) &= \max\{n_2(T) - n_0(T), 0\}.
\end{aligned}$$

The following proposition summarizes the frictionless limit of the equilibrium surplus, $S^\infty(\tau) \equiv \lim_{\alpha\to\infty} S(\tau)$, and the fed funds rate (measured as it is usually calculated by financial analysts) that would prevail in the frictionless economy, $\rho^\infty \equiv \lim_{\alpha\to\infty} \rho(\tau)$.

Proposition 3 *For $\tau \in (0, T]$,*

$$S^\infty(\tau) = \begin{cases} 0 & \textit{if } n_2(T) \neq n_0(T) \\ \frac{T-\tau}{T} e^{-r\tau} S(0) & \textit{if } n_2(T) = n_0(T). \end{cases}$$

For $\tau \in [0, T]$,

$$1 + \rho^\infty = \begin{cases} e^{r\Delta}(U_1 - U_0) & \textit{if } n_2(T) < n_0(T) \\ e^{r\Delta}[\theta(U_2 - U_1) + (1-\theta)(U_1 - U_0)] & \textit{if } n_2(T) = n_0(T) \\ e^{r\Delta}(U_2 - U_1) & \textit{if } n_0(T) < n_2(T). \end{cases} \tag{35}$$

To conclude this section we compare the equilibrium fed funds rate of our theory with the frictionless benchmark. Let Q denote the quantity of reserve balances in the market, i.e., $Q = \sum_{k=0}^{2} kn_k(T) = 1 + n_2(T) - n_0(T)$. The rate ρ^∞ is the frictionless analogue of the over-the-counter fed funds rate $\rho(\tau)$. Generically, the fed funds rate in the frictional market is time-dependent and continuous in Q. In contrast, the frictionless rate ρ^∞ is independent of τ and discontinuous in Q; ρ^∞ jumps from $e^{r\Delta}(U_2 - U_1)$ up to $e^{r\Delta}[\theta(U_2 - U_1) + (1-\theta)(U_1 - U_0)]$ as Q approaches 1 from above, and jumps from $e^{r\Delta}(U_1 - U_0)$ down to $e^{r\Delta}[\theta(U_2 - U_1) + (1-\theta)(U_1 - U_0)]$ as Q approaches 1 from below. In general,

$$\rho^\infty - \rho(\tau) = \begin{cases} e^{r\Delta}\beta(\tau) S(0) & \text{if } Q < 1 \\ 0 & \text{if } Q = 1 \\ -e^{r\Delta}[1 - \beta(\tau)] S(0) & \text{if } 1 < Q. \end{cases}$$

Notice that $\rho(\tau) = \rho^\infty$ in the non-generic case of a "balanced market," i.e., if $n_2(T) = n_0(T)$ (or equivalently, $Q = 1$). In this case the distribution of balances is neutral with respect to borrowers and lenders, and hence their effective bargaining powers, $\beta(\tau)$ and $1-\beta(\tau)$, coincide with their fundamental bargaining powers, θ and $1-\theta$. Generically, however, the frictionless approximation overestimates the OTC fed funds rate if reserves are relatively scarce (i.e., if $Q < 1$ or equivalently, $n_2(T) < n_0(T)$), and underestimates the OTC fed funds rate if reserves are relatively abundant (i.e., if $Q > 1$). Interestingly, these biases which are nil when the market is perfectly balanced, will also tend to be relatively small if the market is very unbalanced. For example, if $n_2(T)$ is very large relative to $n_0(T)$, then the equilibrium path for $\beta(\tau)$ will be very close to 1 throughout most of the trading session ($\beta(\tau)$ will fall sharply toward θ over a very short interval of time right before the end of the trading session). Figure 1 compares the path of the intraday interest rate in an OTC market with the one that would prevail in a frictionless market. The first and last panels correspond to a market with defficient and excess aggregate reserves, i.e., $Q < 1$ and $1 < Q$, respectively, while the middle panel corresponds to a "balanced market" in which $Q = 1$.

6 Positive implications

The performance of the fed funds market as a mechanism that reallocates liquidity among banks can be evaluated by studying the behavior of empirical measures of the fed funds rate and of the effectiveness of the market to channel funds from banks with excess balances to those with shortages. In this section we derive the theoretical counterparts to these empirical measures, and use our theory to identify the determinants of trade volume, the fed funds rate, and trading delays.

6.1 Trade volume

The flow volume of trade at time $T-\tau$ is $\upsilon(\tau)=\alpha\phi(\tau)\,n_0(\tau)\,n_2(\tau)$, which with (15) and (17) can be written as

$$\upsilon(\tau)=\begin{cases}\alpha\phi(\tau)\,\frac{n_0(T)n_2(T)[n_2(T)-n_0(T)]^2 e^{\alpha[n_2(T)-n_0(T)](T-\tau)}}{\left[e^{\alpha[n_2(T)-n_0(T)](T-\tau)}n_2(T)-n_0(T)\right]^2} & \text{if } n_2(T)\neq n_0(T)\\ \alpha\phi(\tau)\left[\frac{n_0(T)}{1+\alpha n_0(T)(T-\tau)}\right]^2 & \text{if } n_2(T)=n_0(T).\end{cases}$$

The volume traded during the whole trading session is $\bar{\upsilon}=\int_0^T\upsilon(\tau)\,d\tau$.

6.2 Fed funds rate

The fed funds rate charged by a lender for a loan extended at time $t=T-\tau$ is $\rho(\tau)$ as given in Proposition 1. Then $1+\bar{\rho}\equiv\bar{R}=\frac{1}{T}\int_0^T R(\tau)\,d\tau$ is a (value-weighted) daily average fed funds rate akin to the *effective federal funds rate* published daily by the Federal Reserve. Let $\bar{\Theta}\equiv\frac{1}{T}\int_0^T\Theta(\tau)\,d\tau$. Then

$$1+\bar{\rho}=e^{r\Delta}\left[\bar{\Theta}\left(U_2-U_1\right)+\left(1-\bar{\Theta}\right)\left(U_1-U_0\right)\right],$$

where

$$\bar{\Theta}=\begin{cases}\bar{\beta} & \text{if } n_2(T)\neq n_0(T)\\ \theta & \text{if } n_2(T)=n_0(T),\end{cases}$$

and

$$\begin{aligned}\bar{\beta}=&\frac{\left\{\alpha\theta\left[n_2(T)-n_0(T)\right]T-(1-\theta)\left[1-e^{-\alpha\theta[n_2(T)-n_0(T)]T}\right]\right\}n_2(T)}{\alpha\theta\left[n_2(T)-n_0(T)\right]T\left[n_2(T)-e^{-\alpha[n_2(T)-n_0(T)]T}n_0(T)\right]}\\ &-\frac{\theta\left[1-e^{-\alpha(1-\theta)[n_2(T)-n_0(T)]T}\right]e^{-\alpha\theta[n_2(T)-n_0(T)]T}n_0(T)}{\alpha(1-\theta)\left[n_2(T)-n_0(T)\right]T\left[n_2(T)-e^{-\alpha[n_2(T)-n_0(T)]T}n_0(T)\right]}.\end{aligned}\tag{36}$$

