A Proofs of Propositions

A.1 Preliminaries

Transition times. Let $\tau(t)$ be the largest of t_1 and the last time $(\pi(t), x(t))$ entered the region Ω_{zlb} from Ω_{ss} before or at time t. More formally,[28]

$$\begin{aligned}
\tau(t) &= \max\{t_1, \tau_{zlb}(t)\}, \\
\tau_{zlb}(t) &= \sup\{s \leq t : (\pi(s), x(s)) \in \partial\Omega \text{ and } \exists\varepsilon > 0 \text{ s.t. } (\pi(s+\varepsilon), x(s+\varepsilon)) \in \Omega_{zlb}\}.
\end{aligned}$$

Let $\eta(t)$ be the largest of t_1 and the last time $(\pi(t), x(t))$ enters the region Ω_{ss} from Ω_{zlb} before or at time t, i.e.,

$$\begin{aligned}
\eta(t) &= \max\{t_1, \eta_{ss}(t)\}, \\
\eta_{ss}(t) &= \sup\{s \leq t : (\pi(s), x(s)) \in \partial\Omega \text{ and } \exists\varepsilon > 0 \text{ s.t. } (\pi(s+\varepsilon), x(s+\varepsilon)) \in \Omega_{ss}\}.
\end{aligned}$$

Note that $\tau(t)$ and $\eta(t)$ are piece-wise constant and thus for all $t \neq \tau(t)$ we have $\dot{\tau}(t) = 0$ and for all $t \neq \eta(t)$ we have $\dot{\eta}(t) = 0$.

Solution to IS and NKPC. Let

$$\begin{aligned}
A_{zlb} &= \begin{bmatrix} 0 & -\frac{1}{\sigma} \\ -\kappa & \rho \end{bmatrix}, \\
A_{ss} &= \begin{bmatrix} \frac{1}{\sigma}\xi_x & \frac{1}{\sigma}(\xi_\pi - 1) \\ -\kappa & \rho \end{bmatrix}.
\end{aligned}$$

The matrix A_{zlb} gives the dynamics of the system of ODEs (1)-(2) when $i(t) = 0$ while the matrix A_{ss} gives the dynamics when $i(t) = r_h + \xi_\pi \pi(t) + \xi_x x(t)$. The matrix A_{zlb} has eigenvalues ϕ_1 and ϕ_2 defined in equations (A.17)-(A.18). The eigenvalues of A_{ss} are

$$\alpha_1 = \frac{1}{2\sigma}\left(\xi_x + \sigma\rho + \sqrt{(\xi_x - \sigma\rho)^2 - 4\kappa\sigma(\xi_\pi - 1)}\right), \tag{A.1}$$

$$\alpha_2 = \frac{1}{2\sigma}\left(\xi_x + \sigma\rho - \sqrt{(\xi_x - \sigma\rho)^2 - 4\kappa\sigma(\xi_\pi - 1)}\right). \tag{A.2}$$

Because stable dynamics always produce indeterminacy (see Lemma 3 in Appendix B.9 for a proof), I restrict all analysis to cases in which either $\det A_{ss} > 0$ and $\operatorname{trace} A_{ss} > 0$, or $\det A_{ss} < 0$. Below, I use d_{exit} and d_{trap} defined in equations (A.21) and (A.22).

[28] Recall that the supremum of the empty set is $-\infty$. If there is no s such that $(x(s), \pi(s)) \in \partial\Omega$ or $\nexists\varepsilon > 0$ s.t. $(x(s+\varepsilon), \pi(s+\varepsilon)) \in \Omega_{zlb}$, then $\tau_{zlb}(t) = -\infty$ and $\tau(t) = t_1$.

For $t \in [0, T)$ the solution to (1)-(2) under the interest rate rule in equation (15) is

$$
\begin{aligned}
x(t) &= -\frac{\phi_2}{(\phi_1 - \phi_2)}\left(d_{exit}(0) - \frac{\phi_2}{\kappa}(r_h - r_l)\right)e^{\phi_1 t} \\
&\quad + \frac{\phi_1}{(\phi_1 - \phi_2)}\left(d_{trap}(0) - \frac{\phi_1}{\kappa}(r_h - r_l)\right)e^{\phi_2 t} - \frac{\rho}{\kappa} r_l, \qquad \text{(A.3)} \\
\pi(t) &= -\frac{\kappa}{(\phi_1 - \phi_2)}\left(d_{exit}(0) - \frac{\phi_2}{\kappa}(r_h - r_l)\right)e^{\phi_1 t} \\
&\quad + \frac{\kappa}{(\phi_1 - \phi_2)}\left(d_{trap}(0) - \frac{\phi_1}{\kappa}(r_h - r_l)\right)e^{\phi_2 t} - r_l. \qquad \text{(A.4)}
\end{aligned}
$$

For $t \in [T, t_1)$ the solution is

$$
\begin{aligned}
x(t) &= -\frac{\phi_2 d_{exit}(t)}{(\phi_1 - \phi_2)}e^{\phi_1(t-T)} + \frac{\phi_1 d_{trap}(t)}{(\phi_1 - \phi_2)}e^{\phi_2(t-T)} - \frac{\rho}{\kappa} r_h \qquad \text{(A.5)} \\
\pi(t) &= -\frac{\kappa d_{exit}(t)}{(\phi_1 - \phi_2)}e^{\phi_1(t-T)} + \frac{\kappa d_{trap}(t)}{(\phi_1 - \phi_2)}e^{\phi_2(t-T)} - r_h \qquad \text{(A.6)}
\end{aligned}
$$

For $t \in [t_1, \infty)$, when $(\pi(t), x(t)) \in \Omega_{zlb}$, the solution is

$$
\begin{aligned}
x(t) &= -\frac{\phi_2 d_{exit}(\tau(t))}{(\phi_1 - \phi_2)}e^{\phi_1(t-\tau(t))} + \frac{\phi_1 d_{trap}(\tau(t))}{(\phi_1 - \phi_2)}e^{\phi_2(t-\tau(t))} - \frac{\rho}{\kappa} r_h, \qquad \text{(A.7)} \\
\pi(t) &= -\frac{\kappa d_{exit}(\tau(t))}{(\phi_1 - \phi_2)}e^{\phi_1(t-\tau(t))} + \frac{\kappa d_{trap}(\tau(t))}{(\phi_1 - \phi_2)}e^{\phi_2(t-\tau(t))} - r_h. \qquad \text{(A.8)}
\end{aligned}
$$

For $t \in [t_1, \infty)$, when $(\pi(t), x(t)) \in \Omega_{ss}$, I distinguish three cases:

Case I: $\xi_\pi \neq 1$ and $4\kappa\sigma(\xi_\pi - 1) \neq (\xi_x - \sigma\rho)^2$;

Case II: $\xi_\pi = 1$ and $4\kappa\sigma(\xi_\pi - 1) \neq (\xi_x - \sigma\rho)^2$;

Case III: $4\kappa\sigma(\xi_\pi - 1) = (\xi_x - \sigma\rho)^2$.

For Case I, the solution is

$$
\begin{aligned}
x(t) &= -\frac{(1-\xi_\pi)\pi(\eta(t)) + (\sigma\alpha_2 - \xi_x)x(\eta(t))}{\sigma(\alpha_1 - \alpha_2)}e^{\alpha_1(t-\eta(t))} \\
&\quad + \frac{(1-\xi_\pi)\pi(\eta(t)) + (\sigma\alpha_1 - \xi_x)x(\eta(t))}{\sigma(\alpha_1 - \alpha_2)}e^{\alpha_2(t-\eta(t))}, \qquad \text{(A.9)} \\
\pi(t) &= \frac{(1-\xi_\pi)\pi(\eta(t)) + (\sigma\alpha_2 - \xi_x)x(\eta(t))}{\sigma(\xi_\pi - 1)(\alpha_1 - \alpha_2)}(\xi_x - \sigma\alpha_1)e^{\alpha_1(t-\eta(t))} \\
&\quad - \frac{(1-\xi_\pi)\pi(\eta(t)) + (\sigma\alpha_1 - \xi_x)x(\eta(t))}{\sigma(\xi_\pi - 1)(\alpha_1 - \alpha_2)}(\xi_x - \sigma\alpha_2)e^{\alpha_2(t-\eta(t))}. \qquad \text{(A.10)}
\end{aligned}
$$

For Case II, the solution is

$$x(t) = x(\eta(t))e^{\frac{1}{\sigma}\xi_x(t-\eta(t))}, \tag{A.11}$$

$$\pi(t) = \frac{\pi(\eta(t))(\xi_x - \sigma\rho) + \kappa\sigma x(\eta(t))}{\xi_x - \sigma\rho} e^{\rho(t-\eta(t))} - \frac{\kappa\sigma x(\eta(t))}{\xi_x - \sigma\rho} e^{\frac{1}{\sigma}\xi_x(t-\eta(t))}. \tag{A.12}$$

For Case III, the solution is

$$\begin{aligned} x(t) &= \left(1 + \frac{1}{2\sigma}(\xi_x - \sigma\rho)(t - t_1)\right) x(\eta(t)) e^{\frac{1}{2}\left(\rho + \frac{1}{\sigma}\xi_x\right)(t-\eta(t))} \\ &\quad + \frac{1}{\kappa}\left(\frac{1}{2\sigma}(\sigma\rho - \xi_x)\right)^2 (t - t_1)\, \pi(\eta(t)) e^{\frac{1}{2}\left(\rho + \frac{1}{\sigma}\xi_x\right)(t-\eta(t))}, \end{aligned} \tag{A.13}$$

$$\begin{aligned} \pi(t) &= -\kappa(t - t_1)\, x(\eta(t)) e^{\frac{1}{2}\left(\rho + \frac{1}{\sigma}\xi_x\right)(t-\eta(t))} \\ &\quad + \left(1 - \frac{1}{2\sigma}(\xi_x - \sigma\rho)(t - t_1)\right) \pi(\eta(t)) e^{\frac{1}{2}\left(\rho + \frac{1}{\sigma}\xi_x\right)(t-\eta(t))}. \end{aligned} \tag{A.14}$$

Saddle path in Ω_{ss}. The saddle path Υ_{ss} is the set of points (π, x) such that

$$\pi = \begin{cases} \frac{(\xi_x - \sigma\alpha_2)}{(1-\xi_\pi)} x & , \text{ if } \det A_{ss} < 0 \text{ and } \xi_\pi \neq 1 \\ \frac{\kappa\sigma}{(\sigma\rho - \xi_x)} x & , \text{ if } \det A_{ss} < 0 \text{ and } \xi_\pi = 1 \\ \frac{\kappa}{\rho} x & , \text{ if } \det A_{ss} = 0 \text{ and } \operatorname{trace} A_{ss} \geq 0 \\ \emptyset & , \text{ otherwise} \end{cases} . \tag{A.15}$$

A.2 Preliminaries 2

The goal of this section is to develop the prerequisite mathematical notation and economic intuition to understand, in the next section, the necessary and sufficient conditions for global determinacy of the optimal equilibrium under the rule in equation (15). As before, this rule can be understood in three stages.

First stage ($0 \leq t < T$). The dynamics of the economy are exactly as in rule (10), since $i(t) = 0$ and there are no other decisions for the central bank to make (recall liftoff occurs after T by assumption). Each point $(\pi(0), x(0))$ maps to one and only one $(\pi(T), x(T))$ and the mapping is unaffected by expectations or outcomes. Figure 11 shows the phase portrait that can be used to understand this mapping. Because the natural rate is negative, the unique steady state for the first-stage dynamics, labeled (π_l, x_l) in the figure, is in the first quadrant and given by $\pi_l = -r_l > 0$ and $x_l = -\rho r_l/\kappa > 0$. While the dynamics and the location of the steady state do not change with private-sector expectations or central bank actions, the specific $(\pi(0), x(0))$ that is actually realized in equilibrium —and hence which path is realized— does depend on them.

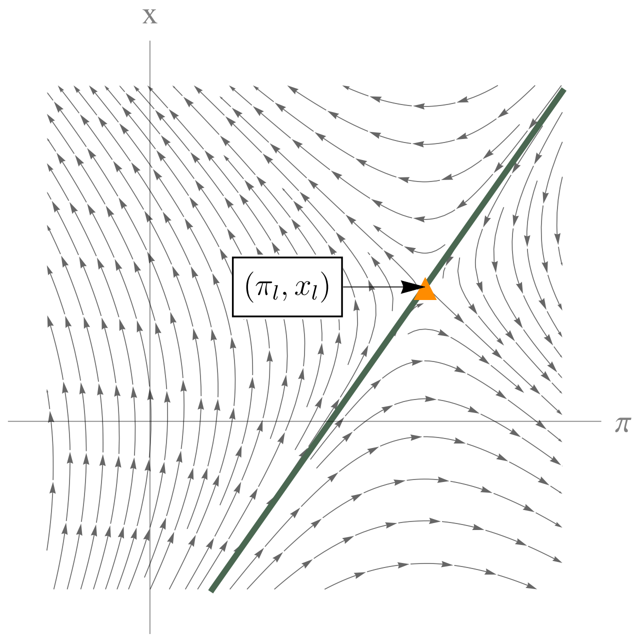

Figure 11: Dynamics of the economy for $t \in [0, T)$, when $i(t) = 0$ for all $\pi(t)$ and $x(t)$ and $r(t) = r_l < 0$. The green line is the saddle path and the orange triangle, labeled (π_l, x_l), is the steady state.

Second stage ($T \leq t < t_1$). The central bank is committed to $i(t) = 0$ between T and $f(R_{t_1})$. The dynamics are identical to those in rule (10), with the only difference being that they are maintained until $f(R_{t_1})$, which depends on R_{t_1}, instead of until the constant liftoff time t^*. In equilibrium, $t_1 = f(R_{t_1})$, so the actual duration of this stage is endogenous and depends not only on the announced liftoff rule but also on private-sector expectations and the realizations of inflation and output in the first stage. Given a known liftoff time t_1, analogous to what happens in the first stage, the dynamics of the economy and the mapping from $(\pi(T), x(T))$ to $(\pi(t_1), x(t_1))$ are always unchanged, while the specific $(\pi(T), x(T))$ and $(\pi(t_1), x(t_1))$ that end up being realized change based on which equilibrium ends up being realized.

Figure 12 shows the phase portrait of $\pi(t)$ and $x(t)$, which reveals saddle dynamics. I denote the stable *zlb saddle path* by

$$\Upsilon_{zlb} = \left\{ (\pi, x) : x = \frac{\phi_1}{\kappa}\pi - \frac{\phi_2}{\kappa} r_h \right\}, \tag{A.16}$$

where

$$\phi_1 = \frac{1}{2}\rho + \frac{1}{2}\sqrt{\rho^2 + 4\frac{\kappa}{\sigma}} > 0, \tag{A.17}$$

$$\phi_2 = \frac{1}{2}\rho - \frac{1}{2}\sqrt{\rho^2 + 4\frac{\kappa}{\sigma}} < 0, \tag{A.18}$$

are the two eigenvalues of the system. The *unstable zlb saddle path* is given by

$$\Psi_{zlb} = \left\{(\pi, x) : x = \frac{\phi_2}{\kappa}\pi - \frac{\phi_1}{\kappa}r_h\right\}, \tag{A.19}$$

which is the saddle path that would be stable if the system evolved backward in time.

The *zlb* saddle path Υ_{zlb}, a line with positive slope, intersects the unstable *zlb* saddle path Ψ_{zlb}, a line with negative slope, at the *zlb steady state*

$$(\pi_{zlb}, x_{zlb}) = \left(-r_h, -\frac{\rho r_h}{\kappa}\right). \tag{A.20}$$

The point (π_{zlb}, x_{zlb}) always lies in the third quadrant of the π-x plane. Neither the location of (π_{zlb}, x_{zlb}) nor the slopes of Υ_{zlb} and Ψ_{zlb} depend on policy choices of the central bank; they are fully specified by the parameters κ, ρ, σ and r_h. In the literature, the steady state (π_{zlb}, x_{zlb}) is variously referred to as the "deflationary steady state," the "liquidity trap steady state," the "expectational trap steady state" or the "unintended steady state."

Two key objects to understanding the behavior of the economy and its determinacy properties are:

$$d_{exit}(t) = x(t) - \frac{\phi_1}{\kappa}\pi(t) + \frac{\phi_2 r_h}{\kappa}, \tag{A.21}$$

$$d_{trap}(t) = x(t) - \frac{\phi_2}{\kappa}\pi(t) + \frac{\phi_1 r_h}{\kappa}. \tag{A.22}$$

The value of d_{exit} is a measure of the distance (with sign) to the stable *zlb* saddle path Υ_{zlb} defined in equation (A.16). Similarly, d_{trap} gives a measure of distance (with sign) from $(\pi(t), x(t))$ to the unstable saddle path Ψ_{zlb} defined in equation (A.19). A d_{exit} closer to zero means the economy is closer to Υ_{zlb} and hence will behave more similarly to an economy that is on Υ_{zlb}, at least for some initial period of time. A d_{exit} closer to zero also implies that the economy at some point gets closer to the unintended steady state (π_{zlb}, x_{zlb}). In fact, $d_{exit}(t) = 0$ indicates that the economy is exactly on Υ_{zlb} at time t and converging toward (π_{zlb}, x_{zlb}). In contrast, a d_{trap} closer to zero implies that the dynamics of the economy look more like those of the unstable saddle path Ψ_{zlb}, pushing the economy further away from (π_{zlb}, x_{zlb}).

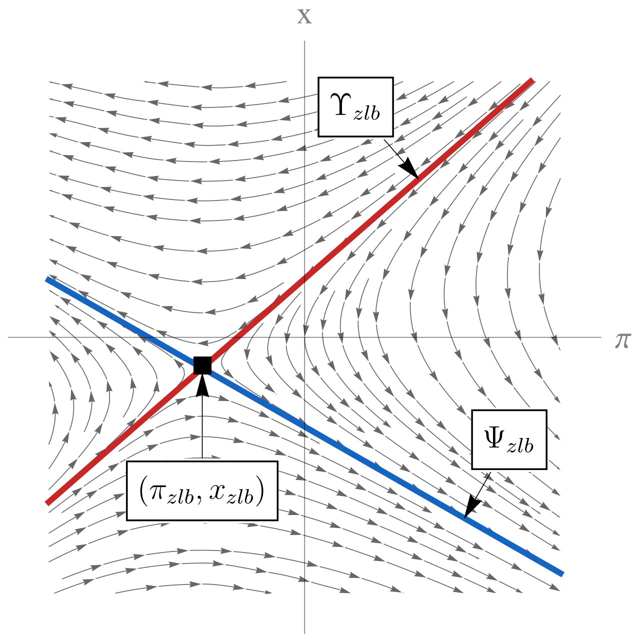

Figure 12: Dynamics of the economy for $t \in [T, t_1)$, when $i(t) = 0$ for all $\pi(t)$ and $x(t)$, and $r(t) = r_h > 0$. The red line, labeled Υ_{zlb}, is the saddle path. If the economy starts on Υ_{zlb}, it converges to the deflationary steady state (π_{zlb}, x_{zlb}). The blue line, labeled Ψ_{zlb}, is the "unstable saddle path." If the economy starts on Ψ_{zlb}, it stays on Ψ_{zlb} and moves away from the deflationary steady state (π_{zlb}, x_{zlb}). Under these dynamics, if the economy is not on Υ_{zlb}, then inflation and output become unbounded.

The dynamics of the economy are a tug of war between two competing forces: one driving the economy into the deflationary steady state (π_{zlb}, x_{zlb}) and another pulling the economy away from it. The strength of these two forces is given by d_{exit} and d_{trap}. Indeed, inflation and output can be written as

$$\begin{bmatrix} x(t) - x_{zlb} \\ \pi(t) - \pi_{zlb} \end{bmatrix} = d_{exit}(t)\ v_{exit} + d_{trap}(t)\ v_{trap}, \tag{A.23}$$

where

$$v_{exit} = \begin{bmatrix} -\frac{\phi_2}{\phi_1 - \phi_2} \\ -\frac{\kappa}{\phi_1 - \phi_2} \end{bmatrix} \text{ and } v_{trap} = \begin{bmatrix} \frac{\phi_1}{\phi_1 - \phi_2} \\ \frac{\kappa}{\phi_1 - \phi_2} \end{bmatrix}, \tag{A.24}$$

are the eigenvectors of the system. The eigenvector v_{exit} is associated with the explosive eigenvalue $\phi_1 > 0$ and lies on the unstable saddle path Ψ_{zlb}. The eigenvector v_{trap} is associated with the stabilizing eigenvalue $\phi_2 < 0$ and lies on the stable saddle path Υ_{zlb}. The eigenvalue v_{trap} is the "trap factor" that drives the economy into the expectational trap steady state (π_{zlb}, x_{zlb}), while v_{exit} is the "exit factor" pulling the economy away from it. After a change of coordinates that makes (π_{zlb}, x_{zlb}) the origin and the eigenvectors of the system the coordinate basis vectors, the vector $(\pi(t), x(t))$ has coordinates $(d_{exit}(t), d_{trap}(t))$:

$$\begin{bmatrix} -\frac{\phi_2}{\phi_1-\phi_2} & \frac{\phi_1}{\phi_1-\phi_2} \\ -\frac{\kappa}{\phi_1-\phi_2} & \frac{\kappa}{\phi_1-\phi_2} \end{bmatrix}^{-1} \begin{bmatrix} x(t) - x_{zlb} \\ \pi(t) - \pi_{zlb} \end{bmatrix} = \begin{bmatrix} d_{exit}(t) \\ d_{trap}(t) \end{bmatrix} \tag{A.25}$$

In other words, projecting $(\pi(t), x(t))$ onto the eigenvalues gives loadings of $(d_{exit}(t), d_{trap}(t))$. This linear two-factor representation of the economy is exact in the sense that there is no residual left once $x(t)$ and $\pi(t)$ are expressed as a linear combination of the factors plus a constant. As already pointed out through various other arguments by Benhabib et al. (2001b), Werning (2012) and others, the current levels of inflation and output are, on their own, not very informative about whether the economy is in a liquidity trap, constrained by the ZLB, at risk of converging to the unintended steady state, or on a desirable policy path. For example, Benhabib et al. (2001b) show that an economy can have inflation and output arbitrarily close to target (in the model I consider here, the target is the intended steady state (π_{ss}, x_{ss})) and yet converge to the unintended steady state (π_{zlb}, x_{zlb}). In contrast, observing a $d_{exit}(t) = 0$ immediately reveals that an economy is headed toward (π_{zlb}, x_{zlb}). In addition, and more important for this paper, the necessary and sufficient conditions for global determinacy in the next section are most simply expressed as functions of $d_{exit}(t)$ and $d_{trap}(t)$, highlighting not just their mathematical convenience but also their economic importance. When a central bank is trying to assess private-sector expectations in order to know what is needed to anchor expectations, the results in this paper suggest that it should focus on the linear combinations of inflation and output given by $d_{exit}(t)$ and $d_{trap}(t)$ rather than on the levels of inflation and output by themselves.