6.3 Distribution of reserve balances

Let $\mu(\tau)$ and $\sigma^2(\tau)$ denote the mean and variance of the cross sectional distribution of reserve balances across banks at time $t=T-\tau$. Then $\mu(\tau)=Q$ and

$$\sigma^2(\tau)=\sigma^2(T)-2\left[2+n_2(T)-n_0(T)\right]\left[n_0(T)-n_0(\tau)\right],$$

where

$$\sigma^2(T) = [3 + n_2(T) - n_0(T)][n_2(T) - n_0(T)] + 2[2 + n_2(T) - n_0(T)] n_0(T).$$

With (15),

$$\sigma^2(\tau) = \begin{cases} \sigma^2(T) - \frac{2[2+n_2(T)-n_0(T)]n_0(T)n_2(T)\left[e^{\alpha[n_2(T)-n_0(T)](T-\tau)}-1\right]}{e^{\alpha[n_2(T)-n_0(T)](T-\tau)}n_2(T)-n_0(T)} & \text{if } n_2(T) \neq n_0(T) \\ \sigma^2(T) - \frac{2\alpha[2+n_2(T)-n_0(T)]n_0(T)^2(T-\tau)}{1+\alpha n_0(T)(T-\tau)} & \text{if } n_2(T) = n_0(T). \end{cases}$$

Since $\dot{n}_0(\tau) > 0$, it follows that $\dot{\sigma}^2(\tau) > 0$, i.e., the variance of the distribution of balances decreases monotonically as the trading session progresses.

7 Central bank policy

In this section we explain the impact of central bank policy (such as changes in the Discount Window rate or the interest rate on reserve balances) on the fed funds rates negotiated between banks throughout the day. To this end, we parametrize $\{U_k\}_{k\in\mathbb{K}}$ so that it captures the basic institutional arrangements currently in place in the United States. We interpret a bank with $k = 1$ as being "on target" (i.e., holding the required level of reserves), a bank with $k = 2$ as being "above target" (i.e., holding excess reserves), and a bank with $k = 0$ as being "below target" (i.e., unable to meet the required level of reserves).

Let $i_f^r \geq 0$ denote the overnight interest rate that a bank earns on required reserves, and let $i_f^e \in [0, i_f^r]$ be the overnight interest rate on excess reserves. The overnight interest rate at which a bank can borrow from the Discount Window is denoted $i_f^w \geq 0$, and $P^w \geq 0$ represents the pecuniary value of the additional costs associated with Discount Window borrowing (such as administrative costs and stigma). Hereafter, we assume:

$$U_0 = -e^{-r\Delta}(i_f^w - i_f^r + P^w) \tag{37}$$

$$U_1 = e^{-r\Delta}(1 + i_f^r) \tag{38}$$

$$U_2 = e^{-r\Delta}(2 + i_f^r + i_f^e). \tag{39}$$

To simplify the exposition, we are postulating that a bank's reserves held overnight at the Federal Reserve (plus or minus any interest earned or due) become available at time $T + \Delta$, i.e., at the same time interbank loans are settled. With (37)-(39), we have

$$\rho(\tau) = \Theta(\tau) i_f^e + [1 - \Theta(\tau)](i_f^w + P^w). \tag{40}$$

According to (40), the fed funds rate is a time-varying weighted average of the lender's end-of-day return on the second unit of balances, i_f^e, and the borrower's end-of-day reservation value for the first unit of balances, $i_f^w + P^w$. The weight on the former at time $T - \tau$ is $\Theta(\tau)$, i.e., the borrower's effective bargaining power at time $T - \tau$.

Proposition 4 *A one percent increase in the overnight interest rate that the central bank pays on excess reserves, i_f^e, causes a $\Theta(\tau)$ percent increase in the fed funds rate at time $T - \tau$. A one percent increase in the overnight cost of a deficient balance, $i_f^w + P^w$, causes a $1 - \Theta(\tau)$ percent increase in the fed funds rate at time $T - \tau$.*

7.1 Discount Window lending rate

Figure 2 (with $t = T - \tau$, on the horizontal axis) shows the time paths for the trade surplus and the fed funds rate for different values of the Discount Window policy rate, i_f^w. The different (annualized) values considered are 0.5%, 0.75%, and 1%. The panels on the top row correspond to the case in which reserve balances are scarce (the initial number of lenders is smaller than the initial number of borrowers), i.e., $n_2(T) = 0.3 < n_0(T) = 0.6$, while the bottom row corresponds to the case with $n_0(T) = 0.3 < n_2(T) = 0.6$, in which reserve balances are abundant. The left panels on the top and bottom rows illustrate that the path of the surplus is shifted up at each point in time when it is more costly to borrow from the Discount Window, an effect driven by the fact that the first-order implication of a larger i_f^w is to reduce the borrower's outside option, making it more valuable for borrowers to trade in order to avoid having to resort to the Window at the end of the day. Naturally, this effect also causes the intraday paths for the interest rate to shift up.

7.2 Interest on reserves

Figure 3 (with $t = T - \tau$, on the horizontal axis) shows the time paths for the trade surplus and the fed funds rate for different values of the interest that the central bank pays on excess reserves, i_f^e. The different (annualized) values considered are 0%, 0.25%, and 0.75%. The panels on the top row correspond to the case in which reserve balances are scarce (the initial number of lenders is smaller than the initial number of borrowers), i.e., $n_2(T) = 0.3 < n_0(T) = 0.6$, while the bottom row corresponds to the case with $n_0(T) = 0.3 < n_2(T) = 0.6$, in which reserve balances are abundant. The left panels on the top and bottom rows illustrate that the path of the surplus is shifted down when the policy makes it more profitable to hold an excess unit of reserves overnight, an effect driven by the fact that the first-order implication of a larger i_f^e is to increase the lender's outside option, making it less valuable for lenders to trade with borrowers who are seeking to avoid having to resort to the Window at the end oft the day. Naturally, this effect also causes the paths for the interest rate to shift up.

7.3 Open market operations

Given the interpretation that a bank with $k = 1$ is "on target" (i.e., holding the required level of reserves), Q can be interpreted as the amount of reserves being held by the consolidated banking sector at the beginning of the trading day, expressed as a fraction of the total target level of reserves that the banking sector as a whole wishes to hold. Thus $Q < 1$ (or equivalently, $n_2(T) < n_0(T)$) can be interpreted as a situation in which the banking sector as a whole is "short of reserves," i.e., banks wish to hold more reserves than are available. In our simple model, changes in Q are a way to capture the effects of unanticipated beginning-of-day open market operations. For example, a small Q can be thought of as having resulted from a large sale of Treasuries earlier in the day.

Figure 4 (with $t = T - \tau$, on the horizontal axis) illustrates the effects of changes in Q on the intraday path of the surplus and the fed funds rate. The figure shows the time paths for the trade surplus and the fed funds rate for two different initial distributions of reserve balances: $\{n_0(T), n_1(T), n_2(T)\} = \{0.7, 0.1, 0.2\}$, and $\{0.3, 0.1, 0.6\}$. These initial distributions imply $Q = 0.5$ and 1.3, respectively. As expected (e.g., from Figure 1), for all $t = T - \tau < T$ the path for the fed funds rate implied by the initial distribution with $Q < 1$ lies above the constant value $\theta i_f^e + (1-\theta)(i_f^w + P^w)$, which is equal to 0.005 (annualized) in this example. Conversely, the path for the fed funds rate implied by the initial distribution with $Q > 1$ lies below $\theta i_f^e + (1-\theta)(i_f^w + P^w)$ for all $t = T - \tau < T$.

In general, since the quantity of reserves in the system is $Q = 1 + n_2(T) - n_0(T)$, it follows that $Q \in [n_1(T), 2 - n_1(T)]$. To simplify the exposition, in this section suppose that $n_1(T) = 0$, so $Q \in [0, 2]$, with $Q = 0$ if and only if $n_2(T) = 1 - n_0(T) = 0$, $Q = 1$ if and only if $n_2(T) = 1 - n_0(T) = 1/2$, and $Q = 2$ if and only if $n_2(T) = 1 - n_0(T) = 1$. Use (36) to define

$$\bar{\beta}(Q) = \frac{\left\{\alpha\theta(Q-1)T - (1-\theta)\left[1 - e^{-\alpha\theta(Q-1)T}\right]\right\} n_2(T)}{\alpha\theta(Q-1)T\left[n_2(T) - e^{-\alpha(Q-1)T} n_0(T)\right]} - \frac{\theta\left[1 - e^{-\alpha(1-\theta)(Q-1)T}\right] e^{-\alpha\theta(Q-1)T} n_0(T)}{\alpha(1-\theta)(Q-1)T\left[n_2(T) - e^{-\alpha(Q-1)T} n_0(T)\right]}.$$

Since $\beta(\tau) \in [0,1]$ for all τ, we know that $\bar{\beta}(Q) \in [0,1]$, but in addition, $\bar{\beta}(0) < \bar{\beta}(1) < \bar{\beta}(2)$, where $\bar{\beta}(0) = \theta \frac{\left[1-e^{-\alpha(1-\theta)T}\right]}{\alpha(1-\theta)T}$, $\bar{\beta}(1) = \theta$, and $\bar{\beta}(2) = 1 - (1-\theta)\frac{1-e^{-\alpha\theta T}}{\alpha\theta T}$. In other words, all else equal, the daily average of the effective bargaining power of borrowers is increasing in the quantity of reserves in the system, Q. Intuitively, since a borrower's outside option of searching for another lender is increasing in Q, the borrower is able to extract a larger share of the surplus in bilateral negotiations with lenders throughout the day when Q is larger. Interestingly, OTC frictions mitigate the magnitude of this effect of market conditions on the effective bargaining power, e.g., generically we have $0 < \bar{\beta}(0)$ and $\bar{\beta}(2) < 1$ (but $\bar{\beta}(0) \to 0$ and $\bar{\beta}(2) \to 1$ as $\alpha \to \infty$). Figure 5 illustrates the equilibrium daily average rate as a function of the quantity of reserves, Q, i.e., $\bar{\rho}(Q) = \bar{\beta}(Q)\, i_f^e + \left[1 - \bar{\beta}(Q)\right](i_f^w + P^w)$. Qualitatively, the locus of equilibrium interest rates that is traced out by changing the total amount of reserves, Q, in Figure 5 is the OTC-theoretical counterpart of the locus of equilibrium interest rates that is traced out by shifting a vertical "supply of reserves" curve along a downward sloping "demand of reserves" curve derived in the context of the traditional static Walrasian "Poole model" (see, e.g., Poole 1968, Ennis and Kiester, 2008).

8 Market structure

In this section we study the effects of changes in θ and α (the two parameters that in our simple model represent the market structure of the fed funds market) on the equilibrium paths for the trade surplus and the fed funds rate.

8.1 Bargaining power

Proposition 5 *Assume $S(0) > 0$. (i) If the initial population of lenders is larger (smaller) than that of borrowers, then the surplus at each point in time is decreasing (increasing) in the borrower's bargaining power for all $\tau > 0$, i.e., $\frac{\partial S(\tau)}{\partial \theta}$ is equal in sign to $n_0(T) - n_2(T)$. (ii) The interest rate $\rho(\tau)$ is decreasing in θ for all $\tau \in [0,T]$.*

The effect of θ on $S(\tau) = 2V_1(\tau) - V_0(\tau) - V_2(\tau)$ in part (i) of Proposition 5 is subtle because a higher θ tends to increase $V_0(\tau)$ (benefits borrowers) and at the same time it tends to decrease $V_2(\tau)$ (hurts lenders). In part *(i)* of the proposition we show that the former effect dominates if and only if $n_0(T) < n_2(T)$, and in this case, the effective discount rate decreases with θ, which implies $S(\tau)$ decreases with θ for all $\tau > 0$. Naturally, an increase in the fundamental bargaining power of the borrower, θ, causes the intraday path for the borrower's effective bargaining power, $\beta(\tau)$ to shift up, which in turn shifts down the whole path of the intraday interest rate.

Proposition 5 is illustrated in Figure 6 (with actual time, $t = T - \tau$, on the horizontal axis), which shows the time paths for the trade surplus and the fed funds rate for different values of the borrower's bargaining power, $\theta = 0.1$, 0.5, and 0.9. The top row of panels corresponds to the case in which the initial number of lenders is smaller than the initial number of borrowers, i.e., $n_2(T) = 0.3 < n_0(T) = 0.6$ (the full parametrization is reported in the caption of the figure). Notice that in this case, reserve balances are relatively scarce. First consider the left panel on the top row. Since $S(0) = 2U_1 - U_2 - U_0$, the trade surplus at the end of the session is the same for all values of θ. For all $t < T$, however, the time-path for the trade surplus is shifted upward as the borrower's bargaining power, θ, increases. The reason is that while for each τ, an increase in θ increases the borrower's outside option, $V_0(\tau)$, and decreases the lender's outside option, $V_2(\tau)$, the fact that $n_2(\tau) < n_0(\tau)$ for all τ, implies that the decrease in the lender's outside option is larger than the increase in the borrower's outside option, so the resulting trade surplus is larger at each point in time along the trading session. The right panel confirms that the path for the fed funds rate is shifted down as the bargaining power of the borrower increases.

The panels on the bottom row correspond to the case in which reserve balances are abundant; since the initial number of borrowers is smaller than the initial number of lenders, i.e., $n_0(T) = 0.3 < n_2(T) = 0.6$, we have $\bar{k} = 1 < 1.3 = Q$. In this case an increase in θ still increases $V_0(\tau)$ and decreases $V_2(\tau)$ for each $\tau \in (0, T]$, but the fact that $n_0(\tau) < n_2(\tau)$ for all τ implies that the decrease in the lender's outside option is smaller than the increase in the borrower's outside option, so the resulting trade surplus is now smaller at each point during the trading session. Again, the right panel confirms that the path for the fed funds rate is shifted down as the bargaining power of the borrower increases.

In order to understand the intraday dynamics of the fed funds rate, it is useful to compare the right panel on the top row with the right panel on the bottom row. In general, the fed funds rate tends to increase over time (i.e., as the end of the trading session approaches) when there are more lenders than borrowers, but it tends to decrease over time when there are more borrowers than lenders, provided θ is not too small.