Third stage ($t \geq t_1$). I split the π-x plane into two disjoint regions defined by whether the ZLB is binding

$$\Omega_{zlb}(R_{t_1}) = \{(x, \pi) : \xi_\pi(R_{t_1})\pi + \xi_x(R_{t_1})x + r_h \leq 0\}, \tag{A.26}$$

$$\Omega_{ss}(R_{t_1}) = \{(x, \pi) : \xi_\pi(R_{t_1})\pi + \xi_x(R_{t_1})x + r_h > 0\}, \tag{A.27}$$

where the subscript zlb in Ω_{zlb} stands for zero lower bound and the subscript ss in Ω_{ss} stands for "intended steady state," as the region $\Omega_{ss}(R_{t_1})$ contains $(\pi_{ss}, x_{ss}) = (0, 0)$, the steady state that the central bank would like the economy to converge to in the long run if the optimal equilibrium is to be achieved. The boundary between the regions $\Omega_{zlb}(R_{t_1})$ and $\Omega_{ss}(R_{t_1})$ is a line

$$\partial\Omega(R_{t_1}) = \{(x, \pi) : \xi_\pi(R_{t_1})\pi + \xi_x(R_{t_1})x + r_h = 0\} \subset \Omega_{zlb}(R_{t_1}). \tag{A.28}$$

Henceforth, I suppress the dependence of ξ_x, ξ_π, Ω_{zlb}, Ω_{ss} and $\partial\Omega$ on R_{t_1} for ease of notation whenever it does not create confusion.

The derivatives of inflation and output with respect to time, $\dot{\pi}(t)$ and $\dot{x}(t)$, inherit the properties of $i(t)$ and are therefore not differentiable on $\partial\Omega$ as a function of time (i.e., $\pi(t)$ and $x(t)$ do not have second derivatives on $\partial\Omega$). However, $\dot{\pi}(t)$ and $\dot{x}(t)$ are always continuous with respect to time, ensuring a continuous path for $(\pi(t), x(t))$.[29] By the second line in equation (15), after t_1,

$$i(t) = 0 \text{ iff } (\pi(t), x(t)) \in \Omega_{zlb}, \tag{A.29}$$

$$i(t) = \xi_\pi \pi(t) + \xi_x x(t) + r_h \text{ iff } (\pi(t), x(t)) \in \Omega_{ss}. \tag{A.30}$$

When equations (A.29) and (A.30) are used in the IS equation and the NKPC, the dynamics of the economy inside each of the two regions Ω_{zlb} and Ω_{ss} are separately given by a system of linear first-order ordinary differential equations (ODEs) in $x(t)$ and $\pi(t)$, each of which is easy to analyze inside its respective region with standard methods. However, when the two regions are analyzed together, the global dynamics are piecewise linear, with a non-differentiable transition at $\partial\Omega$. The behavior of piecewise linear dynamic systems can, in general, exhibit a rich variety of non-linear phenomena such as limit cycles, bifurcations and chaos. The global properties can also be quite different from those of each individual region. For example, it is possible to construct paths that are globally bounded for systems in which each separate region has explosive dynamics.[30] To tackle the non-linearities of the New Keynesian economy at hand, I first analyze the properties of each of the two regions separately and then combine them and analyze the resulting global dynamics. Readers familiar with New Keynesian models without a ZLB should find the analysis of each of the separate regions familiar. The new results arise when I look at both regions together.

First, consider the behavior of the economy in the region Ω_{ss}. Inside Ω_{ss}, there is always a single steady state, $(\pi_{ss}, x_{ss}) = (0, 0)$. The dynamic behavior of the economy depends on the choice of Taylor rule coefficients ξ_π and ξ_x. I focus on Taylor rule coefficients that, absent the ZLB, give either unstable or saddle dynamics, since the central bank would not pick stable dynamics that have no explosive paths, as they always lead to indeterminacy.[31] The Taylor principle is the key concept needed to differentiate between unstable and saddle dynamics, and between *locally* determinate and indeterminate equilibria. The *Taylor principle* is said to *hold* if and only if

[29]In fact, $(\dot{\pi}(t), \dot{x}(t))$ is Lipschitz continuous in $(\pi(t), x(t))$ and continuous in t, guaranteeing the global existence and uniqueness of the continuous solution for $t \geq t_1$.

[30]For example, see Bernardo, Budd, Champneys, and Kowalczyk (2008).

[31]For stable dynamics, it is immediate that there is indeterminacy for any choice of ξ_π, ξ_x and f. I also exclude the knife-edge case in which the dynamics have a line whose points are all steady states, but the dynamics are otherwise explosive. See Lemma 3 in Appendix B.9 for a proof that, in this case, there also is indeterminacy for any choice of ξ_π, ξ_x and f.

$$\kappa\left(\xi_\pi - 1\right) + \rho\xi_x > 0 \quad \text{and} \quad \xi_x + \sigma\rho > 0. \tag{A.31}$$

When $\xi_x = 0$, the Taylor principle is equivalent to $\xi_\pi > 1$, one of its most popular forms. When the Taylor principle holds, if the dynamics of the Ω_{ss} region were extended to the entire plane and nominal interest rates were allowed to be negative, or if I considered a small enough neighborhood of (π_{ss}, x_{ss}) that lies entirely inside Ω_{ss}, the dynamics of the system would be unstable. All paths would be unbounded —or exit the small neighborhood— unless $(\pi(t), x(t)) = (\pi_{ss}, x_{ss}) = (0, 0)$ for all t. Figure 13 shows representative phase portraits of two such economies. In the diagram on the left, the Taylor rule coefficients satisfy $\left(\xi_x - \sigma\rho\right)^2 - 4\kappa\sigma\left(\xi_\pi - 1\right) < 0$ and paths "slowly" spiral outward from the steady state. When the reverse inequality holds, the steady state is instead a source, shown in the diagram on the right. In models without a ZLB, the Taylor principle is a necessary and sufficient condition for local determinacy. When the ZLB is introduced, since there is always a small enough neighborhood of (π_{ss}, x_{ss}) that is contained entirely in Ω_{ss}, the Taylor principle is still a necessary and sufficient condition for local determinacy of the equilibrium with $(\pi(t), x(t)) = (\pi_{ss}, x_{ss})$ for all $t \geq t_1$. As I briefly discussed before and will expand on later, the Taylor principle is, however, neither necessary nor sufficient for *global* determinacy of the optimal equilibrium.

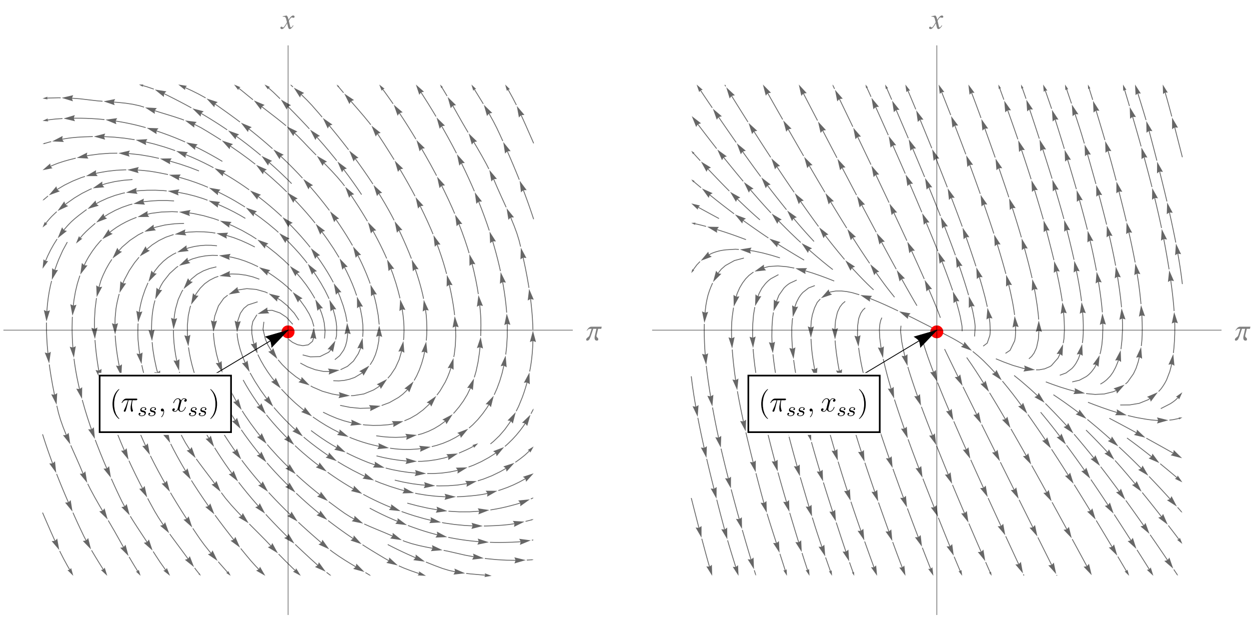

Figure 13: Dynamics of the economy for $t \geq t_1$ when the Taylor principle holds and there is no ZLB. The dynamics of the diagram on the left have imaginary eigenvalues, while the right-hand side have real ones. When there is no ZLB, the Taylor principle is necessary and sufficient for *local* determinacy. Unless the economy starts at $(0, 0)$, inflation and output become unbounded.

The *Taylor principle* is said to *not hold* if and only if

$$\kappa(\xi_\pi - 1) + \rho\xi_x < 0. \tag{A.32}$$

When the Taylor principle does not hold, if the dynamics of the Ω_{ss} region were extended to the entire plane and interest rates were allowed to be negative, or if I considered a small enough neighborhood of (π_{ss}, x_{ss}) that lies entirely in Ω_{ss}, the system would have saddle dynamics. I denote the *ss saddle path* by Υ_{ss}. It is a line through the origin whose slope depends on the Taylor rule coefficients ξ_π and ξ_x.[32] Paths are bounded —or stay in the small neighborhood of (π_{ss}, x_{ss})— if and only if they start on the *ss* saddle path. Figure 14 displays a typical phase diagram when the Taylor principle does not hold.

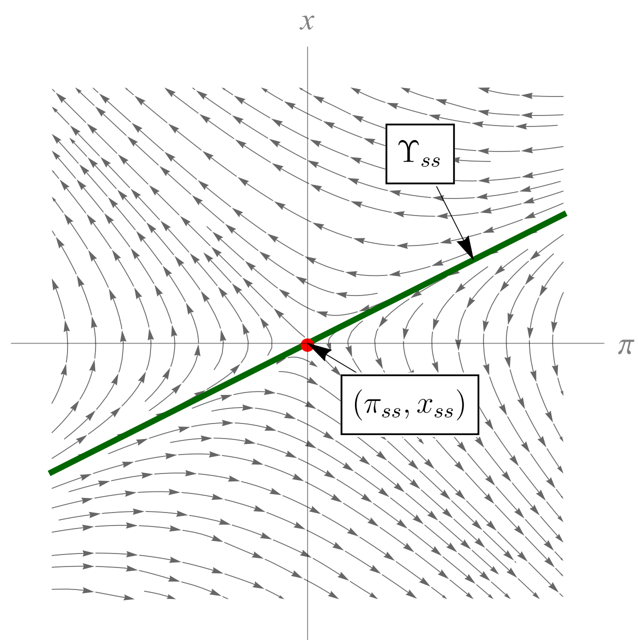

Figure 14: Dynamics of the economy for $t \geq t_1$ when the Taylor principle does not hold and there is no ZLB. Unless the economy starts on its saddle path Υ_{ss}, shown in green, inflation and output become unbounded.

Note that because ξ_π and ξ_x depend on R_{t_1}, whether the Taylor principle holds depends on R_{t_1}. Within the same economy, there can be a subset of (off-equilibrium) paths for which the Taylor principle holds and a different subset for which it does not hold. Instead, when ξ_π and ξ_x are constant, the Taylor principle must hold either for all paths or for no paths. In all cases, the Taylor rule coefficients are fully determined by the time the central bank lifts off and they remain unchanged from then on.

Now consider the behavior in the *zlb* region, Ω_{zlb}. If the dynamics of the Ω_{zlb} region were extended to the entire π-x plane, the dynamics would be identical to those of the second stage analyzed above, with the phase diagram given in Figure 12. Inside Ω_{zlb}, the system always has saddle dynamics with saddles Υ_{zlb} and Ψ_{zlb}.

[32] See equation (A.15) in Appendix A.1 for the explicit formula.

I now put the dynamics of the regions Ω_{ss} and Ω_{zlb} together and describe some of the global properties of the economy. The left panel of Figure 15 shows an example of the global dynamics in which the Taylor principle holds, while the right panel shows an example in which the Taylor principle does not hold. When the Taylor principle holds, the dynamics inside Ω_{ss} look like those in Figure 13. When the Taylor principle does not hold, they look like those in Figure 14. Of course, whether the Taylor principle holds or not, the dynamics in Ω_{zlb} always look like those in Figure 12, as they are not affected by the choice of ξ_x or ξ_π. However, the coefficients ξ_x and ξ_π do have a crucial effect on the ZLB, as they determine the location of the boundary $\partial\Omega$ and, consequently, whether the undesirable deflationary steady state can exist in the economy. If the Taylor principle holds, the *zlb* steady state (π_{zlb}, x_{zlb}) is in Ω_{zlb}; when the Taylor principle does not hold, it is not. To see this, compute

$$\begin{aligned} \xi_\pi \pi_{zlb} + \xi_x x_{zlb} + r_h &= \xi_\pi (-r_h) + \xi_x \left(-\frac{\rho r_h}{\kappa}\right) + r_h, && \text{(A.33)} \\ &= -\frac{r_h}{\kappa} \left(\kappa (\xi_\pi - 1) + \rho \xi_x\right). && \text{(A.34)} \end{aligned}$$

By definition, the sign of this expression determines whether the *zlb* steady state (π_{zlb}, x_{zlb}) is inside or outside Ω_{zlb}. In turn, the sign of $\kappa(\xi_\pi - 1) + \rho\xi_x$ is determined by whether the Taylor principle holds. When the Taylor principle holds, (π_{zlb}, x_{zlb}) is a steady state of the global dynamics. Because of its saddle dynamics, equilibria are locally indeterminate around (π_{zlb}, x_{zlb}). Together with the intended steady state (π_{ss}, x_{ss}), they are the two global steady states of the economy. On the other hand, if the Taylor principle does not hold, (π_{zlb}, x_{zlb}) is in Ω_{ss}. Under the Ω_{ss} dynamics, the point (π_{zlb}, x_{zlb}) is not a steady state. In this case, the only steady state for the global dynamics is the desired one, (π_{ss}, x_{ss}).

The conclusion that following the Taylor principle outside the ZLB induces the existence of a deflationary steady state at the ZLB is similar to one of the results in Benhabib et al. (2001b). They further show that when the Taylor principle holds, the deflationary steady state engenders an infinite number of suboptimal equilibria. As mentioned before, these equilibria can start arbitrarily close to the intended steady state (π_{ss}, x_{ss}) and still converge to (π_{zlb}, x_{zlb}). The same possibility is present in the setup I consider here. To construct equilibria analogous to those in Benhabib et al. (2001b), I use the dynamics for the three stages described above. For the next steps, refer to Figure 16. First pick two numbers q and r such that $r, q > T$ and $r - q > T$. Let $(\pi_b, x_b) = \partial\Omega \cap \Upsilon_{zlb}$.[33] Assume the Taylor principle holds. Using $(\pi(r), x(r)) = (\pi_b, x_b)$ as the starting point, trace the dynamics of $(\pi(t), x(t))$ backward in time using the interest rate specified by equation (A.30) for a length of time

[33] If $\kappa\xi_\pi + \phi_1\xi_x = 0$, $\partial\Omega \cap \Upsilon_{zlb} = \emptyset$. Albeit not a general strategy to eliminate all non-optimal equilibria, picking ξ_x, ξ_π such that $\kappa\xi_\pi + \phi_1\xi_x = 0$ does preclude this particular class of equilibria from forming for any choice of f. This possibility was not present in Benhabib et al. (2001b), as their model did not have both inflation and output as state variables of the economy.

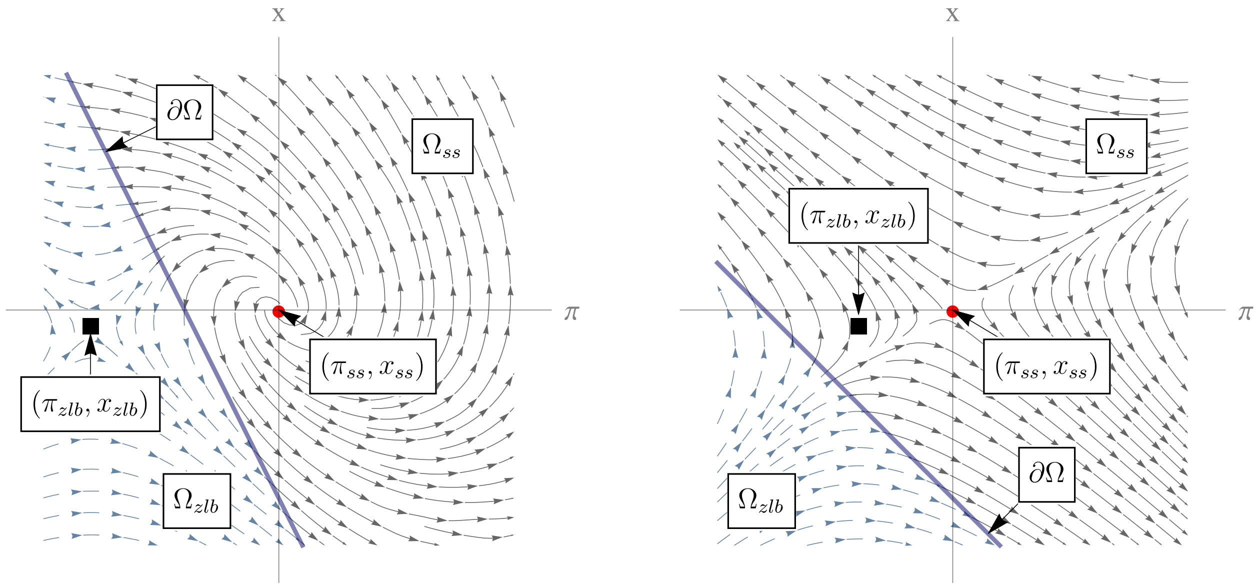

Figure 15: Non-linear dynamics of the economy after liftoff t_1. The central bank follows the Taylor rule $i(t) = \max\{0, \xi_\pi \pi(t) + \xi_x x(t) + r_h\}$. When $\xi_\pi \pi(t) + \xi_x x(t) + r_h > 0$, the economy is in the region Ω_{ss} and follows the solid black flow lines. When $\xi_\pi \pi(t) + \xi_x x(t) + r_h \leq 0$, it is in the region Ω_{zlb} and follows the dashed blue flow lines. The boundary between the two regions is the line $\partial\Omega$. The point (π_{ss}, x_{ss}), shown in red, is always a steady state of the economy. The point (π_{zlb}, x_{zlb}), shown as a black square, is a steady state of the economy if and only if the Taylor principle holds, as in the left panel. When the Taylor principle does not hold, as in the right panel, (π_{zlb}, x_{zlb}) is not in Ω_{zlb} and is therefore not a steady state.

q. As in Benhabib et al. (2001b), these equilibria can get arbitrarily close to the intended steady state: Because the dynamics of $(\pi(t), x(t))$ are unstable when going forward in time, they are stable backward in time and $(\pi(t), x(t))$ converges to (π_{ss}, x_{ss}) as $q \to \infty$.[34] At time $r - q$, trace the dynamics of $(\pi(t), x(t))$ backward in time using $(\pi(r-q), x(r-q))$ as the starting point and $i(t) = 0$ throughout, until $t = 0$, when the path reaches $(\pi(0), x(0))$. Of course, the natural rate is positive after T and negative before T, so the dynamics change from those of the second stage to those of the first. Note that in Figure 16, the gray flow lines in the background reflect the dynamics that prevail for $t \geq t_1$ only. Set $t_1 = r - q > T$. By construction, the path starting at $(\pi(0), x(0))$ reaches (π_b, x_b) at time r when following the interest rate rule in equation (15). Now going forward in time, for $t \geq r$, $(\pi(t), x(t)) \in \Upsilon_{zlb} \subset \Omega_{zlb}$, which means the economy travels on the *zlb* saddle path toward the unintended steady state (π_{zlb}, x_{zlb}). The path constructed is continuous and bounded and has consistent expectations: It is a rational expectations equilibrium. All equilibria in

[34]This result is not immediate, since it may be possible that $(\pi(t), x(t))$ exits Ω_{ss} before getting close to (π_{ss}, x_{ss}) and then follows the Ω_{zlb} dynamics for which (π_{ss}, x_{ss}) is no longer a sink (flowing backward in time). However, I show in Appendix B.6, Lemma 2, item (d) that this never happens. For all q, the path of $(\pi(t), x(t))$ remains entirely in Ω_{ss}.

this class can be obtained by picking different q and r.

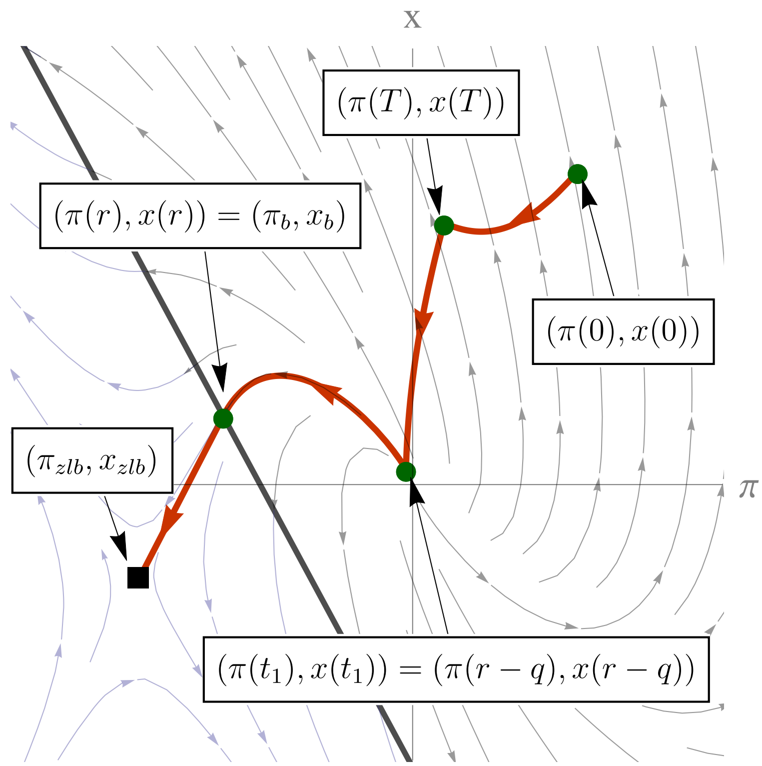

Figure 16: An equilibrium analogous to the one studied by Benhabib et al. (2001b). The flow lines in the background correspond to the dynamics after liftoff, which occurs at t_1. Because the Taylor principle holds, there is a deflationary steady state (π_{zlb}, x_{zlb}), shown as a black square. At time t_1, even though the economy is outside the ZLB and can get arbitrarily close to the "desired" steady state $(\pi_{ss}, x_{ss}) = (0, 0)$, it still converges to the "unintended" steady state (π_{zlb}, x_{zlb}). At time r, the economy enters the ZLB and stays there ($i(t)$=0) forever after.

B Proofs of Propositions

B.1 Proof of Proposition 1

Proof. Assume that rule (11) implements the optimal equilibrium. I show that it also implements a different second equilibrium. Let $R_0^* = (\pi^*(0), x^*(0))$. Because the rule (11) implements the optimal equilibrium and $\max\{0, \xi_\pi(R_0^*)\pi^*(t^*) + \xi_x(R_0^*)x^*(t^*) + r_h\} > 0$, we have $\underline{t} = t^*$.

Consider the path that starts at $(\pi(0), x(0))$ and reaches $(\pi_{ss}, x_{ss}) = (0, 0)$ at $t = t^*$ when following (1), (2) and (11). This path always exists, since we can find it by positioning the economy on $(0, 0)$ at $t = t^*$ and running time backward until $t = 0$ using $i(t) = 0$ throughout.

Since the point $(0,0)$ is a steady state after t^* for any choice of $\xi_\pi(R_0)$ and $\xi_x(R_0)$, $(\pi(t), x(t))$ remains bounded. If $\kappa\sigma\lambda \neq 1$, $(x(\underline{t}), \pi(\underline{t})) = (\pi(t^*), x(t^*)) = (0,0) \neq (\pi^*(t^*), x^*(t^*))$ (see Werning (2012) for a proof that $(0,0) \neq (\pi^*(t^*), x^*(t^*))$ when $\kappa\sigma\lambda \neq 1$). Hence, when $\kappa\sigma\lambda \neq 1$, the path that starts at the $(\pi(0), x(0))$ that reaches $(0,0)$ at $t = t^*$ when $i(t) = 0$ between $t = 0$ and $t = t^*$ constitutes an equilibrium different from the optimal one for any choice of functions $\xi_\pi(R_0)$ and $\xi_x(R_0)$. When $\kappa\sigma\lambda = 1$, the optimal equilibrium happens to have $(x^*(t), \pi^*(t)) = (0,0)$ for all $t \geq t^*$ and the optimal equilibrium is indeed implementable as the unique equilibrium (Appendix B.2 shows how). □

B.2 Case $\kappa\sigma\lambda = 1$ in Proposition 1

The next example shows that when $\kappa\sigma\lambda = 1$, it is indeed possible to implement the optimal equilibrium uniquely with a constant liftoff time. Let $\underline{t}= t^*$ and $\kappa\sigma\lambda = 1$. Pick

$$(\xi_\pi(R_0), \xi_x(R_0)) = \begin{cases} (0,0) & , \text{ if } R_0 \text{ is such that } (\pi(\underline{t}), x(\underline{t})) = (0,0) \text{ or } \rho\pi(\underline{t}) \neq \kappa x(\underline{t}) \\ \left(1, -\frac{\rho}{\kappa}\right) & , \text{ otherwise} \end{cases} \tag{B.1}$$

Note that when the time of liftoff is constant, it is equivalent to write ξ_π and ξ_x as a function of R_0 or as a function of $(\pi(s), x(s))$ for any $s > 0$. Hence, equation (B.1) can be written as

$$(\xi_\pi, \xi_x) = \begin{cases} (0,0) & , \text{ if } (x(\underline{t}), \pi(\underline{t})) = (0,0) \text{ or } \rho\pi(\underline{t}) \neq \kappa x(\underline{t}) \\ \left(1, -\frac{\rho}{\kappa}\right) & , \text{ otherwise} \end{cases} . \tag{B.2}$$

I now show that the rule

$$i(t) = \begin{cases} 0 & , \ 0 \leq t < \underline{t} \\ \max\{0, \xi_\pi \pi(t) + \xi_x x(t) + r(t)\} & , \ \underline{t} \leq t < \infty \end{cases} \tag{B.3}$$

implements the optimal equilibrium as the unique equilibrium of the economy.

When $(\pi(\underline{t}), x(\underline{t})) = (0,0)$, the rule implements the optimal path. Werning (2012) shows that when $\kappa\sigma\lambda = 1$, $(\pi^*(t^*), x^*(t^*)) = (0,0)$. Since $\underline{t}= t^*$ and $(\pi(\underline{t}), x(\underline{t})) = (\pi^*(t^*), x^*(t^*)) = (0,0)$, $(\pi(t), x(t)) = (\pi^*(t), x^*(t))$ for $t < t^*$. By equation (B.2), $\xi_x = \xi_\pi = 0$ and thus $i(t) = i^*(t) = r_h > 0$ for $t \geq t_1$. As $(0,0)$ is a steady state, $(\pi(t), x(t)) = (0,0)$ for all $t \geq t_1$, which shows that $(\pi(t), x(t)) = (\pi^*(t), x^*(t))$ for $t \geq t_1$.

No other equilibrium exists since, for all $R_0 \neq (\pi^*(0), x^*(0))$, continuous paths are unbounded. If $\rho\pi(\underline{t}) \neq \kappa x(\underline{t})$, equation (B.2) gives $(\xi_x, \xi_\pi) = (0,0)$ and, by equation (A.15), the saddle path is $\rho\pi = \kappa x$. It follows that $(\pi(\underline{t}), x(\underline{t})) \notin \Upsilon_{ss}$. In addition, $i(t) = r_h > 0$ for $t \geq \underline{t}$ and thus $(\pi(t), x(t)) \in \Omega_{ss}$ for all $t \geq \underline{t}$ since Ω_{zlb} is empty. The global saddle path dynamics and $(\pi(\underline{t}), x(\underline{t})) \notin \Upsilon_{ss}$ imply that $(\pi(t), x(t))$ explodes as $t \to \infty$.

If $(\pi(\underline{t}), x(\underline{t})) \neq (0,0)$ and $\rho\pi(\underline{t}) = \kappa x(\underline{t})$, $(\xi_x, \xi_\pi) = \left(1, -\frac{\rho}{\kappa}\right)$ implies that the Taylor principle does not hold, since $\kappa(\xi_\pi - 1) + \rho\xi_x = -\kappa < 0$. In addition, $(\pi(\underline{t}), x(\underline{t})) \in \Omega_{ss}$ since $\xi_\pi \pi(\underline{t}) + \xi_x x(\underline{t}) + r(\underline{t}) = r_h > 0$ and $(\pi(\underline{t}), x(\underline{t})) \notin \Upsilon_{ss}$ by equation (A.15). Because the dynamics are saddle path stable and $(\pi(\underline{t}), x(\underline{t}))$ is not on the saddle path, $(\pi(t), x(t))$ either explodes or enters Ω_{zlb} in finite time. By item (c) in Lemma 1 of Appendix B.5, if $(\pi(t), x(t))$ enters Ω_{zlb} it also explodes.

B.3 Constants in the Neo-Fisherian Rule of Section 5.1

To be on the saddle path at time t_1, $\pi(t_1) = \phi x(t_1)$. Using the continuous pasting conditions in equation (B.6) to express $\pi(t_1) = \phi x(t_1)$ in terms of $x(0)$ and $\pi(0)$ gives

$$p(t_1)x(0) + q(t_1)\pi(0) = v(t_1),$$

where

$$\begin{aligned}
p(t_1) &= \kappa\sigma\left((\phi_1 - \kappa\phi)e^{-\phi_1 t_1} - (\phi_2 - \kappa\phi)e^{-\phi_2 t_1}\right), \\
q(t_1) &= \kappa\left((\sigma\phi\phi_2 + 1)e^{-\phi_1 t_1} - (\sigma\phi\phi_1 + 1)e^{-\phi_2 t_1}\right), \\
v(t_1) &= -\left((\kappa + \sigma\phi_1(\rho - \kappa\phi))r_l + \frac{(r_h - r_l)\phi_1}{\phi_1 - \phi_2}\left(\kappa + \sigma\phi_1^2 - \kappa\sigma\phi(\phi_1 - \phi_2)\right)e^{-T\phi_2}\right)e^{-\phi_1 t_1} \\
&\quad + \left((\kappa + \sigma\phi_2(\rho - \kappa\phi))r_l - \frac{(r_h - r_l)\phi_2}{\phi_1 - \phi_2}\left(\kappa + \sigma\phi_2^2 + \kappa\sigma\phi(\phi_1 - \phi_2)\right)e^{-T\phi_1}\right)e^{-\phi_2 t_1} \\
&\quad + \frac{(\sigma\rho^2 + 4\kappa)(\rho - \kappa\phi)r_h}{(\phi_1 - \phi_2)}e^{-\phi_1 t_1}e^{-\phi_2 t_1}.
\end{aligned}$$

Evaluating these functions at $t_1 = t^*$ and $t_1 = t^* + 1$ determines the constants A, B, C and D, E, F, respectively.

B.4 Proof of Proposition ??

By using the explicit solutions in Appendix A.1, the continuous pasting conditions for a path $(\pi(t), x(t))$ with $R_{t_1} = \overline{R}$ and $f(R_{t_1}) = \overline{t_1}$ imply

$$\begin{aligned} x_0 &= \frac{\phi_1 e^{-\phi_2 \overline{t_1}} - \phi_2 e^{-\phi_1 \overline{t_1}}}{\phi_1 - \phi_2} x_1 - \frac{1}{\sigma} \frac{e^{-\phi_1 \overline{t_1}} - e^{-\phi_2 \overline{t_1}}}{\phi_1 - \phi_2} \pi_1 \\ &\quad + \frac{r_h}{\kappa} \frac{\phi_1^2 e^{-\phi_2 \overline{t_1}} - \phi_2^2 e^{-\phi_1 \overline{t_1}}}{\phi_1 - \phi_2} + \left(\frac{r_h - r_l}{\kappa} \right) \frac{\phi_2^2 e^{-T\phi_1} - \phi_1^2 e^{-T\phi_2}}{\phi_1 - \phi_2} - \frac{r_l \rho}{\kappa}, \end{aligned} \tag{B.4}$$

$$\begin{aligned} \pi_0 &= -\kappa \frac{e^{-\phi_1 \overline{t_1}} - e^{-\phi_2 \overline{t_1}}}{\phi_1 - \phi_2} x_1 + \frac{\phi_1 e^{-\phi_1 \overline{t_1}} - \phi_2 e^{-\phi_2 \overline{t_1}}}{\phi_1 - \phi_2} \pi_1 \\ &\quad + r_h \frac{\phi_1 e^{-\phi_2 \overline{t_1}} - \phi_2 e^{-\phi_1 \overline{t_1}}}{\phi_1 - \phi_2} + (r_h - r_l) \frac{\phi_2 e^{-T\phi_1} - \phi_1 e^{-T\phi_2}}{\phi_1 - \phi_2} - r_l. \end{aligned} \tag{B.5}$$

Equations (B.4) and (B.5) are equivalent to continuous pasting at $\overline{t_1}$ if there already is continuous pasting at T. Solving for (π_1, x_1) in equations (B.4) and (B.5) gives, in matrix notation,

$$\begin{bmatrix} x_1 \\ \pi_1 \end{bmatrix} = \frac{e^{(\phi_1+\phi_2)\overline{t_1}}}{\phi_1 - \phi_2} \begin{bmatrix} \phi_1 e^{-\phi_1 \overline{t_1}} - \phi_2 e^{-\phi_2 \overline{t_1}} & \frac{1}{\sigma} \left(e^{-\phi_1 \overline{t_1}} - e^{-\phi_2 \overline{t_1}} \right) \\ \kappa \left(e^{-\phi_1 \overline{t_1}} - e^{-\phi_2 \overline{t_1}} \right) & \phi_1 e^{-\phi_2 \overline{t_1}} - \phi_2 e^{-\phi_1 \overline{t_1}} \end{bmatrix} \begin{bmatrix} x(0) - h\left(\overline{t_1}\right) \\ \pi(0) - m\left(\overline{t_1}\right) \end{bmatrix}, \tag{B.6}$$

where

$$\begin{aligned} h\left(\overline{t_1}\right) &= r_h \frac{\phi_1^2 e^{-\phi_2 \overline{t_1}} - \phi_2^2 e^{-\phi_1 \overline{t_1}}}{\kappa(\phi_1 - \phi_2)} + (r_h - r_l) \frac{\phi_2^2 e^{-T\phi_1} - \phi_1^2 e^{-T\phi_2}}{\kappa(\phi_1 - \phi_2)} - \frac{r_l \rho}{\kappa}, \\ m\left(\overline{t_1}\right) &= r_h \frac{\phi_1 e^{-\phi_2 \overline{t_1}} - \phi_2 e^{-\phi_1 \overline{t_1}}}{(\phi_1 - \phi_2)} + (r_h - r_l) \frac{\phi_2 e^{-T\phi_1} - \phi_1 e^{-T\phi_2}}{(\phi_1 - \phi_2)} - r_l. \end{aligned}$$

Solving for $e^{-\phi_1 \overline{t_1}}$ and $e^{-\phi_2 \overline{t_1}}$ in equation (B.6) and then eliminating $\overline{t_1}$ from one of the equations gives that paths that are already continuous at T are also continuous at $\overline{t_1}$ if and only if

$$0 = \mathcal{P}\left(\overline{R}\right), \tag{B.7}$$

$$\overline{t_1} = \mathcal{T}\left(\overline{R}\right), \tag{B.8}$$

where

$$\mathcal{P}(\overline{R}) = \begin{cases} 1\left\{d_{exit}\left(\overline{t_1}\right) \neq 0 \text{ or } d_{trap}\left(\overline{t_1}\right) \neq 0\right\} & , \text{ if } d_{exit}(0) = \frac{\phi_2}{\kappa}\left(r_h - r_l\right)\left(1 - e^{-T\phi_1}\right) \\ & \quad \text{and } d_{trap}(0) = \frac{\phi_1}{\kappa}\left(r_h - r_l\right)\left(1 - e^{-T\phi_2}\right) \\ d_{exit}\left(\overline{t_1}\right) + 1\left\{d_{trap}\left(\overline{t_1}\right) = 0\right\} & , \text{ if } d_{exit}(0) = \frac{\phi_2}{\kappa}\left(r_h - r_l\right)\left(1 - e^{-T\phi_1}\right) \\ & \quad \text{and } d_{trap}(0) \neq \frac{\phi_1}{\kappa}\left(r_h - r_l\right)\left(1 - e^{-T\phi_2}\right) \\ d_{trap}\left(\overline{t_1}\right) + 1\left\{d_{exit}\left(\overline{t_1}\right) = 0\right\} & , \text{ if } d_{exit}(0) \neq \frac{\phi_2}{\kappa}\left(r_h - r_l\right)\left(1 - e^{-T\phi_1}\right) \\ & \quad \text{and } d_{trap}(0) = \frac{\phi_1}{\kappa}\left(r_h - r_l\right)\left(1 - e^{-T\phi_2}\right) \\ \left(\frac{d_{exit}\left(\overline{t_1}\right)}{d_{exit}(0) + \frac{\phi_2}{\kappa}\left(r_h - r_l\right)\left(e^{-T\phi_1} - 1\right)}\right)^{\phi_2} & , \text{ if } d_{exit}(0) \neq \frac{\phi_2}{\kappa}\left(r_h - r_l\right)\left(1 - e^{-T\phi_1}\right) \\ \quad - \left(\frac{d_{trap}\left(\overline{t_1}\right)}{d_{trap}(0) + \frac{\phi_1}{\kappa}\left(r_h - r_l\right)\left(e^{-T\phi_2} - 1\right)}\right)^{\phi_1} & \quad \text{and } d_{trap}(0) \neq \frac{\phi_1}{\kappa}\left(r_h - r_l\right)\left(1 - e^{-T\phi_2}\right) \end{cases}, \tag{B.9}$$

and

$$\mathcal{T}(\overline{R}) = \begin{cases} [T, \infty) & , \text{ if } d_{exit}(0) = \frac{\phi_2}{\kappa}\left(r_h - r_l\right)\left(1 - e^{-T\phi_1}\right) \\ & \quad \text{and } d_{trap}(0) = \frac{\phi_1}{\kappa}\left(r_h - r_l\right)\left(1 - e^{-T\phi_2}\right) \\ \frac{1}{\phi_2} \log \frac{d_{trap}\left(\overline{t_1}\right)}{d_{trap}(0) + \frac{\phi_1}{\kappa}\left(r_h - r_l\right)\left(e^{-T\phi_2} - 1\right)} & , \text{ if } d_{exit}(0) = \frac{\phi_2}{\kappa}\left(r_h - r_l\right)\left(1 - e^{-T\phi_1}\right) \\ & \quad \text{and } d_{trap}(0) \neq \frac{\phi_1}{\kappa}\left(r_h - r_l\right)\left(1 - e^{-T\phi_2}\right) \\ \frac{1}{\phi_1} \log \frac{d_{exit}\left(\overline{t_1}\right)}{d_{exit}(0) + \frac{\phi_2}{\kappa}\left(r_h - r_l\right)\left(e^{-T\phi_2} - 1\right)} & , \quad \text{otherwise} \end{cases}, \tag{B.10}$$

thereby proving Proposition ??. In the expression for $\mathcal{P}\left(\overline{R}\right)$, I have used the indicator function $1\{E\}$, which is equal to 1 if E is true and zero otherwise.

To derive equations (B.7) and (B.8), which correspond to equations (??) and (??) in the main body of the paper, and to find the explicit expressions for $\mathcal{P}$ and $\mathcal{T}$ shown in equations (B.9)-(B.10), I consider four cases separately.

The first case corresponds to the economy reaching (π_{zlb}, x_{zlb}) at $\overline{t_1}$; the second and third, to the economy reaching, respectively, Υ_{zlb} and Ψ_{zlb} at $\overline{t_1}$; the fourth case considers all remaining $\overline{R}$.