8.2 Trading speed

Figure 7 (with $t = T - \tau$, on the horizontal axis) shows the time paths for the trade surplus and the fed funds rate for different values of the contact rate, $\alpha = 25$, 50, and 100. The panels on the top row correspond to the case in which reserve balances are scarce (the initial number of lenders is smaller than the initial number of borrowers), i.e., $n_2(T) = 0.3 < n_0(T) = 0.6$, while the bottom row corresponds to the case with abundant reserve balances, $n_0(T) = 0.3 < n_2(T) = 0.6$. Traders on the short (long) side of the market benefit (lose) from increases in the contact α. This is explained by the fact that, from the standpoint of the agents on the long side, a faster contact rate has the undesirable effect of taking scarce potential trading partners off the market, which can adversely affect the effective rate at which they are able to trade.[4] For all $t < T$ the time-path for the trade surplus is shifted downward as α increases. In the parametrization illustrated in the top row, an increase in α increases $V_2(\tau)$ for all $\tau \in (0, T]$ and decreases $V_0(\tau)$ for all $\tau \in (0, T]$. However, the former outweights the latter since $n_2(\tau)$ is small relative to $n_0(\tau)$ for all τ. In the parametrization illustrated in the bottom row, an increase in α increases $V_0(\tau)$ for all $\tau \in (0, T]$ and decreases $V_2(\tau)$ for all $\tau \in (0, T]$ and the former effect outweights the latter since $n_0(\tau)$ is small relative to $n_2(\tau)$ for all τ. Together, the dynamics of $V_2(\tau) - V_1(\tau)$ and $S(\tau)$ account for the pattern of interest rates displayed in the right panels of the top and bottom rows. In each case, the right panel shows that traders on the short side of the market benefit from increases in the contact rate. Specifically, when lenders are on the short side, increases in the contact rate take scarce lenders off the market which makes borrowers willing to pay higher rates for the loans. Conversely, when borrowers are on the short side, a faster contact rate takes scarce borrowers off the market making lenders more willing to accept lower rates for the loans.

9 Conclusion

We have presented and analyzed a small-scale version of the dynamic equilibrium over-the-counter theory of trade in the fed funds market developed in Afonso and Lagos (2013). This

[4]In general, the effect of changes in α on equilibrium payoffs can be subtle. For example, in some of our numerical simulations we have found that, if $n_2(T) < n_0(T)$, then $V_0(\tau)$ can be nonmonotonic in α: increasing in α for small values of α, but decreasing in α for large values. If $n_2(T) < n_0(T)$, however, $V_2(\tau)$ is typically increasing in α. We have found the converse to be the case for $n_0(T) < n_2(T)$, i.e., $V_0(\tau)$ is increasing in α, while increases in α from relatively small values tend to shift $V_2(\tau)$ up, while increases in α at large values tend to shift $V_2(\tau)$ down.

version of the model allows closed-form solutions of the relevant endogenous variables, i.e., the equilibrium intraday path for the fed funds rate, trade volume, and the distribution of reserve balances across banks. We have shown how the over-the-counter theory can be fruitfully used to study the effects of changes in the market structure, as well as central bank policies such as open market operations, changes in the Discount Window lending rate, and changes in the interest rate that banks earn for holding reserves.

A Proofs

Proof of Proposition 1. With (15)-(17) and (14), $S(\tau)$ can be written as in the statement of the proposition. Conditions (6) and (7) imply

$$\dot{V}_1(\tau) - \dot{V}_0(\tau) + r[V_1(\tau) - V_0(\tau)] = -\theta\alpha n_2(\tau) S(\tau),$$

a differential equation in $V_1(\tau) - V_0(\tau)$ with boundary condition $V_1(0) - V_0(0) = U_1 - U_0$. The solution to this differential equation is

$$V_1(\tau) - V_0(\tau) = e^{-r\tau}(U_1 - U_0) - \int_0^\tau \theta\alpha n_2(z) S(z) e^{-r(\tau - z)} dz. \tag{41}$$

With (17) and the closed-form expression for $S(\tau)$, the integral on the right side of (41) can be calculated explicitly to obtain

$$V_1(\tau) - V_0(\tau) = \begin{cases} e^{-r\tau}\left[U_1 - U_0 + \frac{\left[1-e^{-\alpha\theta[n_2(T)-n_0(T)]\tau}\right]n_2(T)}{n_2(T)-e^{-\alpha[n_2(T)-n_0(T)]T}n_0(T)} S(0)\right] & \text{if } n_2(T) \neq n_0(T) \\ e^{-r\tau}\left[U_1 - U_0 + \frac{n_0(T)}{1+\alpha n_0(T)T}\alpha\theta\tau S(0)\right] & \text{if } n_2(T) = n_0(T). \end{cases}$$

Finally, the expression for $R(\tau)$ reported in the statement of the proposition is obtained by substituting the analytical expressions for $S(\tau)$ and $V_1(\tau) - V_0(\tau)$ into (21). ■

Lemma 1 *If $n_2(T) < n_0(T)$, then $0 \leq \beta(\tau) \leq \theta$ with $\beta'(\tau) < 0$, and conversely, if $n_0(T) < n_2(T)$, then $\theta \leq \beta(\tau) \leq 1$ with $\beta'(\tau) > 0$. In every case, $\beta(\tau) \in [0, 1]$ for all $\tau \in [0, T]$.*

Proof of Lemma 1. Differentiate $\beta(\tau)$ to obtain

$$-\beta'(\tau) = \theta(1-\theta)\frac{\alpha[n_0(T) - n_2(T)]e^{\alpha[n_0(T)-n_2(T)]\theta\tau}}{n_0(T)e^{\alpha[n_0(T)-n_2(T)]T} - n_2(T)}\left[n_0(T)e^{\alpha[n_0(T)-n_2(T)](T-\tau)} - n_2(T)\right].$$

Clearly, $\beta'(\tau)$ has the same sign as $n_2(T) - n_0(T)$. Since $\beta(0) = \theta$, it follows that $\beta(\tau) \leq \theta$ if $n_2(T) < n_0(T)$, and that $\theta \leq \beta(\tau)$ if $n_0(T) < n_2(T)$. To conclude, verify that $0 \leq \beta(T)$ if $n_2(T) < n_0(T)$, and that $\beta(T) \leq 1$ if $n_0(T) < n_2(T)$, which respectively imply that $0 \leq \beta(\tau)$ for all $\tau \in [0, T]$ if $n_2(T) < n_0(T)$, and that $\beta(\tau) \leq 1$ for all $\tau \in [0, T]$ if $n_0(T) < n_2(T)$. Notice that

$$\begin{aligned} 1 - \beta(T) &= \frac{e^{-\alpha[n_2(T)-n_0(T)]\theta T}\left\{(1-\theta)[n_2(T) - n_0(T)] + n_0(T)\left[1 - e^{-\alpha[n_2(T)-n_0(T)](1-\theta)T}\right]\right\}}{n_2(T) - n_0(T)e^{-\alpha[n_2(T)-n_0(T)]T}} \\ &= 1 - \frac{\left[e^{\alpha[n_0(T)-n_2(T)]\theta T} - 1\right]n_2(T) + \theta e^{\alpha[n_0(T)-n_2(T)]\theta T}[n_0(T) - n_2(T)]}{n_0(T)e^{\alpha[n_0(T)-n_2(T)]T} - n_2(T)}, \end{aligned}$$

so it is immediate from the first expression, that $0 \leq 1-\beta(T)$ if $n_0(T) < n_2(T)$ (with equality only if $\theta = 1$), and from the second expression, that $1-\beta(T) \leq 1$ if $n_2(T) < n_0(T)$ (with equality only if $\theta = 0$). ■