The first case, shown in green in Figure ??, is defined by (π_0, x_0) such that[35]

$$d_{exit}(\pi_0, x_0) = [x_0 - x_{zlb}] - \frac{\phi_1}{\kappa}[\pi_0 - \pi_{zlb}] = \frac{\phi_2}{\kappa}(r_h - r_l)\left(1 - e^{-T\phi_1}\right), \tag{B.11}$$

$$d_{trap}(\pi_0, x_0) = [x_0 - x_{zlb}] - \frac{\phi_2}{\kappa}[\pi_0 - \pi_{zlb}] = \frac{\phi_1}{\kappa}(r_h - r_l)\left(1 - e^{-T\phi_2}\right). \tag{B.12}$$

In the figure, the line defined by equation (B.11) is the black dashed line while the line defined by equation (B.12) is the dashed gray line. This first case corresponds to (π_0, x_0) at the intersection of these two lines. The economy reaches the *zlb* steady-state (π_{zlb}, x_{zlb}) at $t = T$. Since between T and $\overline{t_1}$ the point (π_{zlb}, x_{zlb}) is a steady-state, the economy just sits there for all $t \in [T, \overline{t_1})$. The continuous pasting conditions are

$$d_{exit}(\pi(t), x(t)) = d_{trap}(\pi(t), x(t)) = 0, \tag{B.13}$$

$$d_{exit}(\pi_1, x_1) = d_{trap}(\pi_1, x_1) = 0, \tag{B.14}$$

i.e., $(\pi(t), x(t)) = (\pi_1, x_1) = (\pi_{zlb}, x_{zlb})$. In Figure ??, the lines described in equations (B.13) and (B.14) are shown in the solid black and gray lines, and correspond to Υ_{zlb} and Ψ_{zlb}. Equations (B.13) and (B.14) define the function $\mathcal{P}$ for this case

$$\mathcal{P}(\overline{R}) = 1\left\{d_{exit}(\pi_1, x_1) \neq 0 \text{ or } d_{trap}(\pi_1, x_1) \neq 0\right\},$$

We then have $\mathcal{P}\left(\overline{R}\right) = 0$ if and only if $d_{trap}(\pi_1, x_1)$ and $d_{trap}(\pi_1, x_1)$ are both zero. Graphically, the set of $\overline{R}$ such that $\mathcal{P}(\overline{R}) = 0$ are the two points in Figure ?? where the lines intersect, that is, where the green path begins and ends. Once (B.13) and (B.14) hold, any $\overline{t_1} \geq T$ is consistent with continuous pasting and thus

$$\mathcal{T}(\overline{R}) = [T, \infty).$$

In the three remaining cases, $\mathcal{T}(\overline{R})$ is single-valued and depends on $\overline{R}$.

The second case is defined by the economy reaching some point in the *zlb* saddle path Υ_{zlb} at $\overline{t_1}$, except for (π_{zlb}, x_{zlb}), which was already analyzed. This case, shown in red in Figure ??, occurs when

$$d_{exit}(\pi_0, x_0) = [x_0 - x_{zlb}] - \frac{\phi_1}{\kappa}[\pi_0 - \pi_{zlb}] = \frac{\phi_2}{\kappa}(r_h - r_l)\left(1 - e^{-T\phi_1}\right), \tag{B.15}$$

$$d_{trap}(\pi_0, x_0) = [x_0 - x_{zlb}] - \frac{\phi_2}{\kappa}[\pi_0 - \pi_{zlb}] \neq \frac{\phi_1}{\kappa}(r_h - r_l)\left(1 - e^{-T\phi_2}\right). \tag{B.16}$$

[35] With slight abuse of notation, in this section I write $d_{exit}(\pi_0, x_0)$ instead of $d_{exit}(t)$ to emphasize that $d_{exit}(\pi_0, x_0)$ is not a function of time since (π_0, x_0) is a vector of two numbers, as opposed to $(\pi(0), x(0))$, which is a function of time evaluated at $t = 0$. The same notation applies to d_{trap} and to π_T, x_T, π_1, x_1.

Continuous pasting at T requires

$$d_{exit}\left(\pi(t),x(t)\right) = [x(t)-x_{zlb}] - \frac{\phi_1}{\kappa}\left[\pi(t)-\pi_{zlb}\right] = 0, \tag{B.17}$$

$$T = \frac{1}{\phi_2}\log\left(\frac{d_{trap}\left(\pi_T,x(t)\right)}{d_{trap}\left(\pi_0,x_0\right)+\frac{\phi_1}{\kappa}\left(r_h-r_l\right)\left(e^{-T\phi_2}-1\right)}\right). \tag{B.18}$$

Continuous pasting at $\overline{t_1}$ requires

$$d_{exit}\left(\pi_1,x_1\right) = [x_1-x_{zlb}] - \frac{\phi_1}{\kappa}\left[\pi_1-\pi_{zlb}\right] = 0, \tag{B.19}$$

$$\overline{t_1} = T + \frac{1}{\phi_2}\log\left(\frac{d_{trap}\left(\pi_1,x_1\right)}{d_{trap}\left(\pi(t),x(t)\right)}\right). \tag{B.20}$$

Equations (B.15), (B.16), (B.17) and (B.19) describe the continuous pasting constraints on (π_0, x_0) without any reference to $\overline{t_1}$. Combinations (π_0, x_0) that satisfy these equations can be part of a continuous path for some $\overline{t_1}$. Equations (B.18) and (B.20) then show which particular point is reachable with a specific $\overline{t_1}$. Any continuous path in this case must start in the black dashed line of Figure **??** and be on the solid black line at times T and $\overline{t_1}$. For a specific $\overline{t_1}$, or for a specific point in one of the two lines, only one path is continuous.

Combining equations (B.15)-(B.20) gives

$$\mathcal{P}(\overline{R}) = d_{exit}\left(\pi_1,x_1\right) + 1\left\{d_{trap}\left(\pi_1,x_1\right)=0\right\},$$

$$\mathcal{T}(\overline{R}) = \frac{1}{\phi_2}\log\left(\frac{d_{trap}\left(\pi_1,x_1\right)}{d_{trap}\left(\pi_0,x_0\right)+\frac{\phi_1}{\kappa}\left(r_h-r_l\right)\left(e^{-T\phi_2}-1\right)}\right).$$

The indicator $1\left\{d_{trap}\left(\pi_1,x_1\right)=0\right\}$ in the equation for $\mathcal{P}$ is there to guarantee that equation (B.15) holds. In Figure **??**, the points $\overline{R}$ such that $\mathcal{P}(\overline{R}) = 0$ are given by the dashed and solid black lines, with the exception of the points where the black lines intersect the gray lines.

The third case is similar to the second and is represented by the blue line in Figure **??**. Instead of reaching Υ_{zlb} at $\overline{t_1}$, the economy reaches the unstable *zlb* saddle path Ψ_{zlb} at $\overline{t_1}$, with the exception of (π_{zlb}, x_{zlb}), which was already studied. This case is defined by

$$d_{exit}\left(\pi_0,x_0\right) = [x_0-x_{zlb}] - \frac{\phi_1}{\kappa}\left[\pi_0-\pi_{zlb}\right] \neq \frac{\phi_2}{\kappa}\left(r_h-r_l\right)\left(1-e^{-T\phi_1}\right), \tag{B.21}$$

$$d_{trap}\left(\pi_0,x_0\right) = [x_0-x_{zlb}] - \frac{\phi_2}{\kappa}\left[\pi_0-\pi_{zlb}\right] = \frac{\phi_1}{\kappa}\left(r_h-r_l\right)\left(1-e^{-T\phi_2}\right). \tag{B.22}$$

Continuous pasting at T occurs if and only if

$$\begin{aligned} d_{trap}\left(\pi(t), x(t)\right) &= \left[x(t) - x_{zlb}\right] - \frac{\phi_2}{\kappa}\left[\pi(t) - \pi_{zlb}\right] = 0, && \text{(B.23)} \\ T &= \frac{1}{\phi_1}\log\left(\frac{d_{exit}\left(\pi_T, x(t)\right)}{d_{exit}\left(\pi_0, x_0\right) + \frac{\phi_2}{\kappa}\left(r_h - r_l\right)\left(e^{-T\phi_2} - 1\right)}\right), && \text{(B.24)} \end{aligned}$$

while continuous pasting at t_1 occurs if and only if

$$\begin{aligned} d_{trap}\left(\pi_1, x_1\right) &= \left[x_1 - x_{zlb}\right] - \frac{\phi_2}{\kappa}\left[\pi_1 - \pi_{zlb}\right] = 0, && \text{(B.25)} \\ \overline{t_1} &= T + \frac{1}{\phi_1}\log\left(\frac{d_{exit}\left(\pi_1, x_1\right)}{d_{exit}\left(\pi(t), x(t)\right)}\right). && \text{(B.26)} \end{aligned}$$

It follows that

$$\begin{aligned} \mathcal{P}(\overline{R}) &= d_{trap}\left(\pi_1, x_1\right) + 1\left\{d_{exit}\left(\pi_1, x_1\right) = 0\right\}, \\ \mathcal{T}(\overline{R}) &= \frac{1}{\phi_1}\log\left(\frac{d_{exit}\left(\pi_1, x_1\right)}{d_{exit}\left(\pi_0, x_0\right) + \frac{\phi_2}{\kappa}\left(r_h - r_l\right)\left(e^{-T\phi_2} - 1\right)}\right). \end{aligned}$$

The fourth and last case corresponds to all remaining choices for $\overline{R}$ that can be part of a continuous path. The continuous pasting conditions are

$$\left(\frac{d_{trap}\left(\pi_0, x_0\right) + \frac{\phi_1}{\kappa}\left(r_h - r_l\right)\left(e^{-T\phi_2} - 1\right)}{d_{trap}\left(\pi(t), x(t)\right)}\right)^{\phi_1} = \left(\frac{d_{exit}\left(\pi_0, x_0\right) + \frac{\phi_2}{\kappa}\left(r_h - r_l\right)\left(e^{-T\phi_1} - 1\right)}{d_{exit}\left(\pi(t), x(t)\right)}\right)^{\phi_2}, \tag{B.27}$$

$$T = \frac{1}{\phi_2}\log\frac{d_{trap}\left(\pi(t), x(t)\right)}{d_{trap}\left(\pi_0, x_0\right) + \frac{\phi_1}{\kappa}\left(r_h - r_l\right)\left(e^{-T\phi_2} - 1\right)}, \tag{B.28}$$

for T and

$$\begin{aligned} \left(\frac{d_{trap}\left(\pi_1, x_1\right)}{d_{trap}\left(\pi_T, x(t)\right)}\right)^{\phi_1} &= \left(\frac{d_{exit}\left(\pi_1, x_1\right)}{d_{exit}\left(\pi(t), x(t)\right)}\right)^{\phi_2}, && \text{(B.29)} \\ t_1 &= T + \frac{1}{\phi_2}\log\left(\frac{d_{trap}\left(\pi_1, x_1\right)}{d_{trap}\left(\pi(t), x(t)\right)}\right), && \text{(B.30)} \end{aligned}$$

for $\overline{t_1}$.

Assuming continuous pasting at T, equations (B.27)-(B.29) reveal the set of points

(π_0, x_0, π_1, x_1) that can be reached through continuous paths for some $\overline{t_1}$, which give

$$\mathcal{P}(\overline{R}) = \left(\frac{d_{exit}(\pi_1, x_1)}{d_{exit}(\pi_0, x_0) + \frac{\phi_2}{\kappa}(r_h - r_l)(e^{-T\phi_1} - 1)}\right)^{\phi_2} - \left(\frac{d_{trap}(\pi_1, x_1)}{d_{trap}(\pi_0, x_0) + \frac{\phi_1}{\kappa}(r_h - r_l)(e^{-T\phi_2} - 1)}\right)^{\phi_1} + 1\left\{d_{exit}(\pi_1, x_1) = 0 \text{ or } d_{trap}(\pi_1, x_1) = 0\right\}.$$

Finally, equations (B.28) and (B.30) give

$$\mathcal{T}\left(\overline{R}\right) = \frac{1}{\phi_1} \log \frac{d_{exit}\left(\overline{t_1}\right)}{d_{exit}(0) + \frac{\phi_2}{\kappa}(r_h - r_l)(e^{-T\phi_2} - 1)}.$$

B.5 Proof of Proposition 2

I first prove a lemma and then proceed to the proof of Proposition 2.

Lemma 1. *When the Taylor principle does not hold, the following are true:*

(a) $(\pi_{zlb}, x_{zlb}) \notin \Omega_{zlb}$.

(b) If $(\pi(q), x(q)) \in \Omega_{zlb}$ *for some* $q \geq t_1$, $(\pi(t), x(t))$ *either explodes as* $t \to \infty$ *or exits* Ω_{zlb} *in finite time.*

(c) If $(\pi(s), x(s)) \in \Omega_{ss}$ *for some* $s \geq t_1$ *and* $(\pi(q), x(q)) \in \Omega_{zlb}$ *for some* $q > s$, *then* $(\pi(t), x(t))$ *explodes as* $t \to \infty$.

(d) If $(\pi(q), x(q)) \in \Omega_{zlb}$ *for some* $q \geq t_1$ *and there exist no* $r \geq q$ *such that* $(\pi(r), x(r)) \in \partial\Omega \cap \Upsilon_{ss}$, *then* $(\pi(t), x(t))$ *explodes as* $t \to \infty$.

(e) If $(\pi(q), x(q)) \in \Omega_{zlb}$ *for some* $q \geq t_1$ *and there exist* $r > q$ *such that* $(\pi(r), x(r)) \in \partial\Omega \cap \Upsilon_{ss}$, *then there exist no* $t \in [q, r)$ *such that* $(\pi(t), x(t)) \in \Upsilon_{ss}$.

Proof of Lemma 1. (a) Plugging the steady-state from equation (A.20) into the Taylor rule gives

$$\begin{aligned}
\xi_\pi \pi_{zlb} + \xi_x x_{zlb} + r_h &= \xi_\pi(-r_h) + \xi_x\left(-\frac{\rho}{\kappa} r_h\right) + r_h, \\
&= -\frac{r_h}{\kappa}\left(\kappa(\xi_\pi - 1) + \rho\xi_x\right), \\
&= -\frac{r_h \sigma}{\kappa} \det A_{ss}, \\
&> 0.
\end{aligned}$$

where $\det A_{ss} < 0$ because the Taylor principle does not hold.

(b) The saddle path stable dynamics in Ω_{zlb} and $(\pi_{zlb}, x_{zlb}) \notin \Omega_{zlb}$ immediately imply that paths starting in Ω_{zlb} either explode or exit Ω_{zlb} in finite time.

(c) Let $\hat{n}$ be a unit vector normal to $\partial\Omega$ pointing towards Ω_{ss}. Because $(\pi(t), x(t))$ transitions from Ω_{ss} to Ω_{zlb} and its path is continuous, there exist $\omega \in (s, q]$ such that

$$\begin{aligned} \xi_\pi \pi(\omega) + \xi_x x(\omega) + r_h &= 0, && \text{(B.31)} \\ \hat{n} \cdot (\dot{\pi}(\omega), \dot{x}(\omega)) &\leq 0. && \text{(B.32)} \end{aligned}$$

Equation (B.31) says that $(\pi(\omega), x(\omega)) \in \partial\Omega$. The non-positive dot product in equation (B.32) says that $(\pi(\omega), x(\omega))$ is not moving towards Ω_{ss} (and moving towards Ω_{zlb} when the dot product is negative). Writing out the dot product gives

$$\begin{bmatrix} \frac{\xi_x}{\sqrt{\xi_\pi^2+\xi_x^2}} \\ \frac{\xi_\pi}{\sqrt{\xi_\pi^2+\xi_x^2}} \end{bmatrix}^T \begin{bmatrix} -\frac{1}{\sigma}\left(\pi(\omega) + r_h\right) \\ \rho\pi(\omega) - \kappa x(\omega) \end{bmatrix} = -\frac{\pi(\omega)\left(\xi_x - \sigma\rho\xi_\pi\right) + \xi_x r_h + \kappa\sigma\xi_\pi x(\omega)}{\sigma\sqrt{\xi_\pi^2 + \xi_x^2}} \leq 0,$$

or, simplifying,

$$\pi(\omega)\left(\xi_x - \sigma\rho\xi_\pi\right) + \xi_x r_h + \kappa\sigma\xi_\pi x(\omega) \geq 0. \tag{B.33}$$

The Taylor principle (TP) not holding, equation (B.31), equation (B.33) and $\phi_1, r_h > 0$ imply that

$$\begin{aligned} -r_h \underbrace{\left(\kappa\left(\xi_\pi - 1\right) + \rho\xi_x\right)}_{<0 \text{ as TP does not hold}} - \kappa \underbrace{\left(r_h + \xi_\pi\pi(\omega) + \xi_x x(\omega)\right)}_{=0 \text{ by eq. (B.31)}} \\ + \phi_1 \underbrace{\left(\pi(\omega)\left(\xi_x - \sigma\rho\xi_\pi\right) + \xi_x r_h + \kappa\sigma\xi_\pi x(\omega)\right)}_{\geq 0 \text{ by eq. (B.33)}} > 0. \end{aligned} \tag{B.34}$$

Equation (B.34) is a sufficient condition for $\big(\pi_(t), x_(t)\big)$ to be in Ω_{zlb} for all $t \geq \omega$. To see this, use the dynamics in equations (A.7) and (A.8) to write

$$\xi_\pi\pi(t) + \xi_x x(t) + r_h = W\left(t - \omega\right),$$

where

$$\begin{aligned}
W(t) &= Ae^{\phi_1(t-\omega)} + Be^{\phi_2(t-\omega)} + C, \\
A &= -\left(\phi_1\pi(\omega) - \kappa x(\omega) - \phi_2 r_h\right)\frac{\xi_x - \sigma\phi_1\xi_\pi}{\sigma\phi_1\left(\phi_1 - \phi_2\right)}, \\
B &= \left(\phi_2\pi(\omega) - \kappa x(\omega) - \phi_1 r_h\right)\frac{\xi_x - \sigma\phi_2\xi_\pi}{\sigma\phi_2\left(\phi_1 - \phi_2\right)}, \\
C &= \frac{r_h}{\kappa}\left(\kappa\left(1-\xi_\pi\right) - \rho\xi_x\right).
\end{aligned}$$

By definition, $(\pi(t), x(t)) \in \Omega_{zlb}$ iff $W(t) \leq 0$. Therefore, if $W(t)$ has no zeros for $t > \omega$, $(\pi(t), x(t))$ remains in Ω_{zlb} forever. Since $\phi_2 < 0 < \phi_1$ and $W(\omega) = 0$ by (B.31) and $W'(\omega) \leq 0$ by (B.32), a sufficient condition for $W(u)$ to have no zeros for $u > \omega$ is that $A < 0$. After some manipulations, it can be seen that (B.34) is equivalent to $A < 0$.

By (b), since $(\pi(t), x(t))$ never transitions to Ω_{ss} after ω, it follows that $(\pi(t), x(t))$ explodes as $t \to \infty$.

(d) By (b), if $(\pi(t), x(t))$ does not exit Ω_{zlb}, it explodes. If $(\pi(t), x(t))$ exits Ω_{zlb} at some time η and $(\pi(\eta), x(\eta))$ is not on the ss saddle path, due to the saddle path dynamics inside Ω_{ss}, $(\pi(t), x(t))$ either explodes or returns to Ω_{zlb} in finite time. If $(\pi(t), x(t))$ returns to Ω_{zlb}, by item (c), it explodes.

(e) By equation (A.15), Υ_{ss} is a line through the origin, which can be written as $A\pi - x = 0$ with $A \neq 0$. If $\xi_\pi \neq 1$ then $A = \left(1-\xi_\pi\right)/\left(\xi_x - \sigma\alpha_2\right)$ and if $\xi_\pi = 1$ then $A = \left(\sigma\rho - \xi_x\right)/\kappa\sigma$. Let

$$F(t) = A\pi(t) - x(t). \tag{B.35}$$

The path of $(\pi(t), x(t))$ intersects Υ_{ss} at some time $\bar{t}$ iff $F(\bar{t}) = 0$.

Plugging $\xi_\pi\pi(t) + \xi_x x(t) + r_h = 0$ in the IS and NKPC after t_1, it can be seen that $(\dot{\pi}(t), \dot{x}(t))$ is continuous on $\partial\Omega$ for all $t \geq t_1$. It follows that the right and left derivatives of $F(t)$ are equal at $t = r$. By equations (A.9)-(A.14) and (A.15), $(\pi(t), x(t))$ remains on Υ_{ss} after intersecting it at $t = r$. Hence, the right derivative of $F(t)$ at $t = r$ is zero.

Now I find the left derivative. If $(\pi(t), x(t))$ exits Ω_{zlb} without intersecting $\partial\Omega \cap \Upsilon_{ss}$, by uniqueness inside Ω_{ss}, it won't intersect $\Upsilon_{ss} \cap \Omega_{ss}$. If $(\pi(t), x(t))$ re-enters Ω_{zlb} after being in $\Omega_{ss} \setminus \Upsilon_{ss}$, by item (c), it explodes. Hence, between times q and r, $(\pi(t), x(t)) \in \Omega_{zlb}$ and its dynamics are given by equations (A.7) and (A.8).

Using these dynamics in equation (B.35) gives

$$F(t) = Pe^{\phi_1(t-\tau(t))} + Qe^{\phi_2(t-\tau(t))} + R, \tag{B.36}$$

where

$$\begin{aligned}
P &= \frac{\phi_2 - A\kappa}{\kappa\left(\phi_1 - \phi_2\right)}\left(\phi_2 r_h - \phi_1 \pi(\tau(t)) + \kappa x(\tau(t))\right), \\
Q &= -\frac{\phi_1 - A\kappa}{\kappa\left(\phi_1 - \phi_2\right)}\left(\phi_1 r_h - \phi_2 \pi(\tau(t)) + \kappa x(\tau(t))\right), \\
R &= \frac{r_h\left(\rho - A\kappa\right)}{\kappa}.
\end{aligned}$$

Note that P and Q cannot both be zero. Indeed, $P = Q = 0$ implies $R = 0$, since $F(r) = 0$. But $R = 0$ implies $A = \rho/\kappa$, which in turn implies $(x(\tau(t)), \pi(\tau(t))) = (\pi_{zlb}, x_{zlb}) = (x(q), \pi(q)) \in \Omega_{zlb}$, contradicting item (a). Thus, P and Q cannot both be zero.

Using equation (B.36) to compute the left derivative of $F(t)$ at $t = r$ and setting it equal to the value of the right derivative, which is zero as shown above, gives

$$F'(r) = 0 = \phi_1 Pe^{\phi_1(r-\tau(r))} + \phi_2 Qe^{\phi_2(r-\tau(r))}. \tag{B.37}$$

In other words, the path for $(\pi(t), x(t))$ must be tangent to the ss saddle path Υ_{ss} at $t = r$. Since $\phi_2 < 0 < \phi_1$, equation (B.37) implies that P and Q have the same sign (and the sign is not zero since P and Q cannot both be zero). In turn, P and Q having the same (non-zero) sign implies that

$$F''(t) = \phi_1^2 Pe^{\phi_1(t-\tau(t))} + \phi_2^2 Qe^{\phi_2(t-\tau(t))} \tag{B.38}$$

has the same (non-zero) sign for all $t \in [q, r]$, so $F'(t)$ is strictly monotonic. A continuous and strictly monotonic $F'(t)$ in $t \in [q, r]$, together with $F(r) = F'(r) = 0$, imply that the only solution to $F(t) = 0$ for $t \in [q, r]$ is r. □

Proof of Proposition 2. Consider the following condition:

$$\begin{gathered}
(\pi(t_1), x(t_1)) \in \Omega_{ss} \cap \Upsilon_{ss}, \\
\text{or} \\
(\pi(t_1), x(t_1)) \in \Omega_{zlb} \text{ and } (\pi(r), x(r)) \in \partial\Omega \cap \Upsilon_{ss} \text{ for some } r \in [t_1, \infty).
\end{gathered} \tag{B.39}$$

I first show that condition (B.39) implies paths are not explosive. If $(\pi(t_1), x(t_1)) \in \Omega_{ss} \cap$

Υ_{ss}, then $(\pi(t), x(t)) \in \Upsilon_{ss}$ for all $t \geq t_1$. If $(x(t_1), \pi(t_1)) \in \Omega_{zlb}$ and $(x(r), \pi(r)) \in \partial\Omega \cap \Upsilon_{ss}$ for some $r \in [t_1, \infty)$, $(\pi(t), x(t)) \in \Upsilon_{ss}$ for all $t \geq r$. In either case, the path converges to $(0, 0)$ and therefore does not explode.

To prove the converse, I prove the contrapositive. There are two cases to consider.

Case 1: If $(\pi(t_1), x(t_1)) \notin \Omega_{ss}$ and there exist no $r \in [t_1, \infty)$ such that $(\pi(r), x(r)) \in \partial\Omega \cap \Upsilon_{ss}$, then $(\pi(t), x(t))$ explodes by item (e) of Lemma 1.

Case 2: If $(\pi(t_1), x(t_1)) \notin \Upsilon_{ss}$ and $(\pi(t_1), x(t_1)) \notin \Omega_{zlb}$, $(\pi(t), x(t))$ either explodes or enters Ω_{zlb}. If it enters Ω_{zlb}, it explodes by item (c) of Lemma 1.

Note that cases 1 and 2 above also cover the case in which $(\pi(t_1), x(t_1)) \notin \Upsilon_{ss}$ and there exists no $r \in [t_1, \infty)$ such that $(x_r, \pi_r) \in \partial\Omega \cap \Upsilon_{ss}$. Indeed, if $(\pi(t_1), x(t_1)) \notin \Omega_{ss}$, case 1 applies. And if $(\pi(t_1), x(t_1)) \notin \Omega_{zlb}$, case 2 applies. □

B.6 Proof of Proposition 3

I first prove a lemma and then proceed to the proof of Proposition 3.

Lemma 2. *When the Taylor principle holds, the following are true:*

(a) $(\pi_{zlb}, x_{zlb}) \in \Omega_{zlb}$.

(b) If $(\pi(m), x(m)) \in \Omega_{zlb} \cap \Upsilon_{zlb}$ *with* $m \geq T$, *then* $(\pi(t), x(t)) \in \Omega_{zlb} \cap \Upsilon_{zlb}$ *for all* $t \geq m$.

(c) There exist $(\pi(0), x(0))$ *such that* $(\pi(T), x(T)) \in \Omega_{zlb} \cap \Upsilon_{zlb}$.

(d) If $(\pi(s), x(s)) \in \partial\Omega$ *for some* $s \geq t_1$, *there is no* $p > 0$ *such that* $(\pi(t), x(t)) \in \Omega_{ss}$ *for* $t \in (s, s+p)$ *and* $(\pi(s+p), x(s+p)) \in \Upsilon_{zlb} \cap \partial\Omega$.

(e) If $(\pi(q), x(q)) \in \Omega_{ss}$ *for* $q \geq t_1$ *with* $(\pi(q), x(q)) \neq (\pi_{ss}, x_{ss})$ *and there is no* $p > 0$ *such that* $[(\pi(t), x(t)) \in \Omega_{ss}$ *for* $t \in (q, q+p)$ *and* $(\pi(q+p), x(q+p)) \in \Upsilon_{zlb} \cap \partial\Omega]$, *then* $(\pi(t), x(t))$ *explodes as* $t \to \infty$.

(f) There is no chaos (in the sense of R. Devaney[36]).

Proof of Lemma 2. (a) Plugging the steady-state (A.20) into the Taylor rule gives

$$\begin{aligned}
\xi_\pi \pi_{zlb} + \xi_x x_{zlb} + r_h &= \xi_\pi(-r_h) + \xi_x\left(-\frac{\rho}{\kappa} r_h\right) + r_h, \\
&= -\frac{1}{\kappa} r_h\left(\kappa(\xi_\pi - 1) + \rho\xi_x\right), \\
&= -\frac{r_h \sigma}{\kappa} \det A_{ss}, \\
&< 0.
\end{aligned}$$

[36] See Banks, Brooks, Cairns, Davis, and Stacey (1992) for a definition.

where $\det A_{ss} > 0$ because the Taylor principle holds.

(b) Let (x, π) be a point in the line segment with endpoints $(\pi(m), x(m))$ and (π_{zlb}, x_{zlb}), i.e. (x, π) is in the portion of the *zlb* saddle path between $(\pi(m), x(m))$ and (π_{zlb}, x_{zlb}). Then

$$(x, \pi) = a(\pi(m), x(m)) + (1-a)(\pi_{zlb}, x_{zlb}),$$

for some $a \in [0, 1]$. It follows that

$$\begin{aligned} \xi_\pi \pi + \xi_x x + r_h &= \xi_\pi (a\pi(m) + (1-a)\pi_{zlb}) + \xi_x (ax(m) + (1-a)x_{zlb}) + r_h, \\ &= a(\xi_\pi \pi(m) + \xi_x x(m) + r_h) + (1-a)(\xi_\pi \pi_{zlb} + \xi_x x_{zlb} + r_h), \\ &< 0, \end{aligned} \tag{B.40}$$

where the last line uses that $(\pi(m), x(m))$ and (π_{zlb}, x_{zlb}) are both in Ω_{zlb}. The line segment with endpoints $(\pi(m), x(m))$ and (π_{zlb}, x_{zlb}) is thus entirely in Ω_{zlb}. For $t \in [T, t_1)$, the dynamics of $(\pi(t), x(t))$ are given by (A.5)-(A.6) and thus $(\pi(t), x(t))$ travels along the *zlb* saddle path. For $t \geq t_1$, equation (B.40) implies that $\max\{0, \xi_\pi \pi(t) + \xi_x x(t) + r_h\} = 0$ so that $(\pi(t), x(t))$ follows the same dynamics given by (A.7)-(A.8), which means $(\pi(t), x(t))$ stays on the *zlb* saddle path and travels on it towards (π_{zlb}, x_{zlb}).

(c) Because $(\pi(T), x(T))$ is in Ω_{zlb} and in Υ_{zlb}, it satisfies

$$\begin{aligned} 0 &\geq \xi_\pi \pi(t) + \xi_x x(t) + r_h, \\ x(t) &= \frac{\phi_1}{\kappa}\pi(t) - \frac{\phi_2}{\kappa} r_h, \end{aligned}$$

which is equivalent to

$$\begin{aligned} x(t) &= \frac{\phi_1}{\kappa}\pi(t) - \frac{\phi_2}{\kappa} r_h, && \text{(B.41)} \\ (\kappa \xi_\pi + \xi_x \phi_1)\pi(t) &\leq r_h(\phi_2 \xi_x - \kappa). && \text{(B.42)} \end{aligned}$$

If $\kappa\xi_\pi + \xi_x\phi_1 \neq 0$, it is easy to find $(\pi(t), x(t))$ that satisfies (B.41) and (B.42). If $\kappa\xi_\pi + \xi_x\phi_1 = 0$, equation (B.42) holds because the Taylor principle holds. Any pair $(\pi(t), x(t))$ that satisfies equation (B.41) will be in Ω_{zlb} and in Υ_{zlb}. To find the corresponding $(\pi(0), x(0))$, use the dynamics of $(\pi(t), x(t))$ for $t \in [0, T)$ given by (A.3)-(A.4).

(d) By direct computation, the set of points $(x,\pi) \in \Upsilon_{zlb} \cap \partial\Omega$ are

$$(x,\pi) = \begin{cases} \left(-r_h \frac{\phi_1+\phi_2\xi_\pi}{\kappa\xi_\pi+\phi_1\xi_x}, -r_h \frac{\kappa-\phi_2\xi_x}{\kappa\xi_\pi+\phi_1\xi_x}\right) & \text{if} \quad \kappa\xi_\pi + \phi_1\xi_x \neq 0 \\ \emptyset & \text{if} \quad \kappa\xi_\pi + \phi_1\xi_x = 0 \end{cases}. \tag{B.43}$$

If $\kappa\xi_\pi + \phi_1\xi_x = 0$, there is clearly no $p > 0$ such that $(\pi(s+p), x(s+p)) \in \Upsilon_{zlb} \cap \partial\Omega$. If $\kappa\xi_\pi + \phi_1\xi_x \neq 0$, I analyze three cases according to the three different dynamics that $(\pi(t), x(t))$ can follow in Ω_{ss} given in Section A.1.

Case I. Let $t = s + p$. Then $\eta(t) = s$ and $(\pi(s+p), x(s+p)) \in \Upsilon_{zlb} \cap \partial\Omega$ gives

$$\begin{aligned} x(s+p) = -r_h \frac{\phi_1 + \phi_2\xi_\pi}{\kappa\xi_\pi + \phi_1\xi_x} = &-\frac{(1-\xi_\pi)\,\pi(s) + (\sigma\alpha_2 - \xi_x)\,x(s)}{\sigma\,(\alpha_1 - \alpha_2)} e^{\alpha_1 p} \\ &+ \frac{(1-\xi_\pi)\,\pi(s) + (\sigma\alpha_1 - \xi_x)\,x(s)}{\sigma\,(\alpha_1 - \alpha_2)} e^{\alpha_2 p}, \end{aligned}$$

$$\begin{aligned} \pi(s+p) = -r_h \frac{\kappa - \phi_2\xi_x}{\kappa\xi_\pi + \phi_1\xi_x} = &\frac{(1-\xi_\pi)\,\pi(s) + (\sigma\alpha_2 - \xi_x)\,x(s)}{\sigma\,(\xi_\pi - 1)\,(\alpha_1 - \alpha_2)} (\xi_x - \sigma\alpha_1)\, e^{\alpha_1 p} \\ &- \frac{(1-\xi_\pi)\,\pi(s) + (\sigma\alpha_1 - \xi_x)\,x(s)}{\sigma\,(\xi_\pi - 1)\,(\alpha_1 - \alpha_2)} (\xi_x - \sigma\alpha_2)\, e^{\alpha_2 p}. \end{aligned}$$

Solving for $(\pi(s), x(s))$ as a function of p gives

$$\begin{aligned} x(s)\,(p) = &-\frac{r_h}{\sigma} \frac{(-\kappa + \kappa\xi_\pi + \phi_1\xi_x + \phi_2\xi_x - \sigma\alpha_2\phi_1 - \sigma\alpha_2\phi_2\xi_\pi)}{(\kappa\xi_\pi + \phi_1\xi_x)\,(\alpha_1 - \alpha_2)} e^{-p\alpha_1} \\ &- \frac{r_h}{\sigma} \frac{(\kappa - \kappa\xi_\pi - \phi_1\xi_x - \phi_2\xi_x + \sigma\alpha_1\phi_1 + \sigma\alpha_1\phi_2\xi_\pi)}{(\kappa\xi_\pi + \phi_1\xi_x)\,(\alpha_1 - \alpha_2)} e^{-p\alpha_2}, \end{aligned}$$

$$\begin{aligned} \pi(s)\,(p) = &\frac{r_h}{\sigma} \frac{(\xi_x - \sigma\alpha_1)\,(-\kappa + \kappa\xi_\pi + \phi_1\xi_x + \phi_2\xi_x - \sigma\alpha_2\phi_1 - \sigma\alpha_2\phi_2\xi_\pi)}{(\xi_\pi - 1)\,(\kappa\xi_\pi + \phi_1\xi_x)\,(\alpha_1 - \alpha_2)} e^{-p\alpha_1} \\ &+ \frac{r_h}{\sigma} \frac{(-\xi_x + \sigma\alpha_2)\,(-\kappa + \kappa\xi_\pi + \phi_1\xi_x + \phi_2\xi_x - \sigma\alpha_1\phi_1 - \sigma\alpha_1\phi_2\xi_\pi)}{(\xi_\pi - 1)\,(\kappa\xi_\pi + \phi_1\xi_x)\,(\alpha_1 - \alpha_2)} e^{-p\alpha_2}. \end{aligned}$$

Let

$$F(p) = -\frac{r_h\left(-\xi_x+\sigma\alpha_1\xi_\pi\right)\left(-\kappa+\kappa\xi_\pi+\phi_1\xi_x+\phi_2\xi_x-\sigma\alpha_2\phi_1-\sigma\alpha_2\phi_2\xi_\pi\right)}{\sigma\left(\xi_\pi-1\right)\left(\kappa\xi_\pi+\phi_1\xi_x\right)\left(\alpha_1-\alpha_2\right)}e^{-p\alpha_1}$$
$$-\frac{r_h\left(\xi_x-\sigma\alpha_2\xi_\pi\right)\left(-\kappa+\kappa\xi_\pi+\phi_1\xi_x+\phi_2\xi_x-\sigma\alpha_1\phi_1-\sigma\alpha_1\phi_2\xi_\pi\right)}{\sigma\left(\xi_\pi-1\right)\left(\kappa\xi_\pi+\phi_1\xi_x\right)\left(\alpha_1-\alpha_2\right)}e^{-p\alpha_2}$$
$$+\frac{r_h\sigma\left(\xi_\pi-1\right)\left(\alpha_1-\alpha_2\right)\left(\kappa\xi_\pi+\phi_1\xi_x\right)}{\sigma\left(\xi_\pi-1\right)\left(\kappa\xi_\pi+\phi_1\xi_x\right)\left(\alpha_1-\alpha_2\right)}.$$

Then,

$$F(p) = \xi_\pi\pi(s)\,(p)+\xi_x x(s)\,(p)+r_h,$$

and since $(\pi(s), x(s)) \in \partial\Omega$, it follows that $\xi_\pi\pi(s)(0)+\xi_x x(s)(0)+r_h = 0 = F(0)$. I show there is no $p>0$ that satisfies $F(p)=0$.

First, note that $\pi(s)$ and $x(s)$ are always real, even when α_1 and α_2 are complex. By direct computation, I find that

$$\begin{aligned}
F(0) &= 0. && \text{(B.44)}\\
F'(p) &= 0 \text{ has at most one solution for } p>0, && \text{(B.45)}\\
F'(0) &= \frac{r_h}{\sigma\phi_1}\left(\kappa\left(\xi_\pi-1\right)+\rho\xi_x\right)>0, && \text{(B.46)}\\
\lim_{p\to\infty}F(p) &= r_h>0. && \text{(B.47)}
\end{aligned}$$

$F'(0)>0$ because the Taylor principle holds. Together, equations (B.44)-(B.47) and continuity of $F\left(p\right)$ show that there is no solution to $F(p)=0$ for $p>0$.

Case II. Let $t=s+p$. Then $\eta(t)=s$ and

$$\begin{aligned}
-r_h\frac{\rho}{\kappa+\phi_1\xi_x} &= x(s)e^{\frac{1}{\sigma}\xi_x p}, && \text{(B.48)}\\
-r_h\frac{\kappa-\phi_2\xi_x}{\kappa+\phi_1\xi_x} &= \frac{\pi(s)\left(\xi_x-\sigma\rho\right)+\kappa\sigma x(s)}{\xi_x-\sigma\rho}e^{\rho p}-\frac{\kappa\sigma x(s)}{\xi_x-\sigma\rho}e^{\frac{1}{\sigma}\xi_x p}. && \text{(B.49)}
\end{aligned}$$

Using equation (B.43) and that $\xi_\pi\pi(s)+\xi_x x(s)+r_h=0$, equation (B.49) becomes

$$\begin{aligned}
-r_h\frac{\kappa-\phi_2\xi_x}{\kappa+\phi_1\xi_x} &= \left(\frac{\left(\xi_x-\sigma\rho\right)\xi_x-\kappa\sigma}{\left(\xi_x-\sigma\rho\right)\left(\kappa+\phi_1\xi_x\right)}r_h\rho e^{-\frac{1}{\sigma}\xi_x p}-r_h\right)e^{\rho p}\\
&\quad+\frac{r_h\kappa\sigma\rho}{\left(\xi_x-\sigma\rho\right)\left(\kappa+\phi_1\xi_x\right)}. && \text{(B.50)}
\end{aligned}$$

Solving for $x(s)$ in equation (B.48) and plugging it into equation (B.50) gives

$$x(s) = \xi_x\left(\kappa-\phi_2\xi_x+\sigma\rho\phi_2\right), \qquad \text{(B.51)}$$

and

$$\begin{aligned}\xi_x\left(\kappa-\phi_2\xi_x+\sigma\rho\phi_2\right) &= e^{p\rho}\left(\xi_x-\sigma\rho\right)\left(\kappa+\phi_1\xi_x\right)\\ &\quad +e^{\frac{p}{\sigma}\left(\sigma\rho-\xi_x\right)}\rho\left(-\xi_x^2+\kappa\sigma+\sigma\rho\xi_x\right). \end{aligned}\tag{B.52}$$

I now show that there is no $p>0$ such that equations (B.51)-(B.52) hold. If $\xi_x=\sigma\phi_1$, then $-\xi_x^2+\kappa\sigma+\sigma\rho\xi_x=0$ and equations (B.51)-(B.52) become

$$\begin{aligned} x(s) &= -\rho r_h\frac{e^{-p\phi_1}}{\sigma\phi_1^2+\kappa},\\ 1 &= e^{p\rho}. \end{aligned}$$

The last equation has no solution for $p>0$. If $\xi_x\neq\sigma\phi_1$, and recalling that $\xi_x\neq\sigma\phi_2$ so that $\Upsilon_{zlb}\cap\partial\Omega$ is non-empty, then $-\xi_x^2+\kappa\sigma+\sigma\rho\xi_x\neq 0$, and equations (B.51)-(B.52) become

$$\begin{aligned} x(s) &= -\frac{r_h\rho}{\kappa+\phi_1\xi_x}e^{-\frac{1}{\sigma}\xi_x p},\\ 0 &= \frac{\phi_2\xi_x}{\xi_x-\sigma\phi_1}\left(e^{p\rho}-1\right)+\rho e^{p\rho}\left(e^{-\frac{\xi_x}{\sigma}p}-1\right). \end{aligned}$$

Let

$$F(p)=\frac{\phi_2\xi_x}{\xi_x-\sigma\phi_1}\left(e^{p\rho}-1\right)+\rho e^{p\rho}\left(e^{-\frac{\xi_x}{\sigma}p}-1\right).$$

Compute

$$\begin{aligned} F'(p) &= \frac{\rho\xi_x}{\sigma}\left(e^{-\frac{p}{\sigma}\xi_x}-\frac{\sigma\phi_2}{\left(\xi_x-\sigma\phi_1\right)}e^{-p\rho}\right),\\ F''(p) &= -\frac{\rho\xi_x}{\sigma^2}\left(\xi_x e^{-\frac{p}{\sigma}\xi_x}-\frac{\sigma^2\rho\phi_2}{\left(\xi_x-\sigma\phi_1\right)}e^{-p\rho}\right), \end{aligned}$$

and

$$F(0) = 0, \tag{B.53}$$

$$F'(0) = \frac{\rho\xi_x\left(\xi_x-\sigma\rho\right)}{\sigma\left(\xi_x-\sigma\phi_1\right)}, \tag{B.54}$$

$$F'(p) = 0\Rightarrow e^{\left(\rho-\frac{\xi_x}{\sigma}\right)p}=\frac{\sigma\phi_2}{\left(\xi_x-\sigma\phi_1\right)}, \tag{B.55}$$

$$\lim_{p\to\infty}F(p) = \phi_1\frac{\xi_x-\sigma\rho}{\xi_x-\sigma\phi_1}, \tag{B.56}$$

$$\lim_{p\to\infty}F'(p) = 0. \tag{B.57}$$

If $\xi_x - \sigma\phi_1 > 0$, $F(p)$ is monotonic, which combined with $F(0) = 0$ gives no solutions to $F(p) = 0$ for $p > 0$. If $\xi_x - \sigma\phi_1 < 0$ and $\xi_x - \sigma\rho < 0$, then the unique local maximum occurs for some $p > 0$ and F is positive at that maximum. Using (B.56) and (B.57) then shows that there is no solution to $F(p) = 0$ for $p > 0$. If $\xi_x - \sigma\phi_1 < 0$ and $\xi_x - \sigma\rho > 0$, an analogous argument applies but instead of a unique maximum, there is a unique minimum.