Proof of Proposition 2. Immediate from $\psi(\tau) = \phi(\tau) = 1$ for all τ, which follows from Assumption (A). ■

Proof of Proposition 3. $S^{\infty}(\tau)$ is obtained by letting $\alpha \to \infty$ in the analytical expression for $S(\tau)$ reported in Proposition 1. To obtain $1+\rho^{\infty}(\tau)$, notice that

$$\lim_{\alpha\to\infty} \Theta(\tau) = \begin{cases} 0 & \text{if } n_2(T) < n_0(T) \\ \theta & \text{if } n_0(T) < n_2(T) \\ 1 & \text{if } n_0(T) < n_2(T) \end{cases}$$

and recall that $1+\rho(\tau) = e^{r\Delta}\{\Theta(\tau)(U_2-U_1) + [1-\Theta(\tau)](U_1-U_0)\}$. ■

Proof of Proposition 4. Immediate from (40).

Proof of Proposition 5. (i) Differentiate (14) to get

$$\frac{\partial S(\tau)}{\partial\theta} = -\alpha\tau[n_2(T)-n_0(T)]S(\tau),$$

which has the sign of $n_0(T)-n_2(T)$. Part (ii) follows from the fact that $\frac{\partial\rho(\tau)}{\partial\theta} = -\frac{\partial\Theta(\tau)}{\partial\theta}S(0)e^{r\Delta}$, with

$$\frac{\partial\Theta(\tau)}{\partial\theta} = \begin{cases} \frac{\partial\beta(\tau)}{\partial\theta} & \text{if } n_2(T) \neq n_0(T) \\ 1 & \text{if } n_2(T) = n_0(T), \end{cases}$$

where

$$\frac{\partial\beta(\tau)}{\partial\theta} = \frac{e^{-\alpha\theta[n_2(T)-n_0(T)]\tau}}{n_2(T)-n_0(T)e^{-\alpha[n_2(T)-n_0(T)]T}}\Big\{[n_2(T)-n_0(T)e^{-\alpha[n_2(T)-n_0(T)](T-\tau)}]$$
$$+\alpha\tau[n_2(T)-n_0(T)][(1-\theta)n_2(T)+\theta n_0(T)e^{-\alpha[n_2(T)-n_0(T)](T-\tau)}]\Big\}$$

is positive for all $\tau \in [0,T]$. ■

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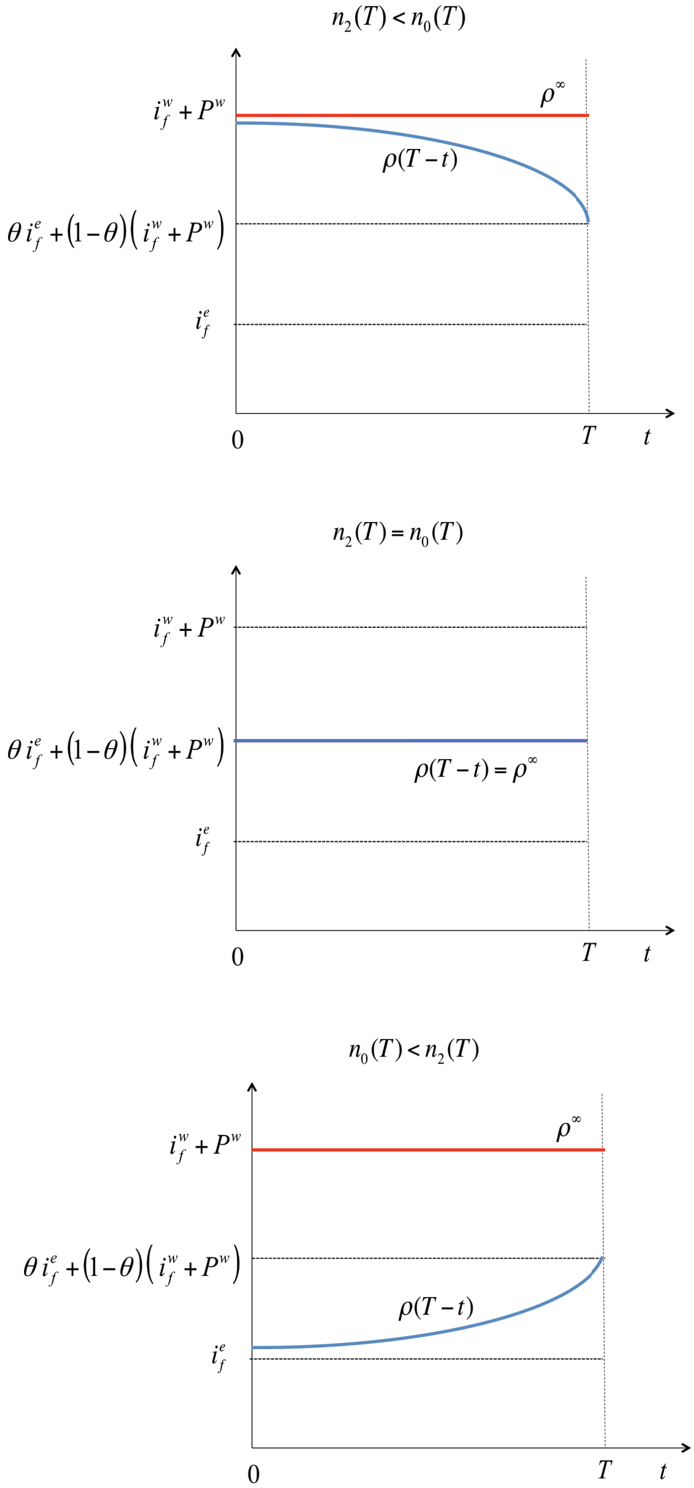


Figure 1: Intraday path of the fed funds rate in an OTC market and in a frictionless market, for different aggregate levels of reserves.

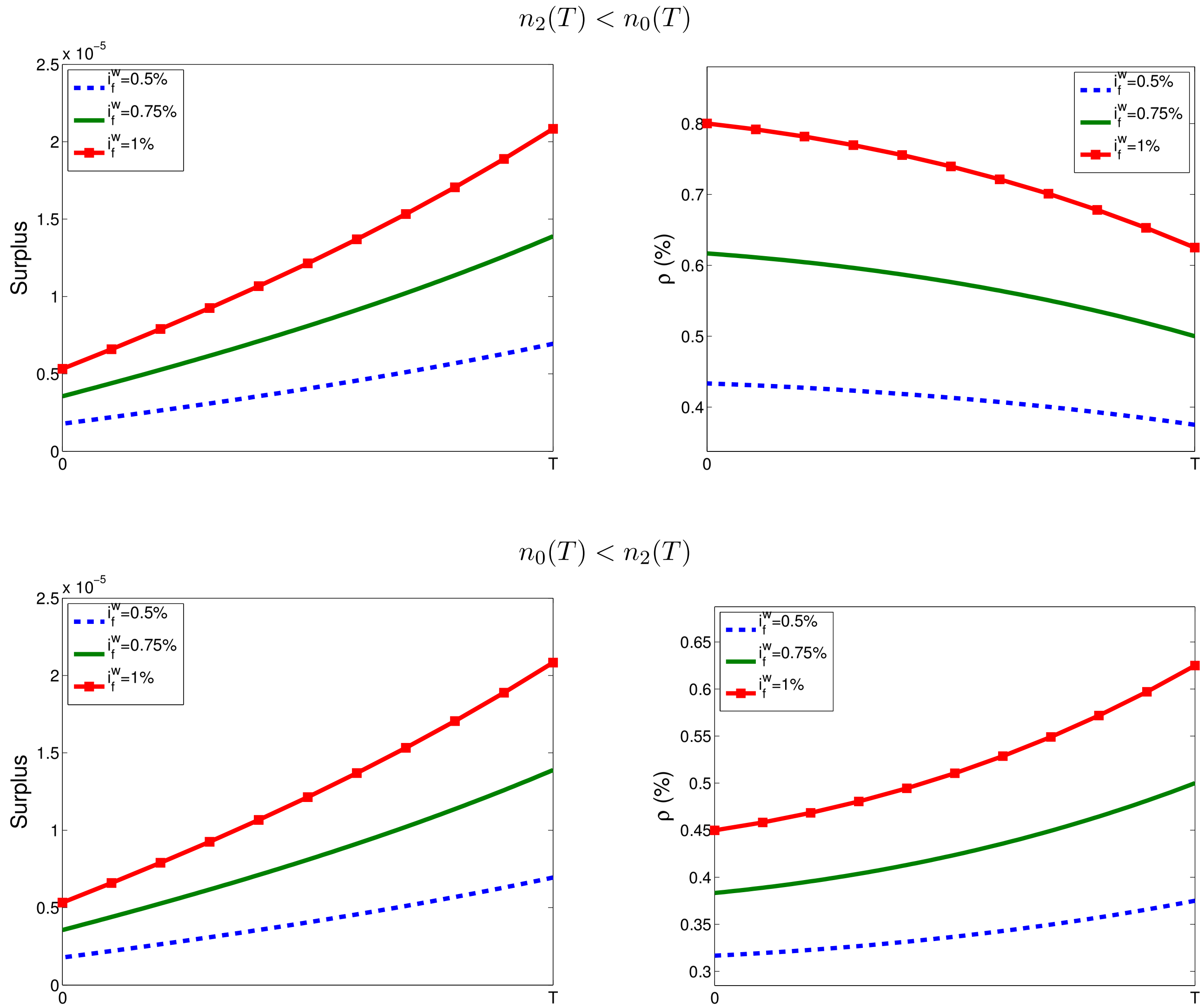


Figure 2: Surplus and fed funds rate for different values of the Discount Window rate: $i_f^w = 0.005/360$, $i_f^w = 0.0075/360$, and $i_f^w = 0.01/360$, when $n_k(T)_{k=0}^2 = \{0.6, 0.1, 0.3\}$ (top row), and when $n_k(T)_{k=0}^2 = \{0.3, 0.1, 0.6\}$ (bottom row). Other parameter values: $\alpha = 50$, $\theta = 1/2$, $T = 2.5/24$, $\Delta = 22/24$, $r = 0.0001/365$, $i_f^r = i_f^e = 0.0025/360$, and $P^w = 0$.

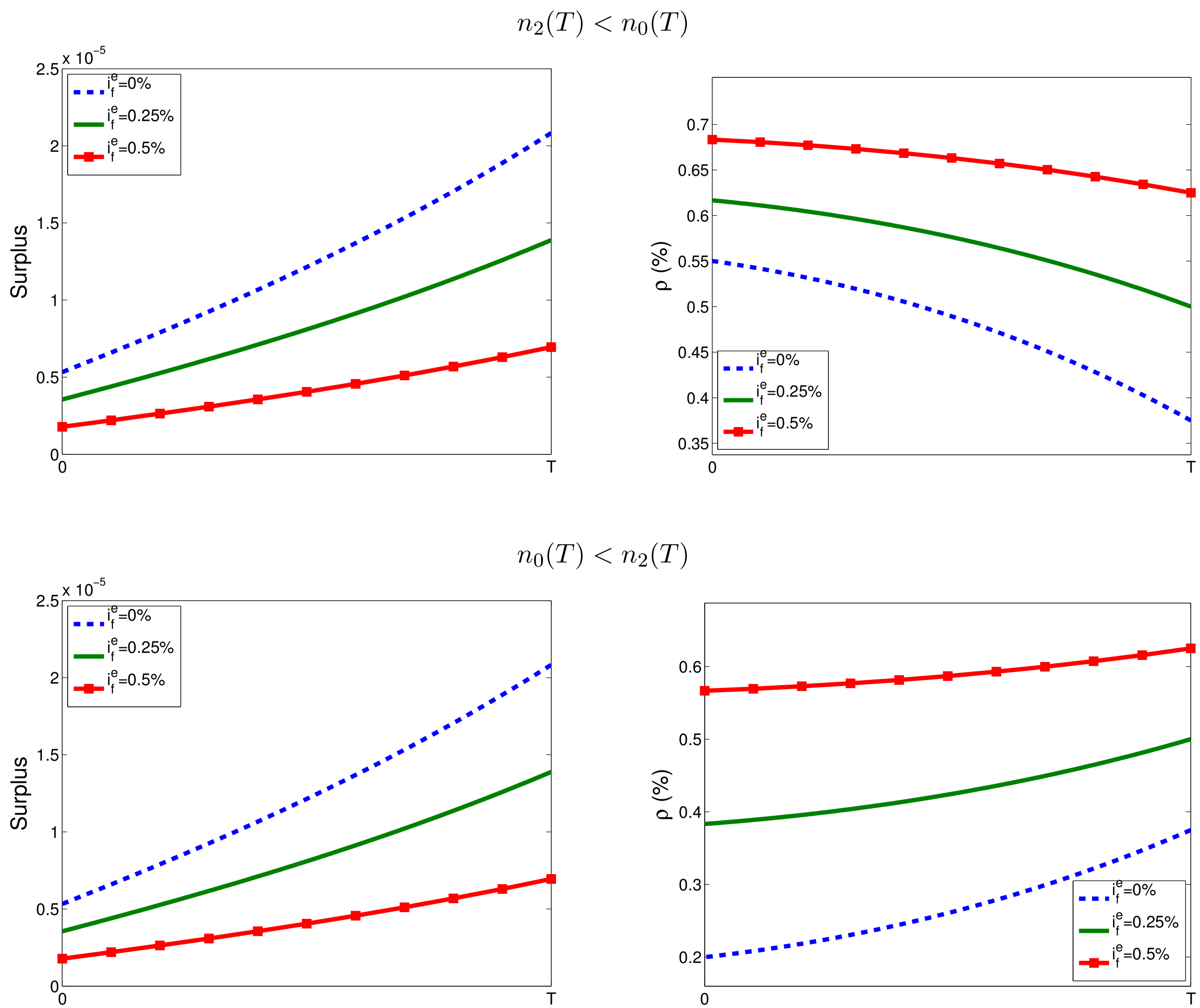


Figure 3: Surplus and fed funds rate for different values of the interest on excess reserves: $i_f^e = 0$, $i_f^e = 0.0025/360$, and $i_f^e = 0.005/360$, when $n_k(T)_{k=0}^2 = \{0.6, 0.1, 0.3\}$ (top row), and when $n_k(T)_{k=0}^2 = \{0.3, 0.1, 0.6\}$ (bottom row). Other parameter values: $\alpha = 50$, $\theta = 1/2$, $T = 2.5/24$, $\Delta = 22/24$, $r = 0.0001/365$, $i_f^r = i_f^e$, $i_f^w = 0.0075/360$, and $P^w = 0$.