Case III. Let $t = s + p$. Then $\eta(t) = s$ and

$$x(p) = \left(\left(1 + \frac{1}{2\sigma}(\xi_x - \sigma\rho)\,p\right)x(s) + \frac{1}{\kappa}\left(\frac{1}{2\sigma}(\sigma\rho - \xi_x)\right)^2 p\pi(s)\right)e^{\frac{1}{2}\left(\rho + \frac{1}{\sigma}\xi_x\right)p}, \tag{B.58}$$

$$\pi(p) = \left(-\kappa p x(s) + \left(1 - \frac{1}{2\sigma}(\xi_x - \sigma\rho)\,p\right)\pi(s)\right)e^{\frac{1}{2}\left(\rho + \frac{1}{\sigma}\xi_x\right)p}. \tag{B.59}$$

Using equation (B.43) and that $\xi_\pi \pi(s) + \xi_x x(s) + r_h = 0$, equations (B.58)-(B.59) become

$$\begin{aligned} -r_h\frac{\phi_1 + \phi_2\xi_\pi}{\kappa\xi_\pi + \phi_1\xi_x} &= \left(1 + \frac{1}{2\sigma}(\xi_x - \sigma\rho)\,p\right)x(s)e^{\frac{1}{2}\left(\rho + \frac{1}{\sigma}\xi_x\right)p} \\ &\quad + \frac{1}{\kappa}\left(\frac{1}{2\sigma}(\sigma\rho - \xi_x)\right)^2 p\left(-\frac{(\xi_x x(s) + r_h)}{\xi_\pi}\right)e^{\frac{1}{2}\left(\rho + \frac{1}{\sigma}\xi_x\right)p}, \end{aligned} \tag{B.60}$$

$$\begin{aligned} -r_h\frac{\kappa - \phi_2\xi_x}{\kappa\xi_\pi + \phi_1\xi_x} &= -\kappa p x(s)e^{\frac{1}{2}\left(\rho + \frac{1}{\sigma}\xi_x\right)p} \\ &\quad + \left(1 - \frac{1}{2\sigma}(\xi_x - \sigma\rho)\,p\right)\left(-\frac{(\xi_x x(s) + r_h)}{\xi_\pi}\right)e^{\frac{1}{2}\left(\rho + \frac{1}{\sigma}\xi_x\right)p}. \end{aligned} \tag{B.61}$$

Combining equations (B.60)-(B.61), I solve for $x(s)$ as a function of p

$$x(s) = \frac{A_0 + A_1 p}{B_0 + B_1 p}, \tag{B.62}$$

where

$$\begin{aligned} A_0 &= -4\sigma r_h\left(\phi_2\xi_x^2 + \sigma^2\rho^2\phi_2 + 4\kappa\sigma\rho - 2\sigma\rho\phi_2\xi_x\right), \\ A_1 &= 2r_h\left(\xi_x^2 - \sigma^2\rho^2\right)(2\kappa - \phi_2\xi_x + \sigma\rho\phi_2), \\ B_0 &= 4\kappa\sigma\left(\xi_x^2 + \sigma^2\rho^2 + 4\kappa\sigma - 2\sigma\rho\xi_x + 4\sigma\phi_1\xi_x\right), \\ B_1 &= (\xi_x + \sigma\rho)(2\kappa - \phi_2\xi_x + \sigma\rho\phi_2)\left(\sigma^2\rho^2 + 4\kappa\sigma - \xi_x^2\right). \end{aligned}$$

Plugging equation (B.62) into equation (B.60), I get

$$F(p) = 0,$$

where

$$F(p) = e^{\frac{1}{2}\left(\rho+\frac{1}{\sigma}\xi_x\right)p} - 1 + \frac{(\xi_x + \sigma\rho)(2\kappa - \phi_2\xi_x + \sigma\rho\phi_2)}{4\kappa\sigma}\left(\frac{\xi_x - \sigma(\phi_1 - \phi_2)}{\xi_x + \sigma(\phi_1 - \phi_2)}\right)p.$$

Since

$$\begin{aligned} F(0) &= 0, \\ F'(0) &= -\frac{\phi_2}{4\kappa\sigma}(\xi_x + \sigma\rho)^2 > 0, \\ F''(p) &= \frac{1}{4}\left(\rho + \frac{1}{\sigma}\xi_x\right)^2 e^{\frac{1}{2}\left(\rho+\frac{1}{\sigma}\xi_x\right)p} > 0, \end{aligned}$$

the equation $F(p) = 0$ has no solution for $p > 0$.

(e) The assumptions required for Theorem 3 in Appendix C, the Poincaré-Bendixson Theorem, hold. Indeed, because

$$\dot{x}(t) = \sigma^{-1}\left(\max\{0, \xi_x x(t) + \xi_\pi \pi(t) + r(t)\} - r(t) - \pi(t)\right), \tag{B.63}$$

$$\dot{\pi}(t) = \rho\pi(t) - \kappa x(t), \tag{B.64}$$

are, as functions of $\pi(t)$ and $x(t)$, continuous and differentiable almost everywhere, they are Lipschitz. The rest of the conditions are easy to check.

I show that the ω-limit set[37] of $(\pi(q), x(q))$ contains no steady-states and is not a periodic orbit. By Theorem 3, $(\pi(t), x(t))$ then explodes.

Because (π_{ss}, x_{ss}) is a locally unstable steady-state (by the Taylor principle) and $(\pi(q), x(q)) \neq (\pi_{ss}, x_{ss})$, the ω-limit set of $(\pi(q), x(q))$ does not contain (π_{ss}, x_{ss}), as $(\pi(t), x(t))$ is bounded away from (π_{ss}, x_{ss}) for all $t \geq q$. Because (π_{zlb}, x_{zlb}) is locally a saddle-path steady-state, the only paths converging to (π_{zlb}, x_{zlb}) as $t \to \infty$ must eventually be in $\Upsilon_{zlb} \cap \Omega_{zlb}$. By hypothesis, $(\pi(\tau(t)), x(\tau(t))) \notin \Upsilon_{zlb} \cap \partial\Omega$, where recall $\tau(t)$ is the time of first entry into Ω_{zlb} after t. By item (d), if $(\pi(t), x(t))$ enters Ω_{zlb} a second time after $\tau(t)$ (of course, by first visiting Ω_{ss}) it is not through $\Upsilon_{zlb} \cap \partial\Omega$. It follows that the ω-limit set of $(\pi(q), x(q))$ does not contain (π_{zlb}, x_{zlb}), as the orbit of $(\pi(q), x(q))$ never intersects $\Upsilon_{zlb} \cap \Omega_{zlb}$.

[37] See Appendix C for definitions of ω-limit sets and other concepts needed to state the Poincaré-Bendixson Theorem.

I now show that there are no closed orbits. The divergence of $(\dot{\pi}(t), \dot{x}(t))$ computed in the distribution sense is

$$\operatorname{div}(\dot{\pi}(t), \dot{x}(t)) = \frac{\partial \dot{x}}{\partial x} + \frac{\partial \dot{\pi}}{\partial \pi} = \begin{cases} \rho & , \quad \text{if } (\pi(t), x(t)) \in \Omega_{zlb} \setminus \partial\Omega \\ \frac{\xi_x}{2\sigma} + \rho & , \quad \text{if } (\pi(t), x(t)) \in \partial\Omega \\ \frac{1}{\sigma}\xi_x + \rho & , \quad \text{if } (\pi(t), x(t)) \in \Omega_{ss} \end{cases},$$

where $\Omega_{zlb} \setminus \partial\Omega$ denotes the interior of Ω_{zlb}.

The Taylor principle and $\rho > 0$ imply that $\operatorname{div}(\pi(t), x(t)) > 0$ for all $(\pi(t), x(t))$. By Theorem 2, there are no closed orbits[38].

(f) The result that there is no chaos is a direct consequence of Theorem 3, which tightly restricts the behavior of bounded solutions to two cases, none of which is chaotic. For continuous systems, strange attractors and other chaotic behavior can only emerge when the dimension of the phase space is three or more. Note that the concept of chaos I consider here is different from chaos in the sense of Li and Yorke (1975) used in Benhabib et al. (2002), which is more appropriate for a discrete time setting.

□

Proof of Proposition 3. Consider the following condition:

$$\begin{aligned} (\pi(t_1), x(t_1)) &= (\pi_{ss}, x_{ss}), \\ &\text{or} \\ (\pi(t_1), x(t_1)) &\in \Omega_{zlb} \cap \Upsilon_{zlb}, \\ &\text{or} \\ (\pi(t_1), x(t_1)) &\in \overline{\Omega}_{ss} \text{ and } (\pi(r), x(r)) \in \partial\Omega \cap \Upsilon_{zlb} \text{ for some } r \in (t_1, \infty). \end{aligned} \tag{B.65}$$

I first prove that if condition (B.65) holds, then $(\pi(t), x(t))$ is bounded. I consider three cases.

Case 1: If $(\pi(t_1), x(t_1)) = (\pi_{ss}, x_{ss})$, then $(\pi(t), x(t))$ is bounded because (π_{ss}, x_{ss}) is a steady-state.

[38] The version of the Poincaré-Bendixson theorem I have used is stronger than needed since our vector field is continuous (but non-differentiable) in $\partial\Omega$ while the theorem allows for discontinuities across the boundary between regions.

In addition, I have used one particular generalized derivative, the "derivative in the distribution sense." However, since the vector field under consideration is continuous, any generalized derivative (such as viscosity solutions) would still give a finite value for $(\dot{\pi}(t), \dot{x}(t))$. When the value of $(\dot{\pi}(t), \dot{x}_t)$ is finite along $\partial\Omega$, because $\partial\Omega$ has measure zero, its value does not contribute to the line integral along a closed loop. By Green's theorem, it then does not matter which concept of generalized derivative I use for this particular purpose.

Case 2: If $(\pi(t_1), x(t_1)) \in \Omega_{zlb} \cap \Upsilon_{zlb}$, then item (b) of Lemma 2 shows, by picking $m = t_1$, that $(\pi(t), x(t)) \in \Omega_{zlb} \cap \Upsilon_{zlb}$ for all $t \geq t_1$. The dynamics in equations (A.7)-(A.8) then show $(\pi(t), x(t)) \to (\pi_{zlb}, x_{zlb})$.

Case 3: If $(\pi(t_1), x(t_1)) \in \overline{\Omega}_{ss}$ and $(\pi(r), x(r)) \in \partial\Omega \cap \Upsilon_{zlb}$ for some $r \in (t_1, \infty)$, item (b) of Lemma 2 shows, by picking $m = r$, that $(\pi(t), x(t)) \in \Omega_{zlb} \cap \Upsilon_{zlb}$ for all $t \geq r$. The dynamics in equations (A.7)-(A.8) then show $(\pi(t), x(t)) \to (\pi_{zlb}, x_{zlb})$.

To prove the converse, I prove the contrapositive. Assume $(\pi(t_1), x(t_1)) \neq (\pi_{ss}, x_{ss})$ and $(\pi(t_1), x(t_1)) \notin \Omega_{zlb} \cap \Upsilon_{zlb}$. I consider two cases.

Case 1: $(\pi(t_1), x(t_1)) \notin \overline{\Omega}_{ss}$. Because of the saddle path dynamics in Ω_{zlb}, if $(\pi(t_1), x(t_1)) \notin \Upsilon_{zlb}$, then $(\pi(t), x(t))$ either explodes or enters Ω_{ss} in finite time. If it enters Ω_{ss} by intersecting $\partial\Omega$ at some time $r > t_1$, item (d) of Lemma 2 shows that there is no $p > 0$ such that $(\pi(t), x(t)) \in \Omega_{ss}$ for $t \in (r, r+p)$ and $(\pi(r+p), x(r+p)) \in \Upsilon_{zlb} \cap \partial\Omega$. Then item (e) of Lemma 2 shows that $(\pi(t), x(t))$ explodes.

Case 2: There is no $r \in (t_1, \infty)$ such that $(\pi(r), x(r)) \in \partial\Omega \cap \Upsilon_{zlb}$. If $(\pi(t_1), x(t_1)) \notin \overline{\Omega}_{ss}$, case 1 shows $(\pi(t), x(t))$ explodes. If $(\pi(t_1), x(t_1)) \in \overline{\Omega}_{ss}$, given that $(\pi(t_1), x(t_1)) \neq (\pi_{ss}, x_{ss})$, $(\pi(t), x(t))$ either explodes or enters Ω_{zlb}. By assumption, if it enters Ω_{zlb}, it does not intersect Υ_{zlb}. This means $(\pi(t), x(t))$ is eventually in the interior of Ω_{zlb} but not in Υ_{zlb}. The same logic applied in case 1 shows that $(\pi(t), x(t))$ explodes. $\square$

B.7 Proof of Proposition 4

Assume the rule implements the optimal equilibrium, i.e. $\{x(t), \pi_t, i_t\} = \{x^*(t), \pi^*(t), i^*(t)\}$ when the central bank follows the rule in equation (15). Werning (2012) shows that $i^*(t) = (1 - \kappa\sigma\lambda)\,\pi^*(t) + r(t) > 0$ for $t \geq t^*$. It follows that $f(R^*) \leq t^*$. In addition, $f(R^*) \geq s$ for all $s \leq t^*$ such that $(1 - \kappa\sigma\lambda)\,\pi^*(t) + r(t) > 0$, since otherwise the rule (15) would prescribe $i_t > 0$ while $i^*(t) = 0$. Pick $s = t^*$ to get $f(R^*) \geq t^*$ since $(1 - \kappa\sigma\lambda)\,\pi^*_{t^*} + r_{t^*} > 0$. Because $f(R^*) \leq t^*$ and $f(R^*) \geq t^*$, it follows that $f(R^*) = t^*$, and equation (19) holds.

To prove (20), I use $f(R^*) = t^*$ to get that for all $t \geq t^*$

$$\begin{aligned} \max\{0, \xi_\pi(R^*)\pi^*(t) + \xi_x(R^*)x^*(t) + r(t)\} &= \xi_\pi(R^*)\pi^*(t) + \xi_x(R^*)x^*(t) + r_h, \\ &= (1 - \kappa\sigma\lambda)\,\pi^*(t) + r_h, \end{aligned} \tag{B.66}$$

since otherwise $i_t = i^*(t)$ would not hold. If $\kappa\sigma\lambda \neq 1$, use $x^*(t) = \phi\pi^*(t)$ in equation (B.66) and then equation (20) follows immediately, as $\pi^*(t) \neq 0$ for $t \in [t^*, \infty)$. If $\kappa\sigma\lambda = 1$, any $\xi_\pi(R^*), \xi_x(R^*)$ implement the optimal equilibrium as $(0, 0)$ is a steady-state for all $\xi_\pi(R^*), \xi_x(R^*)$.

Now assume that equations (19)-(20) hold. I show rule (15) implements the optimal equilibrium. When $(\pi_0, x_0) = (x_0^*, \pi_0^*)$, clearly $i_t = i^*(t) = 0$ and $(\pi(t), x(t)) = (x^*(t), \pi^*(t))$ for $t < t^*$. Because $(\pi_t, x(t))$ and $(x^*(t), \pi^*(t))$ are continuous as a function of time and their paths coincide in $[t^* - \varepsilon, t^*)$ for any $\varepsilon > 0$, $(x(t^*), \pi(t^*)) = (x^*(t^*), \pi^*(t^*))$. As $x^*(t^*) = \phi\pi^*(t^*)$ for $t = t^*$,

$$x(t^*) = \phi\pi(t^*). \tag{B.67}$$

If $\kappa\sigma\lambda = 1$, $(x(t^*), \pi(t^*)) = (x^*(t^*), \pi^*(t^*)) = (0, 0)$, because $(0, 0)$ is a steady-state, $(\pi_t, x(t)) = (x^*(t), \pi^*(t)) = (0, 0)$ for all $t \geq t^*$ and any $\xi_\pi(R^*), \xi_x(R^*)$. In addition, if $(\pi_t, x(t)) = (0, 0)$,

$$i_t = \max\{0, \xi_\pi(R^*)\pi(t) + \xi_x(R^*)x(t) + r(t)\} = r_h = i^*(t)$$

for all $t \geq t^*$.

When $\kappa\sigma\lambda \neq 1$, using equations (A.9)-(A.14), it can be checked by direct computation that $(\pi(t), x(t)) = (x^*(t), \pi^*(t))$ for all $t \geq t^*$ where $(x^*(t), \pi^*(t))$ is given by

$$\begin{aligned} x^*(t) &= x_1^* \exp\left(-\frac{\kappa\lambda}{\phi}(t - t_1)\right), &\text{(B.68)}\\ \pi^*(t) &= \pi_1^* \exp\left(-\frac{\kappa\lambda}{\phi}(t - t_1)\right). &\text{(B.69)}\end{aligned}$$

The following relations may be helpful for the computations: If $\xi_\pi(R^*) < \kappa\sigma\lambda + \sigma\phi\rho + 1$, then

$$\begin{aligned} \alpha_1 &= \rho + \frac{1 - \xi_\pi(R^*)}{\sigma\phi}, &\text{(B.70)}\\ \alpha_2 &= -\frac{\kappa\lambda}{\phi}. &\text{(B.71)}\end{aligned}$$

If $\xi_\pi(R^*) > \kappa\sigma\lambda + \sigma\phi\rho + 1$, then

$$\begin{aligned} \alpha_1 &= -\frac{\kappa\lambda}{\phi}, &\text{(B.72)}\\ \alpha_2 &= \rho + \frac{1 - \xi_\pi(R^*)}{\sigma\phi}. &\text{(B.73)}\end{aligned}$$

If $\xi_\pi(R^*) = \kappa\sigma\lambda + \sigma\phi\rho + 1$

$$\alpha_1 = \alpha_2 = -\frac{\kappa\lambda}{\phi}. \tag{B.74}$$

B.8 Proof of Proposition 5

I first assume the rule implements no equilibrium with $R \neq R^*$ and prove items (a)-(c) hold.

Item (a): By Proposition 3, if the Taylor principle holds, there exist continuous bounded paths with $(\pi(t_1), x(t_1)) \in \Omega_{zlb} \cap \Upsilon_{zlb}$. Since for these paths $(\pi(t), x(t)) \to (\pi_{zlb}, x_{zlb})$, they constitute non-optimal equilibria. By item (b) of Lemma 2, $(\pi(t), x(t)) \in \Omega_{zlb} \cap \Upsilon_{zlb}$ for all $t \geq t_1$, irrespective of the choice of t_1. It follows that the only way to preclude these type of equilibria is to have the Taylor principle not hold for $(\pi(t_1), x(t_1)) \in \Omega_{zlb} \cap \Upsilon_{zlb}$.

Item (b): If there are continuous paths that satisfy the hypotheses of items i., ii. or iii., they are bounded by Propositions 2 and 3 and constitute non-optimal equilibria. Thus, all paths that satsify the hypothesis in items i., ii. and iii. must be discontinuous, which implies equation (21) holds.

Item (c): If $\partial\Omega \cap \Upsilon_{ss} = \emptyset$, then the item is vacuously true. If $\partial\Omega \cap \Upsilon_{ss}$ is non-empty then

$$(\pi(r), x(r)) = \begin{cases} \left(r_h \frac{(\xi_\pi - 1)}{\xi_x - \sigma\alpha_2\xi_\pi}, -r_h \frac{(\xi_x - \sigma\alpha_2)}{\xi_x - \sigma\alpha_2\xi_\pi}\right) & , \text{ if } \det A_{ss} < 0 \text{ and } \xi_\pi = 1 \\ \left(-r_h \frac{(\xi_x - \sigma\rho)}{\xi_x^2 - \kappa\sigma - \sigma\rho\xi_x}, \kappa\sigma \frac{r_h}{\xi_x^2 - \sigma\rho\xi_x - \kappa\sigma}\right) & , \text{ if } \det A_{ss} < 0 \text{ and } \xi_\pi \neq 1 \end{cases} \tag{B.75}$$

Assume that the Taylor principle does not hold for $(x(t_1), \pi_{t_1}) \in \Omega_{zlb}$ and that there exist some $r \in (t_1, \infty)$ such that $(\pi(r), x(r)) \in \partial\Omega_{zlb} \cap \Upsilon_{ss}$. I show that if equation (21) does not hold, then there exist a non-optimal equilibrium. By assumption, $t_1 \in [T, r)$. Let P be the set of points in the continuous path between $(\pi(T), x(T))$ and $(\pi(r), x(r))$, which can be obtained by running the system dynamics backward in time while respecting continuous pasting. By the dynamics in equations (A.7)-(A.8) and equation (B.75), the time q at which $(\pi(t_1), x(t_1)) \in \Omega_{zlb}$ reaches $(\pi(r), x(r))$ while following a continuous path is

$$q = \frac{1}{\phi_1} \log\left(\frac{d_{exit}(r)}{d_{exit}(t_1)}\right) \tag{B.76}$$

Note that q is not necessarily equal to r since the hypotheses of item (c) do not require that paths are continuous. Because equation (22) does not hold, $T \leq t_1 \leq q$ and thus $(\pi(t_1), x(t_1)) \in P$. By Theorem 2, the continuous path going through $(\pi(t_1), x(t_1))$ and $(\pi(r), x(r))$ is bounded for $t \geq t_1$. Using the continuous pasting conditions in Section ??, the path can be continuously extended from $(\pi(t_1), x(t_1))$ to $(\pi(0), x(0))$ to get a continuous bounded path for all $t \geq 0$. This equilibrium is non-optimal since no optimal path has $(\pi(t_1), x(t_1)) \in \Omega_{zlb} \setminus \partial\Omega$.