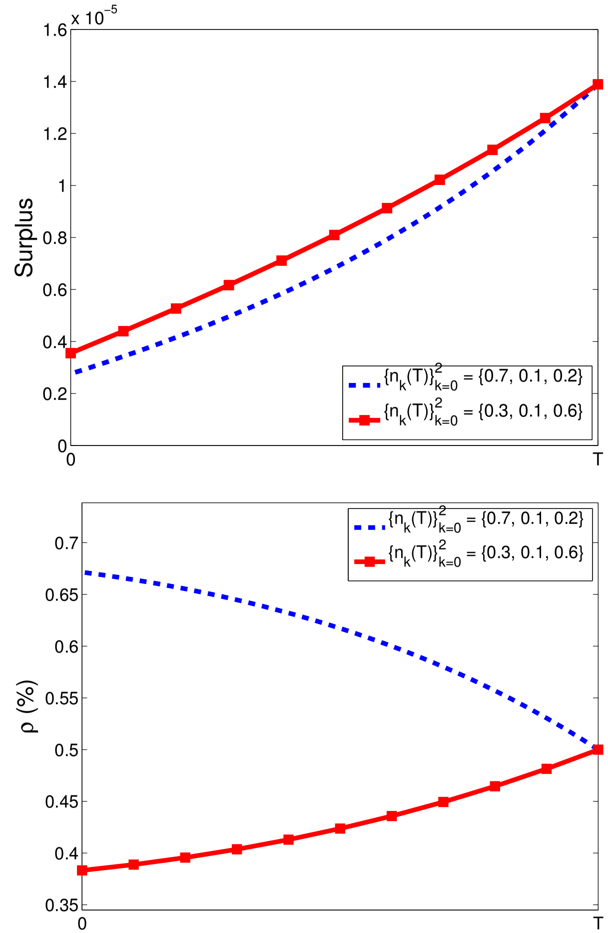


Figure 4: Surplus and fed funds rate for different values of the initial distribution of balances $n_k(T)_{k=0}^2$. Parameter values: $\alpha = 50$, $\theta = 1/2$, $T = 2.5/24$, $\Delta = 22/24$, $r = 0.0001/365$, $i_f^e = i_f^r = 0.0025/360$, $i_f^w = 0.0075/360$, and $P^w = 0$.

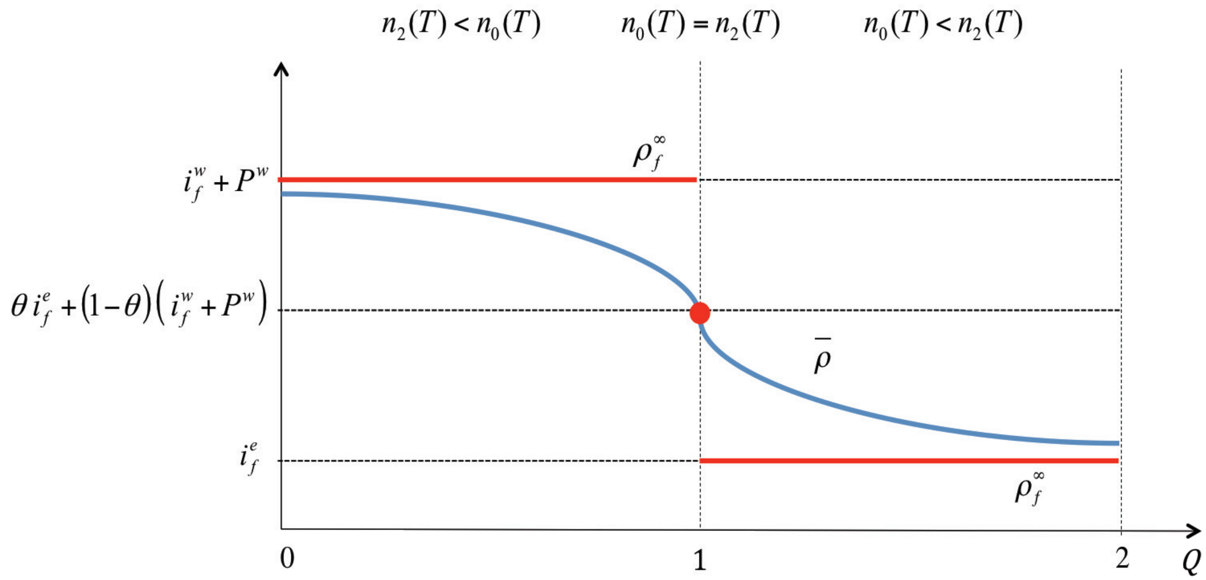
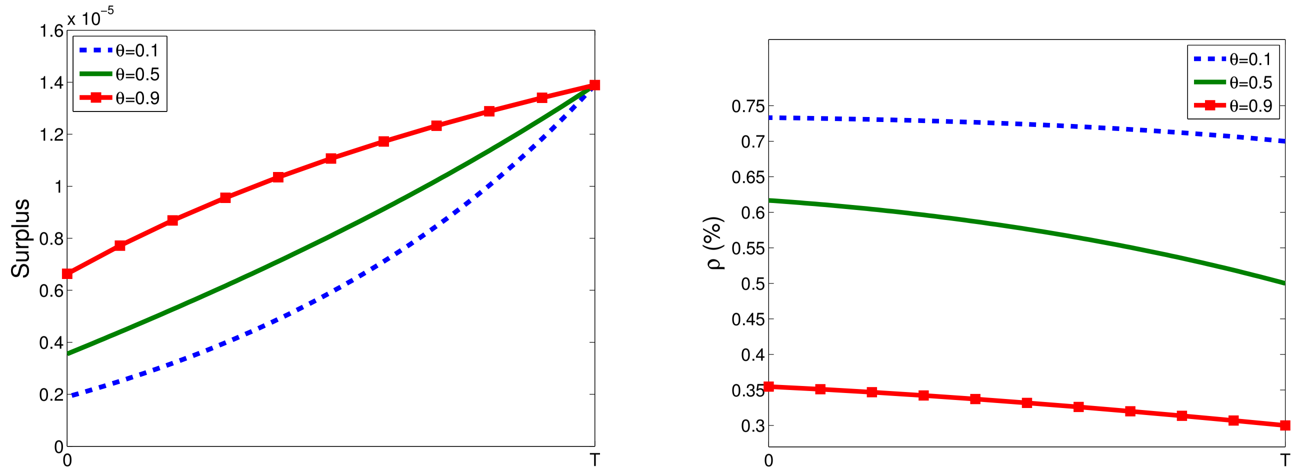


Figure 5: Daily average fed funds rate as a function of the aggregate quantity of reserves.