Conversely, I now assume items (a)-(c) hold and prove that the rule implements no equilibria with $R_{t_1} \neq R^*$. By Proposition 3 and item (a), there are no equilibria with $(x(t_1), \pi(t_1)) \in \Omega_{zlb} \cap \Upsilon_{zlb}$. By Propositions 2 and 3, items i.-iii. and the continuous pasting conditions in Section ??, there are no equilibria when: The Taylor principle holds for $(\pi(t_1), x(t_1)) \in \Omega_{ss}$ and there exist some $r \in (t_1, \infty)$ such that $(x_r, \pi_r) \in \partial\Omega \cap \Upsilon_{zlb}$, the

Taylor principle holds for $(\pi(t_1), x(t_1)) = (\pi_{ss}, x_{ss})$, or the Taylor principle does not hold for $(x(t_1), \pi(t_1)) \in \overline{\Omega}_{ss} \cap \Upsilon_{ss}$. Item (e) of Proposition 2 and item (c) imply that there is no continuous path from $(\pi(t_1), x(t_1)) \in \Omega_{zlb}$ to $(\pi(r), x(r)) \in \partial\Omega \cap \Upsilon_{ss}$ that remains bounded after r, and thus there are no equilibria when the Taylor principle does not hold for $(\pi(t_1), x(t_1)) \in \Omega_{zlb}$ and there exist some $r \in (t_1, \infty)$ such that $(\pi(r), x(r)) \in \partial\Omega_{zlb} \cap \Upsilon_{ss}$. By Propositions 2 and 3, all other cases lead to paths that are discontinuous or unbounded.

B.9 Proof of Proposition 6

Item (a). Assume $f(R_{t_1})$ is constant in its first two arguments. I show there always exist a non-optimal equilibrium. Denote the value of $f(\cdot, \cdot, 0, 0)$ by $\hat{t}$ (because f is constant in its first two arguments, $f(a, b, 0, 0) = \hat{t}$ for all a, b). Define $(\hat{\pi}_0, \hat{x}_0)$ by

$$\hat{x}_0 = \frac{r_h}{\kappa} \frac{\phi_1^2 e^{-\phi_2 \hat{t}} - \phi_2^2 e^{-\phi_1 \hat{t}}}{\phi_1 - \phi_2} + \left(\frac{r_h - r_l}{\kappa}\right) \frac{\phi_2^2 e^{-T\phi_1} - \phi_1^2 e^{-T\phi_2}}{\phi_1 - \phi_2} - \frac{r_l \rho}{\kappa}, \tag{B.77}$$

$$\hat{\pi}_0 = r_h \frac{\phi_1 e^{-\phi_2 \hat{t}} - \phi_2 e^{-\phi_1 \hat{t}}}{\phi_1 - \phi_2} + (r_h - r_l) \frac{\phi_2 e^{-T\phi_1} - \phi_1 e^{-T\phi_2}}{\phi_1 - \phi_2} - r_l. \tag{B.78}$$

The continuous path starting at $(\hat{\pi}_0, \hat{x}_0)$ reaches $(0, 0)$ at time $\hat{t}$ by equations (B.4) and (B.5). Since $(0, 0)$ is a steady-state, $(\pi(t), x(t)) = (0, 0)$ for all $t \geq \hat{t}$. The path for $(\pi(t), x(t))$ is continuous, bounded and follows the IS, the NKPC and the interest rate rule: It is an equilibrium. If $\kappa\sigma\lambda \neq 1$, the equilibrium is not optimal.

Item (b). Consider a rule with

$$\begin{aligned} \xi_\pi(R_{t_1}) &= 1 - \kappa\sigma\lambda \\ \xi_x(R_{t_1}) &= 0 \\ f(R^*) &= t^* \\ f(R_{t_1}) &= \tau(\pi(0), x(0)) \end{aligned}$$

for some function $\tau : \mathbb{R}^2 \to \mathbb{R}$. I use Proposition 5 to show that there exists a choice of τ compatible with the optimal equilibrium being a unique equilibrium.

The choice of ξ_x, ξ_π implies

$$\alpha_1 = \frac{1}{2}\left(\rho + \sqrt{\rho^2 + 4\kappa^2\lambda}\right) = \kappa\phi > 0 \tag{B.79}$$

$$\alpha_2 = \frac{1}{2}\left(\rho - \sqrt{\rho^2 + 4\kappa^2\lambda}\right) < 0 \tag{B.80}$$

$$\alpha_1\alpha_2 = -\kappa^2\lambda < 0 \tag{B.81}$$

$$\alpha_1 + \alpha_2 = \rho \tag{B.82}$$

The Taylor principle never holds, as $\kappa(\xi_\pi - 1) + \rho\xi_x = -\kappa^2\sigma\lambda < 0$.

Item (a) of Proposition 5 is true because the Taylor principle does not hold. Subitems (b)i. and (b)iii. of Proposition 5 do not apply, since the Taylor principle does not hold.

I now analyze subitem (b)ii. and item (c) of Proposition 5, and show they can be satisfied with an appropriate choice of τ.

First, consider item (b)ii.. Because $(\pi(t_1), x(t_1)) \in \Upsilon_{ss}$,

$$\pi(t_1) = \frac{1}{\phi}x(t_1) \tag{B.83}$$

and because $(\pi(t_1), x(t_1)) \in \overline{\Omega}_{ss}$

$$(1 - \kappa\sigma\lambda)\,\pi(t_1) + r_h \geq 0.$$

Consider the four cases of the continuous pasting condition in equation B.7 given by equation (B.9).

Case 1: If $d_{exit}(t_1) = 0$ and $d_{trap}(t_1) = 0$, $x(t_1) = x_{zlb} = -\frac{1}{\kappa}r_h\rho$ and $\pi(t_1) = \pi_{zlb} = -r_h$, which contradicts equation (B.83) and hence there is no equilibrium for this case.

Case 2: If $d_{exit}(t_1) = 0$ and $d_{trap}(t_1) \neq 0$,

$$x(t_1) = \frac{\phi_1}{\kappa}\pi(t_1) - \frac{r_h\phi_2}{\kappa}. \tag{B.84}$$

If $\sigma\lambda\kappa = 1$, there is no $(\pi(t_1), x(t_1))$ that satisfies equations (B.84) and (B.83) simultaneously. If $\sigma\lambda\kappa \neq 1$, equations (B.84) and (B.83) imply

$$\begin{aligned} x(t_1) &= r_h\frac{\phi - \sigma\lambda\phi_2}{(\kappa\sigma\lambda - 1)}, \\ \pi(t_1) &= \frac{r_h}{\phi}\frac{\phi - \sigma\lambda\phi_2}{(\kappa\sigma\lambda - 1)} \end{aligned}$$

But then, since $\phi_2 < 0$,

$$\xi_x x(t_1) + \xi_\pi\pi(t_1) + r_h = (1 - \kappa\sigma\lambda)\,\pi(t_1) + r_h = \frac{\lambda\sigma\phi_2 r_h}{\phi} < 0$$

contradicts that $(\pi(t_1), x(t_1)) \in \overline{\Omega}_{ss}$ and thus there is no equilibrium for this case.

Case 3: If $d_{exit}(t_1) \neq 0$ and $d_{trap}(t_1) = 0$,

$$x(t_1) = \frac{\phi_2}{\kappa}\pi(t_1) - \frac{r_h\phi_1}{\kappa}. \tag{B.85}$$

If $\sigma\lambda\kappa = 1$, there is no $(\pi(t_1), x(t_1))$ that satisfies equations (B.84) and (B.83) simultaneously.

If $\sigma\lambda\kappa \neq 1$, equations (B.84) and (B.83) imply

$$x(t_1) = r_h \frac{\phi - \sigma\lambda\phi_1}{(\kappa\sigma\lambda - 1)}, \tag{B.86}$$

$$\pi(t_1) = \frac{r_h}{\kappa\sigma\lambda - 1}\frac{(\phi - \sigma\lambda\phi_1)}{\phi}. \tag{B.87}$$

The pasting condition in equations (B.8) and (B.10), and equation (B.87), give

$$\begin{aligned}
\mathcal{T}(R_{t_1}) &= -\frac{1}{\phi_1}\log\frac{\pi(0) + \left(r_l + (r_h - r_l)\,\frac{\phi_1 e^{-T\phi_2} - \phi_2 e^{-T\phi_1}}{\phi_1 - \phi_2}\right)}{(\pi(t_1) + r_h)} && \text{(B.88)}\\
&= -\frac{1}{\phi_1}\log\frac{\pi(0) + \left(r_l + (r_h - r_l)\,\frac{\phi_1 e^{-T\phi_2} - \phi_2 e^{-T\phi_1}}{\phi_1 - \phi_2}\right)}{\left(\frac{r_h}{\kappa\sigma\lambda - 1}\frac{(\phi - \sigma\lambda\phi_1)}{\phi} + r_h\right)} && \text{(B.89)}\\
&= \mathcal{T}(\pi(0), x(0)) && \text{(B.90)}
\end{aligned}$$

Setting

$$\tau(\pi(0), x(0)) \neq \mathcal{T}(\pi(0), x(0))$$

precludes any equilibrium for this case.

Case 4: If $d_{exit}(t_1) \neq 0$ and $d_{trap}(t_1) \neq 0$, using equation (B.83), the continuous pasting condition in equations (B.7) and (B.9) is

$$\begin{aligned}
&-\frac{1}{\phi_1}\log\frac{x(0) - \frac{\phi_1}{\kappa}\pi(0) + \frac{\phi_2 r_h}{\kappa} + \frac{\phi_2}{\kappa}(r_h - r_l)\left(e^{-T\phi_1} - 1\right)}{\left(\phi - \frac{\phi_1}{\kappa}\right)\pi(t_1) + \frac{\phi_2 r_h}{\kappa}}\\
&\qquad = -\frac{1}{\phi_2}\log\frac{x(0) - \frac{\phi_2}{\kappa}\pi(0) + \frac{\phi_1 r_h}{\kappa} + \frac{\phi_1}{\kappa}(r_h - r_l)\left(e^{-T\phi_2} - 1\right)}{\left(\phi - \frac{\phi_2}{\kappa}\right) + \frac{\phi_1 r_h}{\kappa}} && \text{(B.91)}
\end{aligned}$$

Let

$$\begin{aligned}
H(\pi(t_1), \pi(0), x(0)) = &-\frac{1}{\phi_1}\log\frac{x(0) - \frac{\phi_1}{\kappa}\pi(0) + \frac{\phi_2 r_h}{\kappa} + \frac{\phi_2}{\kappa}(r_h - r_l)\left(e^{-T\phi_1} - 1\right)}{\left(\phi - \frac{\phi_1}{\kappa}\right)\pi(t_1) + \frac{\phi_2 r_h}{\kappa}}\\
&+ \frac{1}{\phi_2}\log\frac{x(0) - \frac{\phi_2}{\kappa}\pi(0) + \frac{\phi_1 r_h}{\kappa} + \frac{\phi_1}{\kappa}(r_h - r_l)\left(e^{-T\phi_2} - 1\right)}{\left(\phi - \frac{\phi_2}{\kappa}\right)\pi(t_1) + \frac{\phi_1 r_h}{\kappa}}
\end{aligned}$$

Then $G(\pi(t_1), \pi(0), x(0)) = 0$ iff equation (B.91) holds. Since

$$\frac{\partial H(\pi(t_1), \pi(0), x(0))}{\partial \pi(t_1)} = 0 \iff r_h = (\kappa\sigma\lambda - 1)\,\pi(t_1)$$

the implicit function theorem implies that we can write $H(\pi(t_1), \pi(0), x(0)) = 0$ as

$$\pi(t_1) = G(\pi(0), x(0))$$

for some function G, except when

$$\kappa\sigma\lambda - 1 \neq 0 \text{ and } \pi(t_1) = \frac{r_h}{(\kappa\sigma\lambda - 1)} \tag{B.92}$$

If equation (B.92) does not hold, then the continuous pasting conditions are given by

$$\begin{aligned}
\pi(t_1) &= G(\pi(0), x(0)) \\
\mathcal{T}(R_{t_1}) &= -\frac{1}{\phi_1} \log \frac{x(0) - \frac{\phi_1}{\kappa}\pi(0) + \frac{\phi_2 r_h}{\kappa} + \frac{\phi_2}{\kappa}(r_h - r_l)\left(e^{-T\phi_1} - 1\right)}{\left(\phi - \frac{\phi_1}{\kappa}\right)\pi(t_1) + \frac{\phi_2 r_h}{\kappa}} \\
&= -\frac{1}{\phi_1} \log \frac{x(0) - \frac{\phi_1}{\kappa}\pi(0) + \frac{\phi_2 r_h}{\kappa} + \frac{\phi_2}{\kappa}(r_h - r_l)\left(e^{-T\phi_1} - 1\right)}{\left(\phi - \frac{\phi_1}{\kappa}\right) g(\pi_0, x_0) + \frac{\phi_2 r_h}{\kappa}} \\
&= \mathcal{T}(\pi(0), x(0))
\end{aligned}$$

If equation (B.92) holds, $H(\pi(t_1), \pi(0), x(0)) = H\left(\frac{r_h}{\kappa\sigma\lambda - 1}, \pi(0), x(0)\right)$. If there is no $(\pi(0), x(0))$ so that $H\left(\frac{r_h}{(\kappa\sigma\lambda-1)}, \pi(0), x(0)\right) = 0$, then there are no equilibria since no path is continuous. If there exists $(\pi(0), x(0))$ such that $H\left(\frac{r_h}{(\kappa\sigma\lambda-1)}, \pi(0), x(0)\right) = 0$, continuous pasting gives

$$\begin{aligned}
\mathcal{T}(R_{t_1}) &= -\frac{1}{\phi_1} \log \frac{x(0) - \frac{\phi_1}{\kappa}\pi(0) + \frac{\phi_2 r_h}{\kappa} + \frac{\phi_2}{\kappa}(r_h - r_l)\left(e^{-T\phi_1} - 1\right)}{\left(\phi - \frac{\phi_1}{\kappa}\right)\frac{r_h}{(\kappa\sigma\lambda-1)} + \frac{\phi_2 r_h}{\kappa}} \\
&= \mathcal{T}(\pi(0), x(0))
\end{aligned}$$

In either case (when equation (B.92) holds and when it does not hold), setting

$$\tau(\pi(0), x(0)) \neq \mathcal{T}(\pi(0), x(0))$$

precludes any equilibrium for case 4. This concludes the analysis of item (b)ii. of Proposition 5.

Now consider item (c) of Proposition 5. If $\kappa\sigma\lambda = 1$, $\Upsilon_{ss} \cap \partial\Omega_{zlb} = \emptyset$ and thus there are no equilibria. If $\kappa\sigma\lambda \neq 1$, $(\pi(r), x(r)) \in \Upsilon_{ss} \cap \partial\Omega_{zlb}$ implies

$$x(r) = \frac{1}{2\kappa}\frac{r_h}{\kappa\sigma\lambda - 1}\left(\rho + \sqrt{4\lambda\kappa^2 + \rho^2}\right) \tag{B.93}$$

$$\pi(r) = \frac{r_h}{\kappa\sigma\lambda - 1} \tag{B.94}$$

Using (B.93)-(B.94) and that $(\pi(t), x(t)) \in \Omega_{zlb}$ for $t \in [t_1, r]$, the continuous pasting equations (B.4) and (B.5) imply that

$$x(0) = \frac{\phi_1 e^{-\phi_2 r} - \phi_2 e^{-\phi_1 r}}{\phi_1 - \phi_2}\left(\frac{1}{2\kappa}\frac{r_h}{\kappa\sigma\lambda - 1}\left(\rho + \sqrt{4\lambda\kappa^2 + \rho^2}\right)\right)$$
$$- \frac{1}{\sigma}\frac{e^{-\phi_1 r} - e^{-\phi_2 r}}{\phi_1 - \phi_2}\frac{r_h}{\kappa\sigma\lambda - 1} + \frac{r_h}{\kappa}\frac{\phi_1^2 e^{-\phi_2 r} - \phi_2^2 e^{-\phi_1 r}}{\phi_1 - \phi_2}$$
$$+ \left(\frac{r_h - r_l}{\kappa}\right)\frac{\phi_2^2 e^{-T\phi_1} - \phi_1^2 e^{-T\phi_2}}{\phi_1 - \phi_2} - \frac{r_l \rho}{\kappa},$$

$$\pi(0) = -\kappa\frac{e^{-\phi_1 r} - e^{-\phi_2 r}}{\phi_1 - \phi_2}\left(\frac{1}{2\kappa}\frac{r_h}{\kappa\sigma\lambda - 1}\left(\rho + \sqrt{4\lambda\kappa^2 + \rho^2}\right)\right)$$
$$+ \frac{\phi_1 e^{-\phi_1 r} - \phi_2 e^{-\phi_2 r}}{\phi_1 - \phi_2}\frac{r_h}{\kappa\sigma\lambda - 1} + r_h\frac{\phi_1 e^{-\phi_2 r} - \phi_2 e^{-\phi_1 r}}{\phi_1 - \phi_2}$$
$$+ (r_h - r_l)\frac{\phi_2 e^{-T\phi_1} - \phi_1 e^{-T\phi_2}}{\phi_1 - \phi_2} - r_l.$$

Solving for r gives

$$r = \nu(\pi(0), x(0))$$

where

$$\begin{aligned} A &= \frac{1}{\kappa}\rho r_l + \frac{1}{\kappa}(r_h - r_l)\frac{\phi_1^2 e^{-T\phi_2} - \phi_2^2 e^{-T\phi_1}}{\phi_1 - \phi_2} \\ B &= r_l + (r_h - r_l)\frac{\phi_1 e^{-T\phi_2} - \phi_2 e^{-T\phi_1}}{\phi_1 - \phi_2} \end{aligned}$$

are two constants and

$$\nu(\pi(0), x(0)) = -\frac{1}{\phi_2}\log\left(\kappa\frac{\sigma\phi\phi_2 + 1}{r_h(\rho - \kappa\phi)}(x(0) + A) - \frac{\phi_2 + \kappa\phi + \sigma\rho\phi\phi_2}{r_h(\rho - \kappa\phi)}(\pi(0) + B)\right)$$

is a function of $x(0)$ and $\pi(0)$ only (not of $x(t_1)$, $\pi(t_1)$ or t_1). Setting

$$\tau(\pi(0), x(0)) > \nu(\pi(0), x(0))$$

precludes all equilibria for the case in which (c) of Proposition 5 applies.

Item (c). The rule in the last item has constant Taylor rule coefficients.

B.10 Proof of Proposition 7

I start with a Lemma.

Lemma 3. *If* $\det A_{ss}(r) = 0$ *for some* $R = (\pi_0, x_0)$*, then there exist a non-optimal equilibrium.*

Proof of Lemma 3. When $\det A_{ss}(r) = 0$, $(\pi_{zlb}, x_{zlb}) \in \partial\Omega$. A continuous path with $(\pi(t), x(t)) \in \Upsilon_{zlb} \cap \Omega_{zlb}$ is bounded for any choice of f, since $(\pi(t), x(t)) \in \Omega_{zlb}$ for all t and it converges to (π_{zlb}, x_{zlb}), which is a steady-state of the economy. $\square$

Now I prove Proposition 7. Item (a). By Proposition 5, if items (a)-(c) hold but with equation (21) replaced with (23), then there is no equilibrium with $R \neq R^*$, since (23) implies (21).

Conversely, assume there is no equilibrium with $R \neq R^*$. I show equation (23) holds. To do so, I first show that the Intermediate Value Theorem is applicable and then use it to show equation (23) holds. Let

$$\Theta = \left\{ R \in \mathbb{R}^4 : \mathcal{P}(r) = 0 \text{ and } R \neq R^* \right\}$$

Because f, ξ_x and ξ_π are continuous, their restriction to Θ are also continuous. In addition, Θ is path-connected because the solution to the ODE (1)-(2) is continuous with respect to time, the mapping from $(\pi(0), x(0))$ to $(\pi(t_1), x(t_1))$ is a continuous bijection for a fixed t_1, $f(R_{t_1}) = t_1$ is continuous in R_{t_1}, and the exclusion of R^* from Θ does not destroy path-connectedness because it is a zero-dimensional set while the dimension of Θ is 3. Because f, ξ_x and ξ_π are continuous in Θ and Θ is path-connected, we can apply the Intermediate Value Theorem.

Assume, for the sake of contradiction, that there exists $R_{low} \in \Theta$ with $f(R_{low}) < \mathcal{T}(R_{low})$. The inequality $f(R_{low}) < \mathcal{T}(R_{low})$ implies $f(r) < \mathcal{T}(r)$ for all $R \in \Theta$ since otherwise, by the Intermediate Value Theorem, there would be some $R_0 \in \Theta$ with $f(R_0) = \mathcal{T}(R_0)$, contradicting that there is no equilibrium with $R \neq R^*$. Consider the point $R_T = (\hat{\pi}_0, \hat{x}_0, \hat{\pi}_1, \hat{x}_1)$ defined by

$$\begin{aligned}
\hat{x}_0 &= \frac{r_h}{\kappa} \frac{\phi_1^2 e^{-\phi_2 T} - \phi_2^2 e^{-\phi_1 T}}{\phi_1 - \phi_2} + \left(\frac{r_h - r_l}{\kappa} \right) \frac{\phi_2^2 e^{-T\phi_1} - \phi_1^2 e^{-T\phi_2}}{\phi_1 - \phi_2} - \frac{r_l \rho}{\kappa}, \\
\hat{\pi}_0 &= r_h \frac{\phi_1 e^{-\phi_2 T} - \phi_2 e^{-\phi_1 T}}{\phi_1 - \phi_2} + (r_h - r_l) \frac{\phi_2 e^{-T\phi_1} - \phi_1 e^{-T\phi_2}}{\phi_1 - \phi_2} - r_l, \\
\hat{x}_1 &= 0, \\
\hat{\pi}_1 &= 0.
\end{aligned}$$

By the continuous pasting conditions in equations (B.4)-(B.5), $R_T \in \Theta$ and $f(R_T) = T = \mathcal{T}(R_T)$, contradicting that $f(r) < \mathcal{T}(r)$ for all $R \in \Theta$.

Item (b). Since $\xi_\pi(R_{t_1})$ and $\xi_x(R_{t_1})$ are continuous, then either the Taylor principle holds for all R_{t_1}, or the Taylor principle does not hold for all R_{t_1}. To see this, assume for the sake of contradiction that there exists R_{TP} that satisfies the Taylor principle and $R_{no\text{-}TP}$ that does not. Then

$$\begin{aligned}\det A_{ss}(R_{TP}) &= \kappa\left(\xi_\pi(R_{TP})-1\right)+\rho\xi_x(R_{TP})>0,\\ \det A_{ss}(R_{no\text{-}TP}) &= \kappa\left(\xi_\pi(R_{no\text{-}TP})-1\right)+\rho\xi_x(R_{no\text{-}TP})<0.\end{aligned}$$

By the Intermediate Value Theorem, there exist an R_0 such that $\det A_{ss}(R_0)=\kappa\left(\xi_\pi(R_0)-1\right)+\rho\xi_x(R_0)=0$. By Lemma 3, there exist a non-optimal equilibrium.

By Proposition 4, when $\kappa\sigma\lambda\neq 1$, the Taylor principle does not hold for R^*. Because the Taylor principle does not hold for one R, then it does not hold for all R.
Item (c). By item (a) of Proposition 6, the rule cannot be purely forward-looking.

I show that if the rule is purely backward-looking, that is, if f, ξ_x and ξ_π are all constant in their last two arguments, then there exists an equilibrium with $R\neq R^*$. By item (b) of Proposition 7 just proved above, the Taylor principle never holds. I look for an equilibrium with

$$\pi(t_1)=c(R_{t_1})x(t_1) \tag{B.95}$$

with the function $c(R_{t_1})>0$ defined by equation (A.15). Because ξ_x and ξ_π are continuous in R_{t_1} and constant in $x(t_1)$, $\pi(t_1)$, so is c. To see that c is continuous when $\xi_\pi=1$, compute

$$\begin{aligned}\lim_{\xi_\pi\to 1}c(R_{t_1}) &= \lim_{\xi_\pi\to 1}\frac{(\xi_x-\sigma\alpha_2)}{(1-\xi_\pi)}\\ &= \lim_{\xi_\pi\to 1}-\frac{1}{2\left(\xi_\pi-1\right)}\left(\xi_x-\sigma\rho+\sqrt{\xi_x^2+\sigma^2\rho^2+4\kappa\sigma-4\kappa\sigma\xi_\pi-2\sigma\rho\xi_x}\right)\\ &= \lim_{\xi_\pi\to 1}\kappa\sigma\left(\xi_x^2+\sigma^2\rho^2+4\kappa\sigma-4\kappa\sigma\xi_\pi-2\sigma\rho\xi_x\right)^{-\frac{1}{2}}\\ &= \frac{\kappa\sigma}{|\xi_x-\sigma\rho|}\\ &= \frac{\kappa\sigma}{\sigma\rho-\xi_x}\end{aligned}$$

The third line follows by L'Hospital's Rule; in a small enough neighborhood of $\xi_\pi=1$, the Taylor principle not holding implies $\xi_x<0$ and thus both numerator and denominator in the second line go to zero as $\xi_\pi\to 1$. The last line follows because $\xi_x<0$ when $\xi_\pi=1$, again because the Taylor principle does not hold. When $\xi_\pi\neq 1$, c is continuous by equation (A.15).

Let

$$\begin{aligned}
M(\pi(0), x(0)) &= \frac{\phi_1 e^{-\phi_2 f} - \phi_2 e^{-\phi_1 f}}{\phi_1 - \phi_2} c - \frac{1}{\sigma} \frac{e^{-\phi_1 f} - e^{-\phi_2 f}}{\phi_1 - \phi_2} \\
N(\pi(0), x(0)) &= -\kappa \frac{e^{-\phi_1 f} - e^{-\phi_2 f}}{\phi_1 - \phi_2} c + \frac{\phi_1 e^{-\phi_1 f} - \phi_2 e^{-\phi_2 f}}{\phi_1 - \phi_2} \\
P(\pi(0), x(0)) &= \frac{r_h}{\kappa} \frac{\phi_1^2 e^{-\phi_2 f} - \phi_2^2 e^{-\phi_1 f}}{\phi_1 - \phi_2} \\
Q(\pi(0), x(0)) &= r_h \frac{\phi_1 e^{-\phi_2 f} - \phi_2 e^{-\phi_1 f}}{\phi_1 - \phi_2} \\
A &= \left(\frac{r_h - r_l}{\kappa} \right) \frac{\phi_2^2 e^{-T\phi_1} - \phi_1^2 e^{-T\phi_2}}{\phi_1 - \phi_2} - \frac{r_l \rho}{\kappa} \\
B &= (r_h - r_l) \frac{\phi_2 e^{-T\phi_1} - \phi_1 e^{-T\phi_2}}{\phi_1 - \phi_2} - r_l
\end{aligned}$$

The functions M, N, P and Q are continuous and depend only on $x(0), \pi(0)$ (and not on $x(t_1), \pi(t_1)$) because f and c are continuous and constant in $\pi(t_1)$, $x(t_1)$. The continuous pasting conditions in equations (B.4)-(B.5) give

$$x(0) = M\pi(t_1) + P + A \tag{B.96}$$

$$\pi(0) = N\pi(t_1) + Q + B \tag{B.97}$$

If $M \neq 0$ and $N \neq 0$, the last two equations give

$$\pi(t_1) = \frac{x(0) - P - A}{M} \tag{B.98}$$

$$\pi(0) = \frac{N}{M} (x(0) - P - A) + Q + B \tag{B.99}$$

Fix $x(0)$ to $\hat{x}_0 = x^*(0) + \varepsilon$ with $\varepsilon > 0$. The right hand-side of equation (B.99) is a function of $\pi(0)$ only. It is bounded above and below, as $f \in [T, \infty)$ and

$$\begin{aligned}
& \lim_{f \to \infty} \frac{N}{M} (\hat{x}_0 - P - A) + Q + B, \\
= \; & \frac{\kappa\sigma (\rho - 2\phi_1)(A - \hat{x}_0) + B(2\kappa + \sigma\rho\phi_1)}{\phi_1 - \phi_2} \lim_{f \to \infty} \frac{c}{(c\sigma\phi_1 + 1)} \\
& - \frac{(2\kappa + \sigma\rho\phi_2)(A - \hat{x}_0) + B(\rho - 2\phi_1)}{(\phi_1 - \phi_2)} \lim_{f \to \infty} \frac{1}{(c\sigma\phi_1 + 1)},
\end{aligned}$$

is finite since $c > 0$. The left-hand side of equation (B.99), on the other hand, tends to $\pm\infty$ as $\pi(0) \to \pm\infty$. This means, since N, M, Q and B are continuous in $\pi(0)$, that there is at least one $\pi(0)$, say $\hat{\pi}_0$, that satisfies equation (B.99). Plugging $(\hat{\pi}_0, \hat{x}_0)$ into equations (B.95) and (B.98) give values for $(\pi(t_1), x(t_1))$, say $(\hat{\pi}_1, \hat{x}_1)$. By construction, the path defined by $(\hat{\pi}_0, \hat{x}_0)$ is continuous. Picking ε small enough guarantees that $(\hat{\pi}_1, \hat{x}_1) \in \Omega_{ss}$, since $(\pi^*(t^*), x^*(t^*)) \in \Omega_{ss}$ is bounded away from $\partial\Omega$. Equation (B.95) implies $(\hat{\pi}_1, \hat{x}_1) \in \Upsilon_{ss}$. Proposition 2 then shows the path defined by $(\hat{\pi}_0, \hat{x}_0)$ is bounded and hence an equilibrium. Because $\varepsilon \neq 0$, the equilibrium is not the optimal equilibrium.

If $M = 0$ and $N \neq 0$, the continuous pasting conditions (B.4)-(B.5) give

$$x(0) = P + A \tag{B.100}$$

$$\pi(t_1) = \frac{\pi(0) - (Q + B)}{N} \tag{B.101}$$

But $M = 0$ implies

$$e^{-f\phi_2} = \frac{c\sigma\phi_2 + 1}{c\sigma\phi_1 + 1} e^{-f\phi_1}$$

and thus

$$\begin{aligned}
\lim_{f\to\infty} P + A &= \lim_{f\to\infty} \frac{r_h}{\kappa} \frac{\phi_1^2 e^{-\phi_2 f} - \phi_2^2 e^{-\phi_1 f}}{\phi_1 - \phi_2} + A \\
&= \lim_{f\to\infty} \frac{r_h (\rho - c\kappa)}{\kappa + c\kappa\sigma\phi_1} e^{-f\phi_1} + A \\
&= A
\end{aligned}$$

is finite. An argument analogous to the one used for the case in which $M \neq 0$ and $N \neq 0$ shows the existence of a non-optimal equilibrium. The case $M \neq 0$ and $N = 0$ can be treated the same way and $M = N = 0$ cannot happen.

C Non-Linear dynamics – Poincaré-Bendixson Theorem

Assume $f : \mathbb{R}^2 \to \mathbb{R}^2$. Consider the two-dimensional system

$$\dot{x}(t) = f(x(t)). \tag{C.1}$$

Let $\phi_t(p)$ be a solution to (C.1) for $t \geq 0$ with initial condition $x_0 = p$. We assume that for each p, there is a unique solution $\phi(t, p)$. This is the case, for example, if f is Lipschitz.

The *positive semi-orbit* of f through p is defined as

$$\gamma^+(p) = \left\{ x \in \mathbb{R}^2 : x = \phi_t(p) \text{ for some } t \in [0, \infty) \right\}.$$

Similarly, the *negative semi-orbit* though p is

$$\gamma^{-}(p) = \left\{ x \in \mathbb{R}^2 : x = \phi_t(p) \text{ for some } t \in (-\infty, 0] \right\}.$$

The *orbit* of f through p is the union

$$\gamma(p) = \gamma^{+}(p) \cup \gamma^{-}(p).$$

A *periodic solution* is one for which $\phi_{t+T}(p) = \phi_t(p)$ for some $T > 0$ and all $t \in R$. A *periodic orbit* is the orbit $\gamma(p)$ of periodic solution $\phi_t(p)$.

The *ω-limit set* of p, denoted by $\omega(p)$, is the set

$$\omega(p) = \left\{ x \in \mathbb{R}^2 : \exists \{t_k\}_{k=0}^{\infty},\ t_k \in R \text{ with } t_k \to \infty \text{ such that } \phi_{t_k}(p) \to x \text{ as } k \to \infty \right\}.$$

Consider the following four assumptions:

(a) Ω is an open domain in $\mathbb{R}^2$, divided into a finite number of open sub-domains Ω_i such that $\bigcup \overline{\Omega}_i = \overline{\Omega}$.

(b) If $\overline{\Omega}_i$ and $\overline{\Omega}_j$ are not disjoint and $i \neq j$, then $\overline{\Omega}_i \cap \overline{\Omega}_j = \Gamma_{ij}$, where Γ_{ij} (joint boundaries) are piecewise smooth.

(c) f is Lipschitz in all sub-domains Ω_i and possibly discontinuous along Γ_{ij}.

(d) The vector field f defines a direction at each point in Ω. In particular, at every point of Γ_{ij} the vector field $f(x)$ specifies into which Ω_i the flow is directed.

Theorem 1 (Extension of the Poincaré-Bendixson theorem)**.** *Consider the planar autonomous system (C.1). Let the conditions 1-4 be satisfied and let f be bounded in Ω. Suppose that K is a compact region in Ω, containing no fixed points of (C.1). If the solution of (C.1) is in K for all $t \geq t_0$, then (C.1) has a closed orbit in K.*

Theorem 2 (Extension of the Bendixson criterion)**.** *Consider the planar autonomous system (C.1). Let the conditions 1-4 be satisfied and let f be bounded in the simply connected region Ω and C^1 in each Ω_i. If* $\operatorname{div} f$ *(the divergence of f calculated in the distribution sense) is of the same sign and is not identically zero in Ω, then (C.1) has no closed orbit in Ω.*

Remark The requirement that f is bounded is too strong; it suffices that

$$\iint\limits_{D} \operatorname{div} f \text{ and } \int\limits_{C} f \cdot n \, ds$$

are well-defined (in the distribution sense) for all smooth closed curves C, where D is the region enclosed by C and n is a unit vector normal to C.

A proof of both theorems can be found in Melin (2005). Compared to the classical Poincaré-Bendixson theorem, Melin (2005) allows for some discontinuities in f.

We have cited the theorems exactly as they appear in Melin (2005). However, in this context, it is perhaps more familiar for economists to refer to points for which $f = 0$ as steady-states instead of fixed points and to periodic orbits instead of closed orbits.

We now prove an immediate consequence of this "extended" Poincaré-Bendixson theorem.

Theorem 3. *Assume Theorem 1 holds. If a solution φ_t is bounded for all $t \geq 0$, then either*

(a) $\omega(\varphi)$ contains a steady-state

or

(b) $\omega(\varphi)$ is a periodic orbit

Proof. First, note that because φ is bounded, $\omega(\varphi)$ is non-empty. Indeed, consider a sequence $x_i = \varphi_{t_i}(x)$ for some x. The sequence $\{x_i\}$ is bounded and infinite, so there exist a convergent subsequence. If such convergent subsequence converges to p, then $p \in \omega(\varphi)$ and thus $\omega(\varphi)$ is non-empty.

If $\omega(\varphi)$ contains a steady-state, item (a) obtains. If $\omega(\varphi)$ contains no steady-states (no fixed points), then Theorem 1 implies that $\omega(\varphi)$ is a periodic orbit, corresponding to item (b) (note that because φ is bounded we can always find a compact set K that contains it). $\square$

D BSGU Equilibria

The conclusion that following the Taylor principle outside the ZLB induces the existence of a deflationary steady state at the ZLB is similar to one of the results in Benhabib et al. (2001b). They further show that when the Taylor principle holds, the deflationary steady state engenders an infinite number of suboptimal equilibria. As mentioned before, these equilibria can start arbitrarily close to the intended steady state (π_{ss}, x_{ss}) and still converge to (π_{zlb}, x_{zlb}). The same possibility is present in the setup I consider here. To construct equilibria analogous to those in Benhabib et al. (2001b), I use the dynamics for the three stages described above. For the next steps, refer to Figure 16. First pick two numbers q and r such that $r, q > T$ and $r - q > T$. Let $(\pi_b, x_b) = \partial\Omega \cap \Upsilon_{zlb}$.[39] Assume the Taylor principle

[39] If $\kappa\xi_\pi + \phi_1\xi_x = 0$, $\partial\Omega \cap \Upsilon_{zlb} = \emptyset$. Albeit not a general strategy to eliminate all non-optimal equilibria, picking ξ_x, ξ_π such that $\kappa\xi_\pi + \phi_1\xi_x = 0$ does preclude this particular class of equilibria from forming for any choice of f. This possibility was not present in Benhabib et al. (2001b), as their model did not have both inflation and output as state variables of the economy.

holds. Using $(\pi(r), x(r)) = (\pi_b, x_b)$ as the starting point, trace the dynamics of $(\pi(t), x(t))$ backward in time using the interest rate specified by equation (A.30) for a length of time q. As in Benhabib et al. (2001b), these equilibria can get arbitrarily close to the intended steady state: Because the dynamics of $(\pi(t), x(t))$ are unstable when going forward in time, they are stable backward in time and $(\pi(t), x(t))$ converges to (π_{ss}, x_{ss}) as $q \to \infty$.[40] At time $r - q$, trace the dynamics of $(\pi(t), x(t))$ backward in time using $(\pi(r - q), x(r - q))$ as the starting point and $i(t) = 0$ throughout, until $t = 0$, when the path reaches $(\pi(0), x(0))$. Of course, the natural rate is positive after T and negative before T, so the dynamics change from those of the second stage to those of the first. Note that in Figure 16, the gray flow lines in the background reflect the dynamics that prevail for $t \geq t_1$ only. Set $t_1 = r - q > T$. By construction, the path starting at $(\pi(0), x(0))$ reaches (π_b, x_b) at time r when following the interest rate rule in equation (15). Now going forward in time, for $t \geq r$, $(\pi(t), x(t)) \in \Upsilon_{zlb} \subset \Omega_{zlb}$, which means the economy travels on the *zlb* saddle path toward the unintended steady state (π_{zlb}, x_{zlb}). The path constructed is continuous and bounded and has consistent expectations: It is a rational expectations equilibrium. All equilibria in this class can be obtained by picking different q and r.

[40] This result is not immediate, since it may be possible that $(\pi(t), x(t))$ exits Ω_{ss} before getting close to (π_{ss}, x_{ss}) and then follows the Ω_{zlb} dynamics for which (π_{ss}, x_{ss}) is no longer a sink (flowing backward in time). However, I show in Appendix B.6, Lemma 2, item (d) that this never happens. For all q, the path of $(\pi(t), x(t))$ remains entirely in Ω_{ss}.

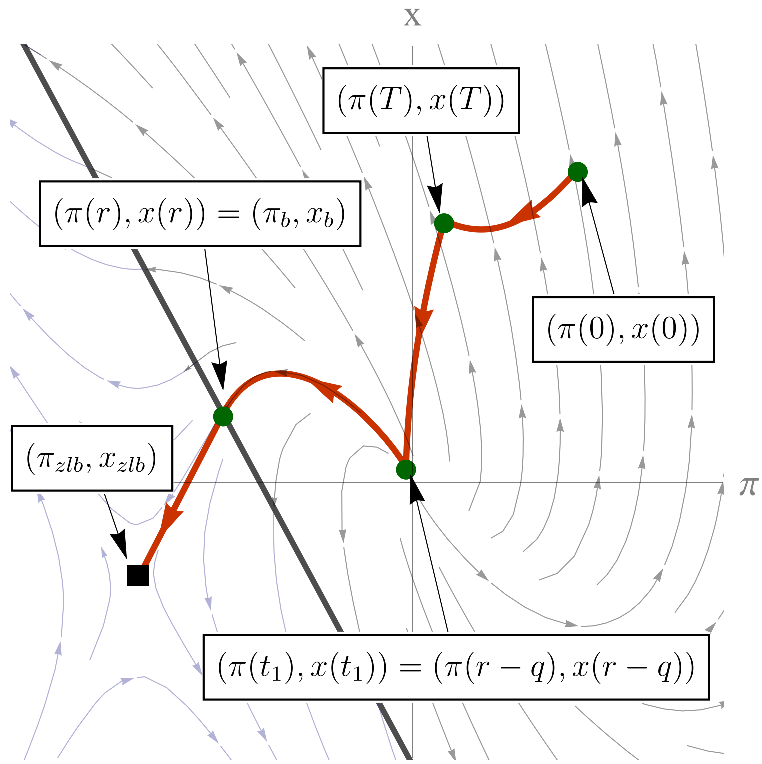

Figure 17: An equilibrium analogous to the one studied by Benhabib et al. (2001b). The flow lines in the background correspond to the dynamics after liftoff, which occurs at t_1. Because the Taylor principle holds, there is a deflationary steady state (π_{zlb}, x_{zlb}), shown as a black square. At time t_1, even though the economy is outside the ZLB and can get arbitrarily close to the "desired" steady state $(\pi_{ss}, x_{ss}) = (0, 0)$, it still converges to the "unintended" steady state (π_{zlb}, x_{zlb}). At time r, the economy enters the ZLB and stays there ($i(t)$=0) forever after.