$n_2(T) < n_0(T)$



$n_0(T) < n_2(T)$

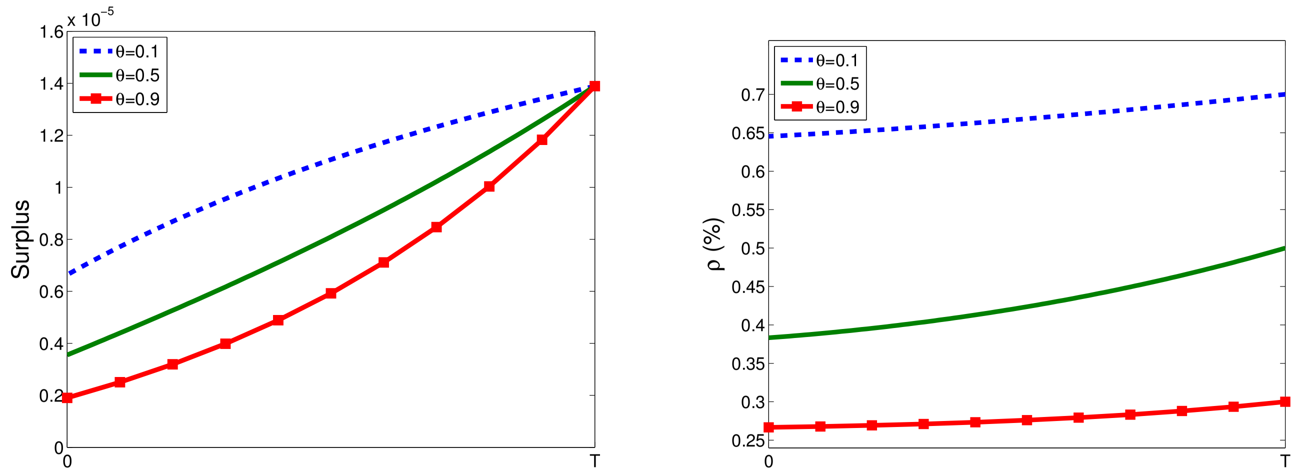


Figure 6: Surplus and fed funds rate for different values of the bargaining power: $\theta = 0.1$, $\theta = 0.5$, and $\theta = 0.9$, when $n_k(T)_{k=0}^2 = \{0.6, 0.1, 0.3\}$ (top row), and when $n_k(T)_{k=0}^2 = \{0.3, 0.1, 0.6\}$ (bottom row). Other parameter values: $\alpha = 50$, $T = 2.5/24$, $\Delta = 22/24$, $r = 0.0001/365$, $i_f^r = i_f^e = 0.0025/360$, $i_f^w = 0.0075/360$, and $P^w = 0$.

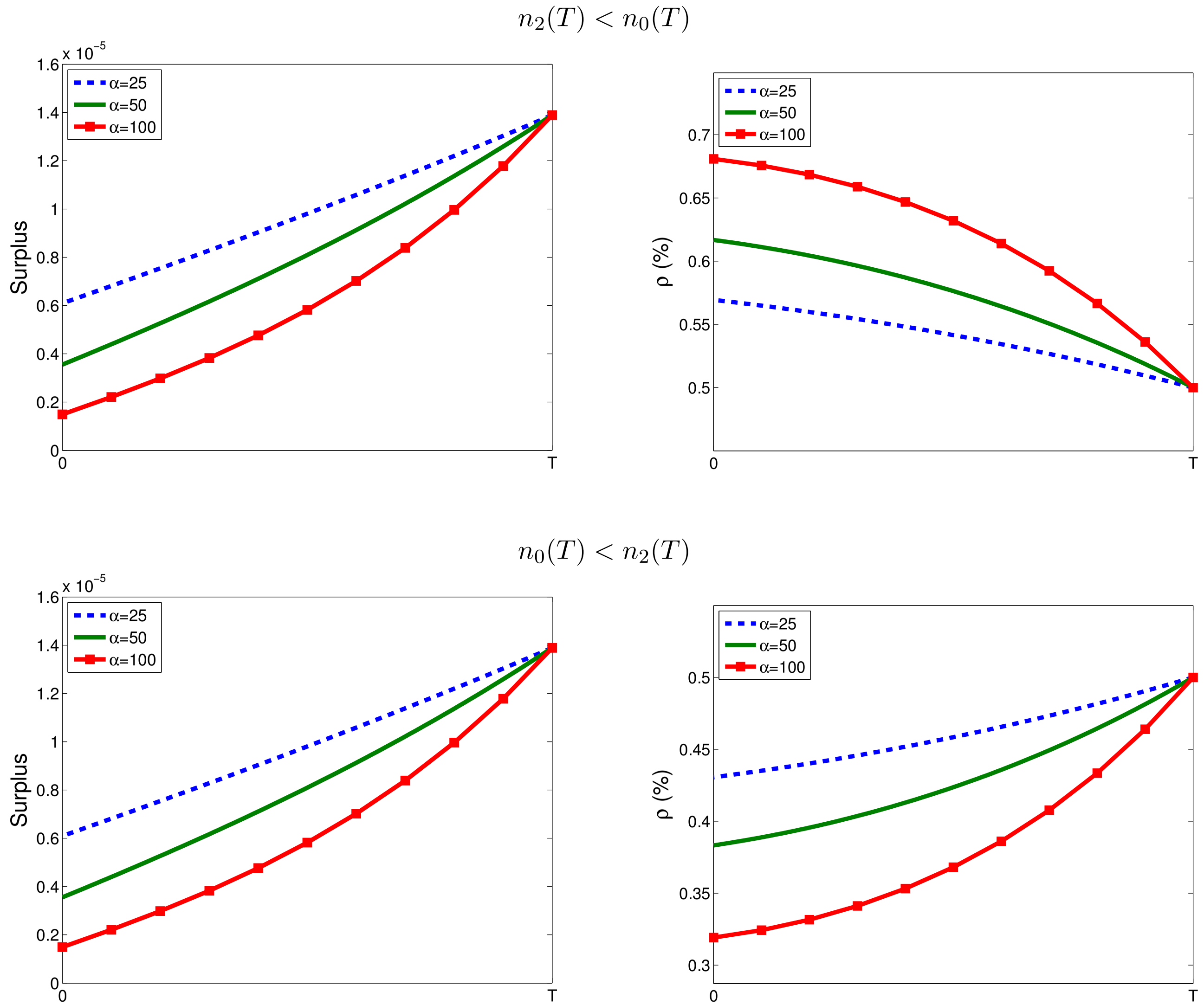


Figure 7: Surplus and fed funds rate for different values of the frequency of meetings: $\alpha = 25$, $\alpha = 50$, and $\alpha = 100$, when $n_k(T)_{k=0}^2 = \{0.6, 0.1, 0.3\}$ (top row), and when $n_k(T)_{k=0}^2 = \{0.3, 0.1, 0.6\}$ (bottom row). Other parameter values: $\theta = 1/2$, $T = 2.5/24$, $\Delta = 22/24$, $r = 0.0001/365$, $i_f^r = i_f^e = 0.0025/360$, $i_f^w = 0.0075/360$, and $P^w = 0$.